A De Bruijn – Erdős Theorem in Graphs?

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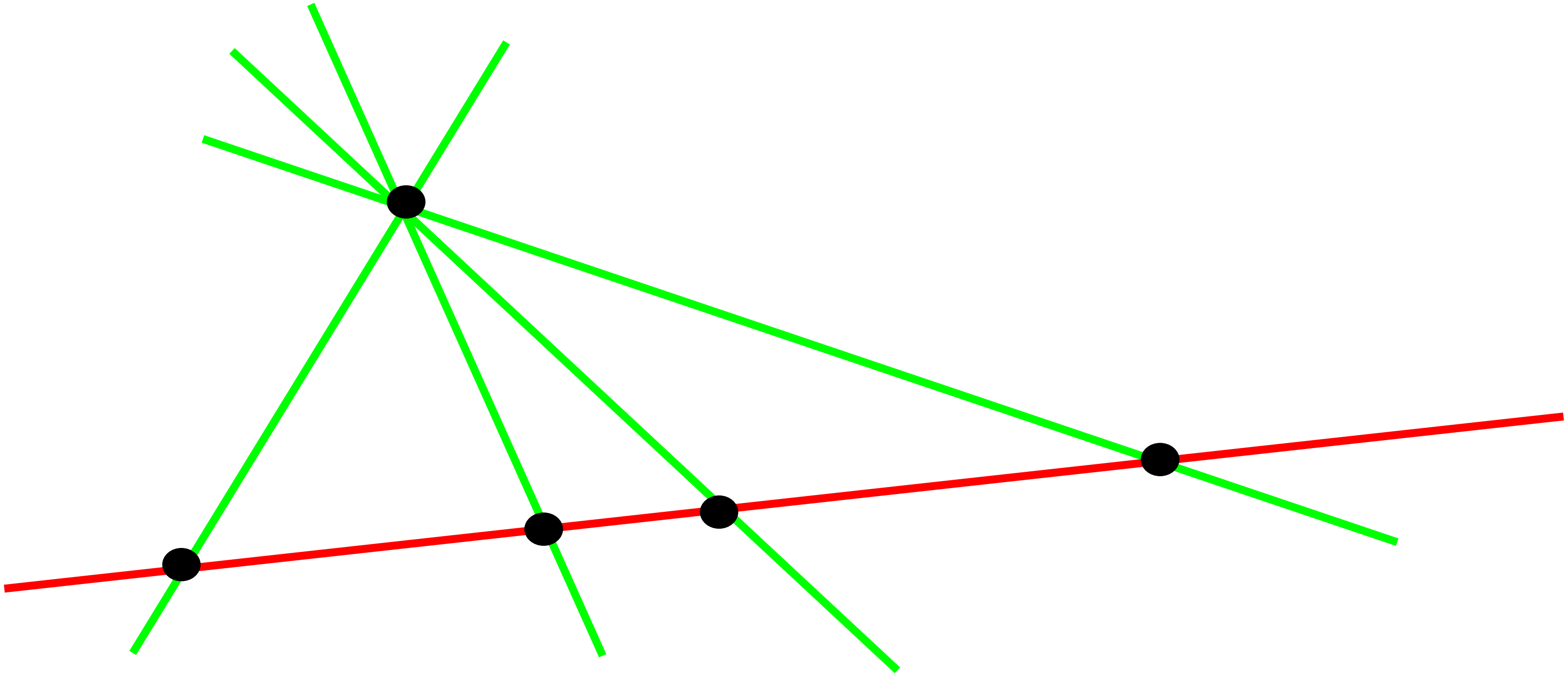

