Event Structures, Median Graphs and CAT(0) Cube Complexes

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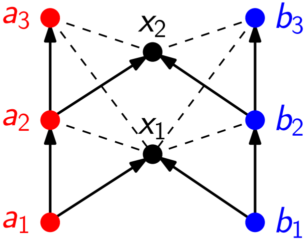
Joint work with Victor Chepoi

(Prime) Event Structures

An event structure is a triple $\mathcal{E} = (