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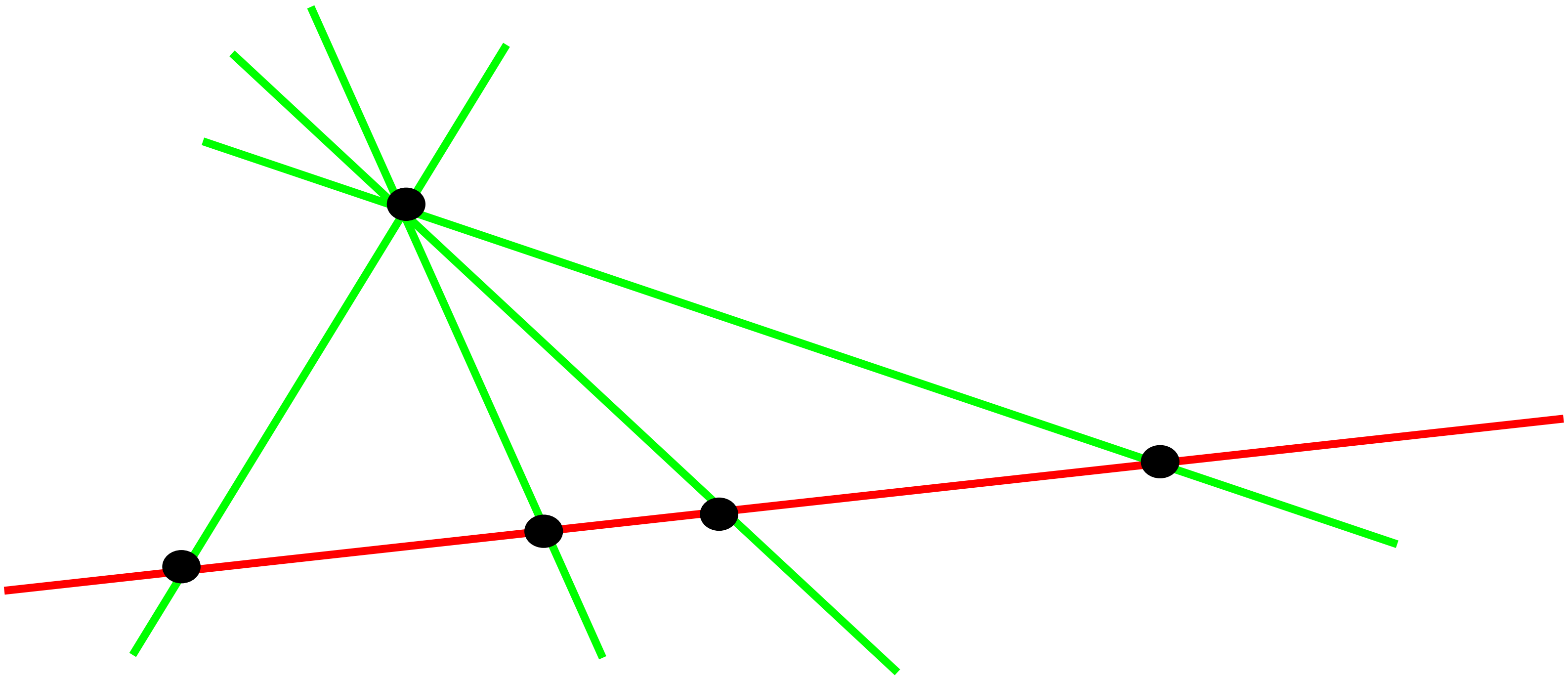

near-pencil

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Three point collinearity, *American Mathematical Monthly* **50**, p. 65

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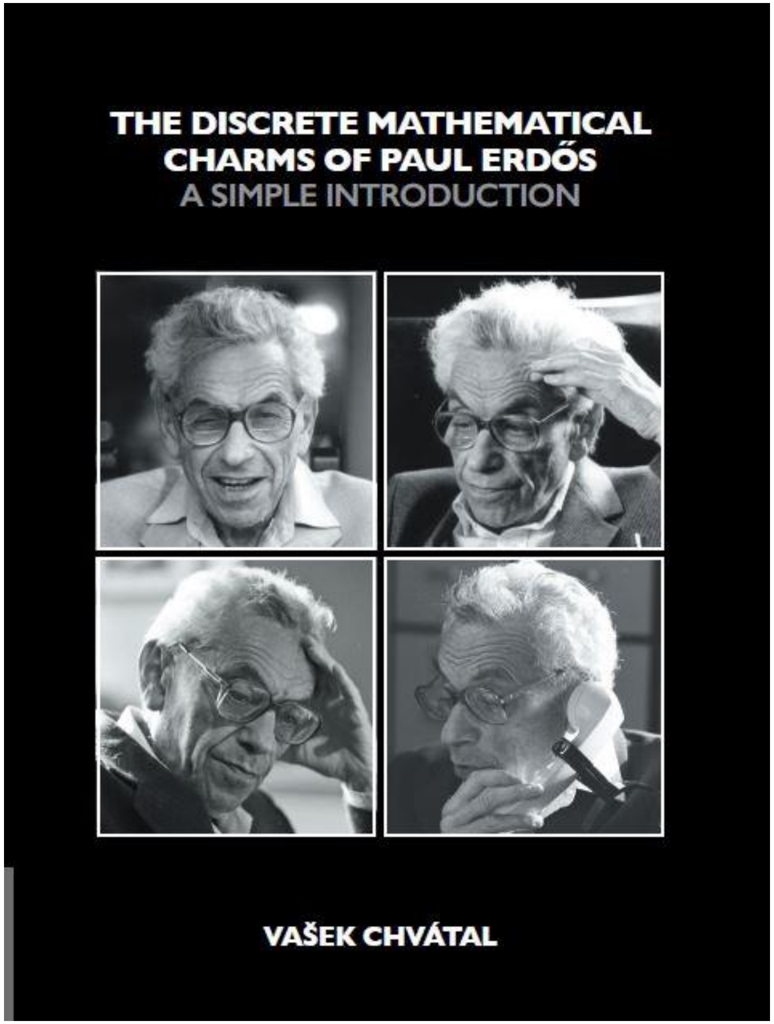
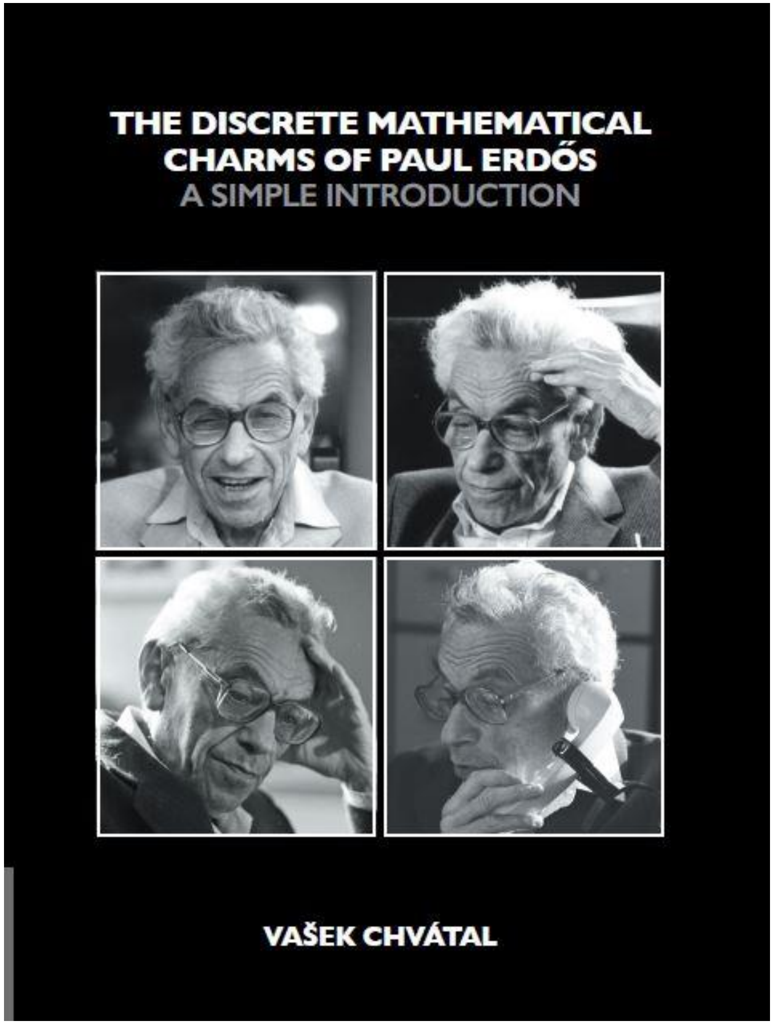
but often referred to (incorrectly, of course) as a "de Bruijn-Erdős Theorem"

because it is a special case of a (far more general) theorem in

On a combinatorial problem, *Indag. Math.* **10** (1948), 421--423



Cambridge University Press, August 2021



2 DISCRETE GEOMETRY AND SPINOFFS

2.1 The Happy Ending Theorem

2.2 The Sylvester–Gallai Theorem

2.3 A De Bruijn–Erdős Theorem

2.4 Other Proofs of the De Bruijn–Erdős Theorem

- 2.4.1 Hanani
- 2.4.2 Motzkin
- 2.4.3 Ryser
- 2.4.4 Basterfield, Kelly, Conway

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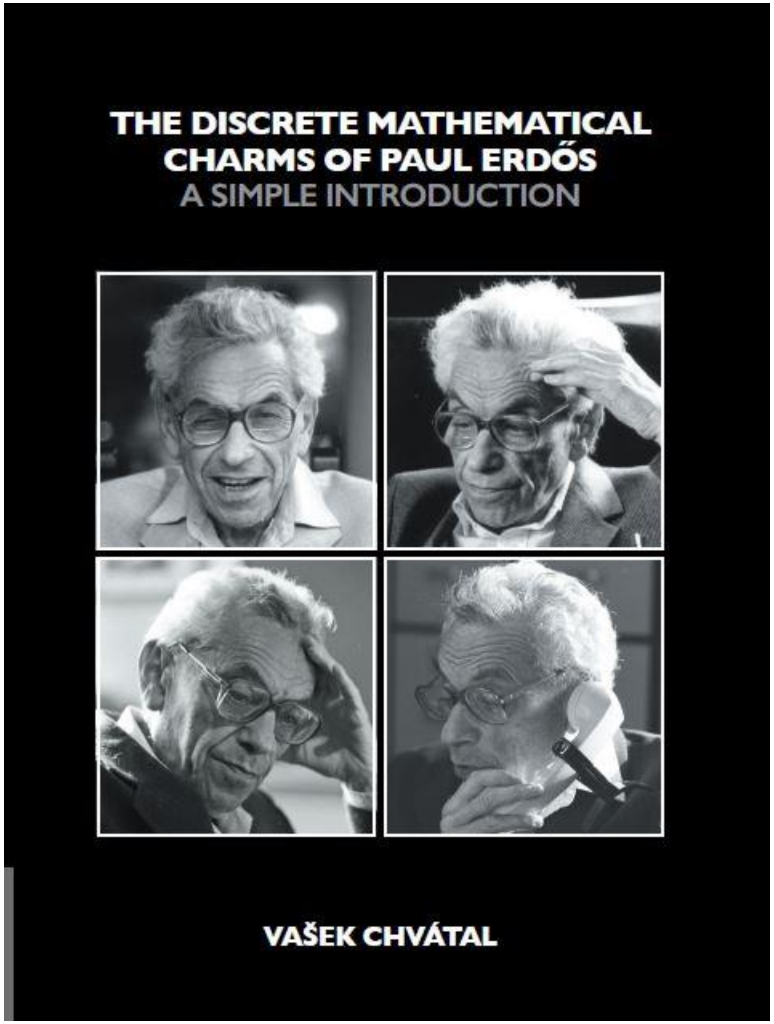

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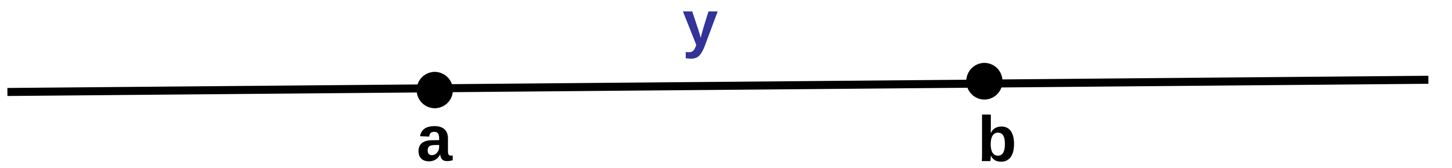
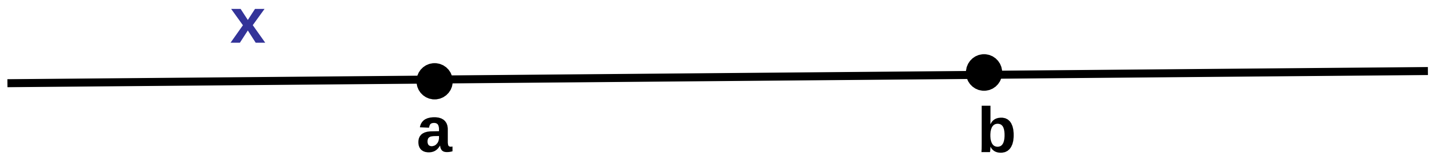
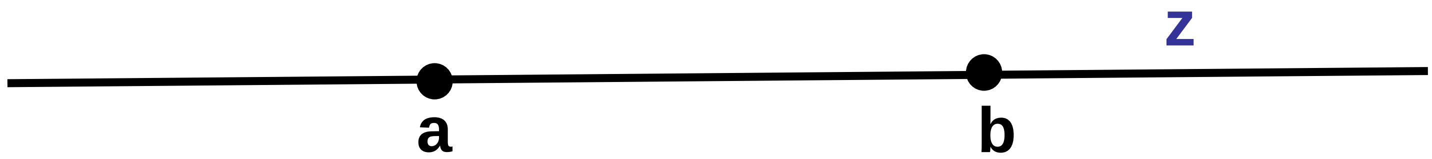
Question (Xiaomin Chen and V.C. 2006):

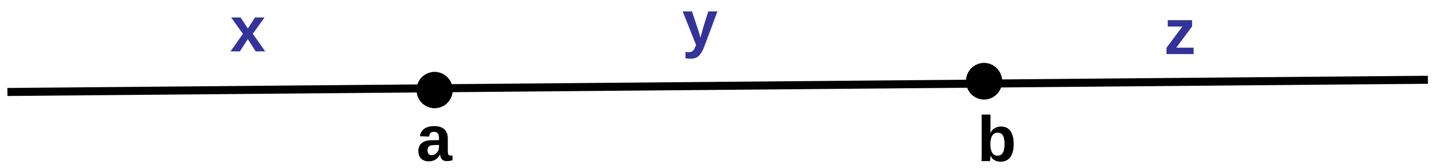
True or false? In every metric space on n points ($n > 1$), there are at least n distinct lines or else some line consists of all these n points.

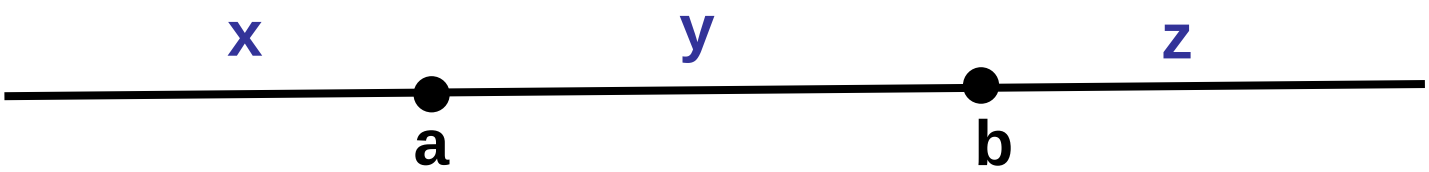

a
b


y
a
b


x
a
b

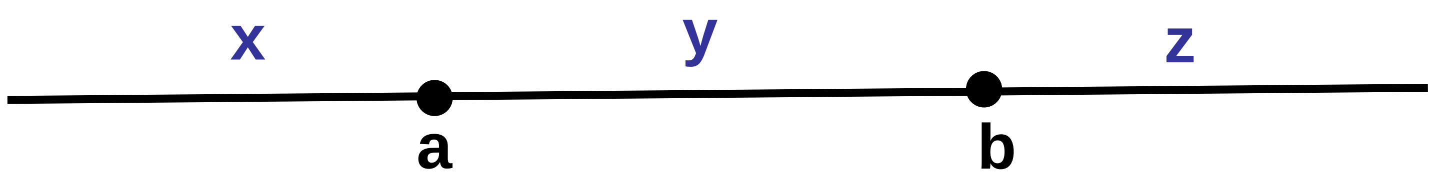

z
a
b


x
y
z
a
b



Observation

Line *ab* consists of
all points *x* such that *dist(x,a)*+*dist(a,b)*=*dist(x,b)*,
all points *y* such that *dist(a,y)*+*dist(y,b)*=*dist(a,b)*,
all points *z* such that *dist(a,b)*+*dist(b,z)*=*dist(a,z)*.



Observation

Line *ab* consists of
all points *x* such that *dist(x,a)*+*dist(a,b)*=*dist(x,b),*
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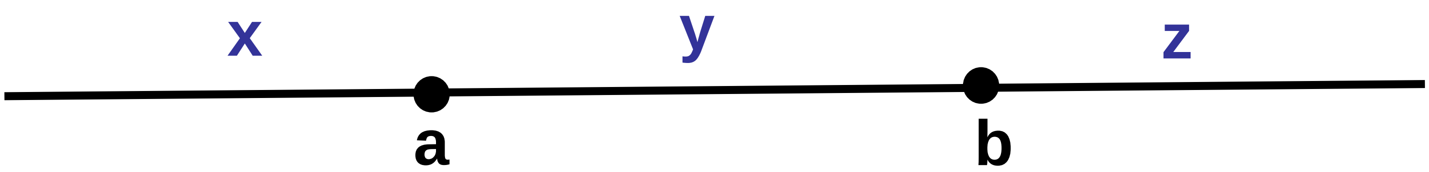
all points *z* such that *dist(a,b)*+*dist(b,z)*=*dist(a,z).*

This can be taken for a definition of a line L(*ab*)

in an arbitrary metric space



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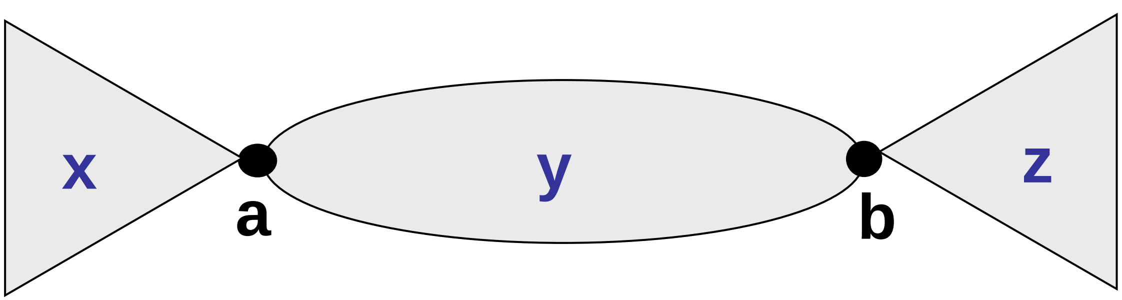

Lines in graphs can be exotic

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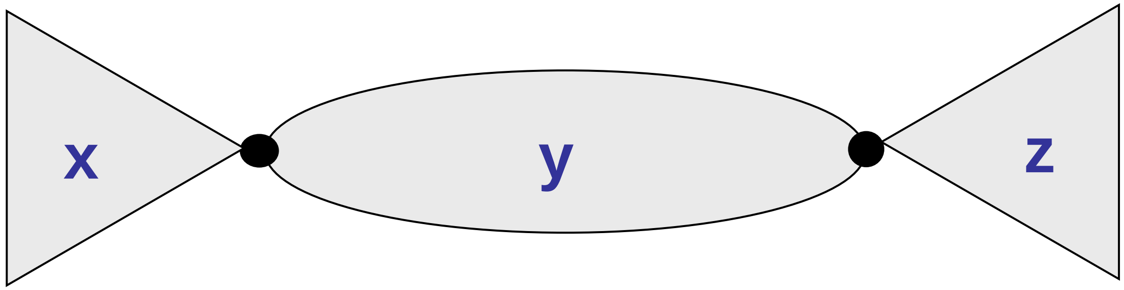
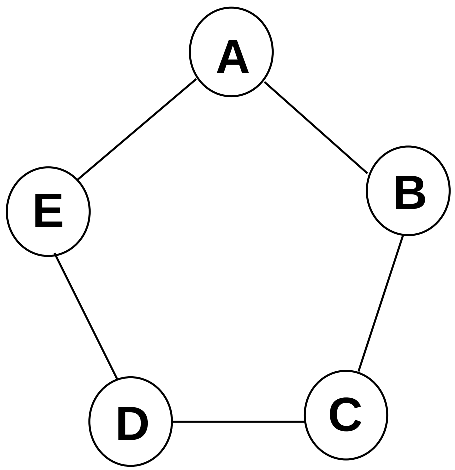
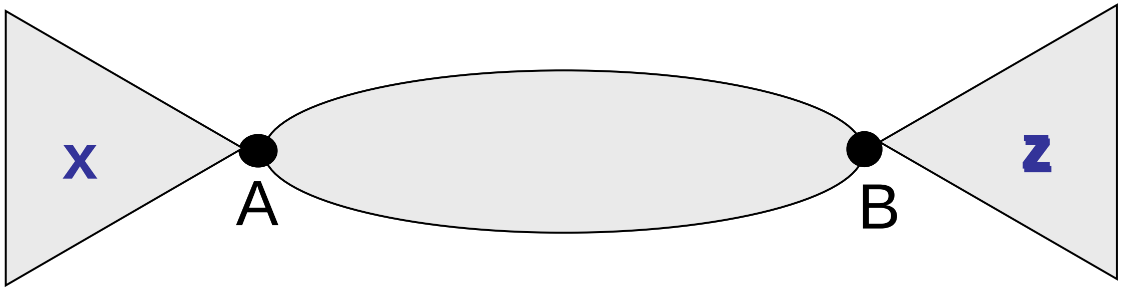
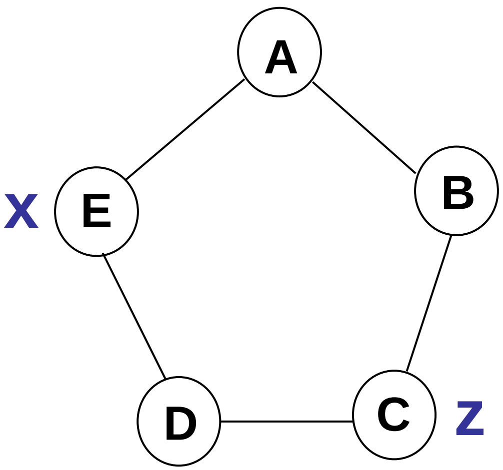
One line can hide another!

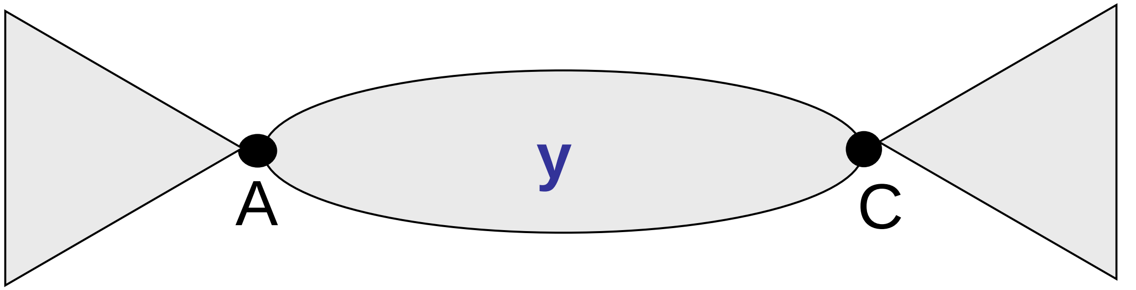

x
y
z

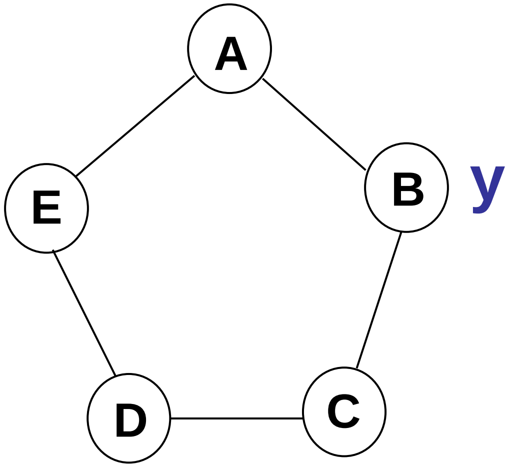

A
B
C
D
E



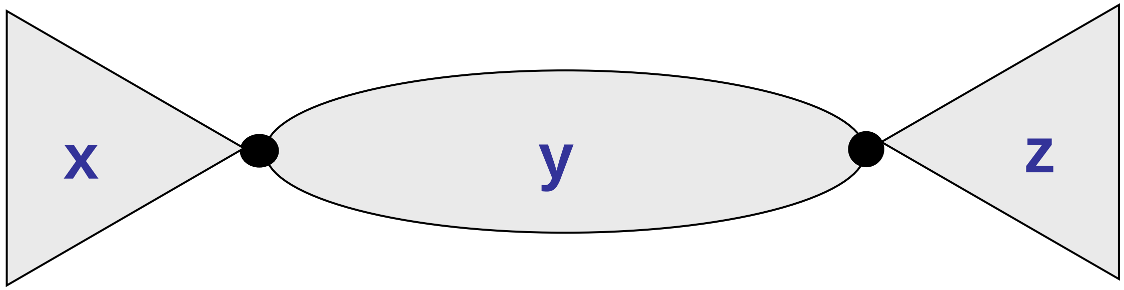


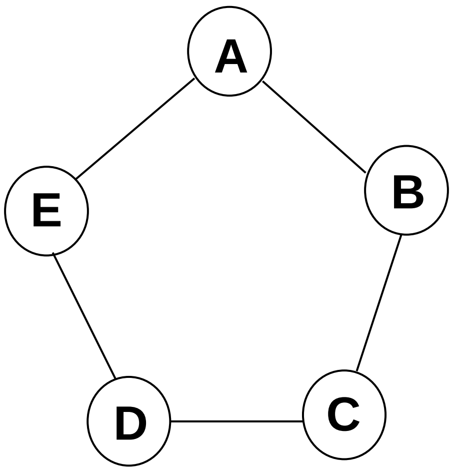
$L(AB) = \{E,A,B,C\}$





L(AC) = {A,B,C}





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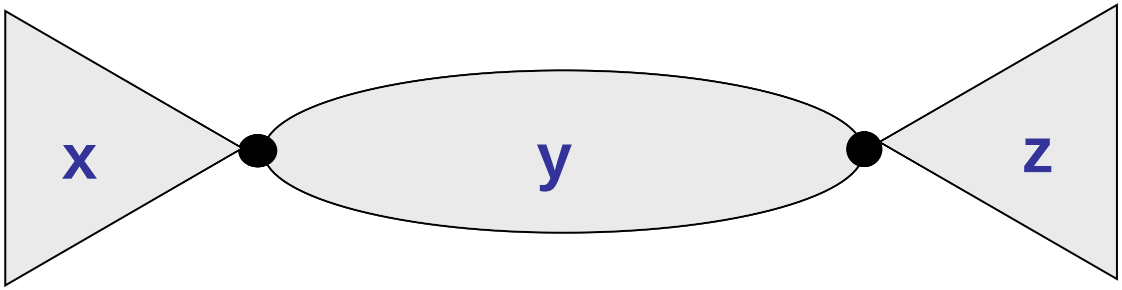

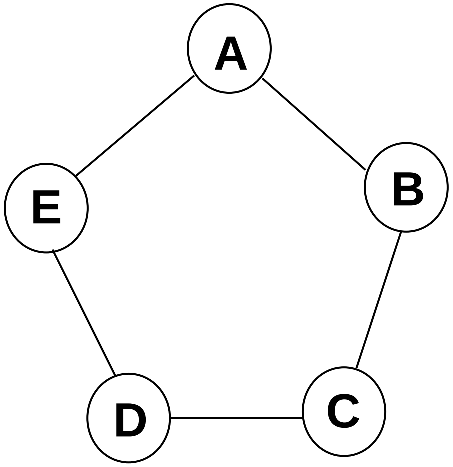

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L(AC) = {A,B,C}

One line can hide another!

Theorem (Xiaomin Chen, Guangda Huzhang, Peihan Miao, Kuan Yang 2015):

In almost all graphs, no line is a proper superset of another.

True or false? In every metric space on n points ($n > 1$), there are at least n distinct lines or else some line consists of all these n points.

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Theorem (Pierre Aboulker, Xiaomin Chen, Guangda Huzhang, Rohan Kapadia, Cathryn Supko 2014):

In all connected graphs on n vertices ($n > 1$), there are $\Omega(n^{4/7})$ distinct lines or else some line consists of all n vertices.

True or false? In all connected graphs on n vertices ($n > 1$), there are at least n distinct lines or else some line consists of all these n vertices.

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A partial answer (Xiaomin Chen, Ehsan Chiniforooshan; Laurent Beaudou, Giacomo Kahn, Matthieu Rosenfeld 2018):

True in all *bisplit* graphs.

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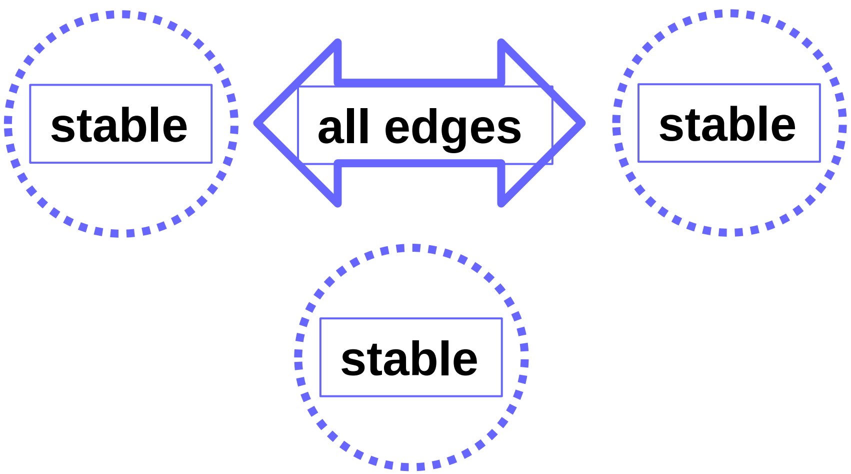
Bisplit graphs



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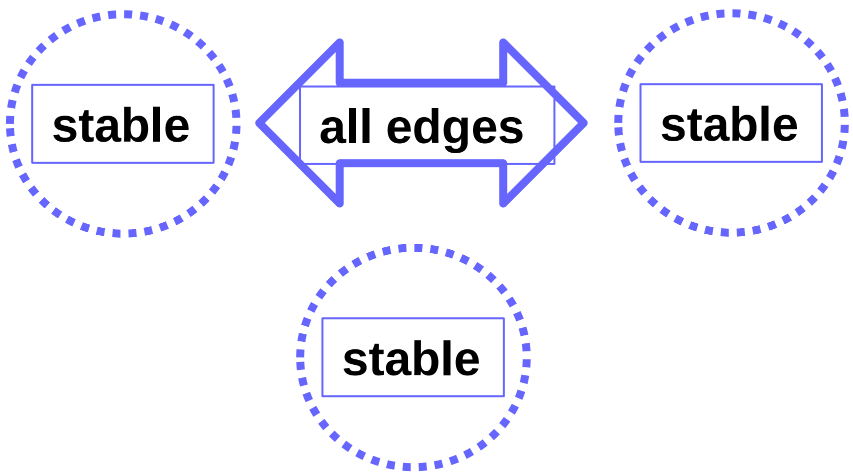

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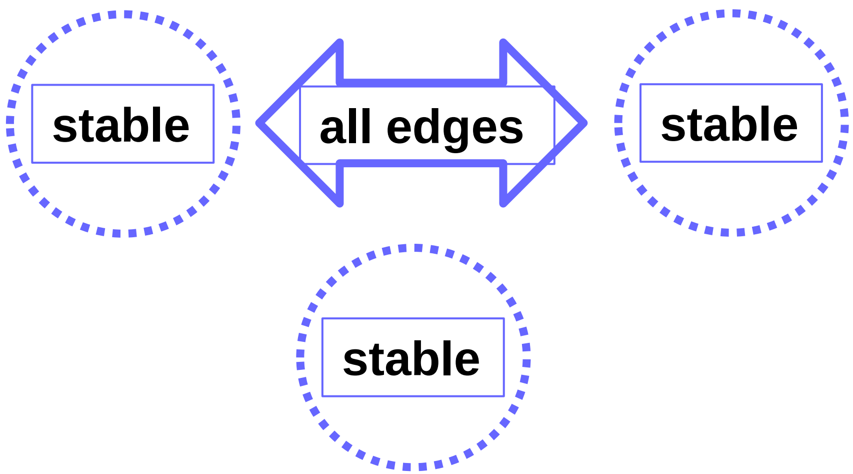

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True in all (house, hole)-free graphs.

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True in all hole-free (a.k.a. *weakly triangulated*) graphs?

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True in all (house, C_5)-free graphs?

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True in all complete multipartite graphs.

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True or false? In all connected graphs on n vertices ($n > 1$), complete multipartite graphs minimize the number of distinct lines.

If the answer to this last question happens to be ‘true’, then the conjectured lower bound n on the number of distinct lines (inherited from the plane geometry theorem of Erdős) is misleading and can be strengthened in the context of graphs!

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Yori Zwols: True for $n \leq 12$.

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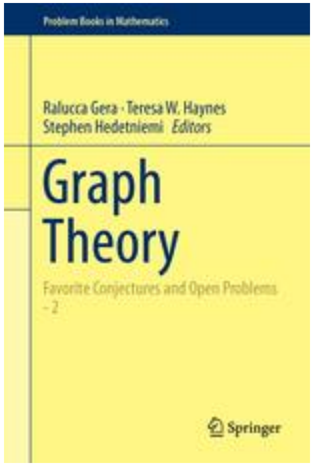
Yori Zwols: True for $n \leq 12$.

Each of $K(3,3,4)$, $K(1,3,3,3)$ and the complement of the Petersen graph has *15* lines.



Graph Theory
Favorite Conjectures and Open Problems - 2
Editors: **Gera**, Ralucca, **Haynes**, Teresa W., **Hedetniemi**, Stephen (Eds.)

Chapter 13:
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Happy birthday, Victor!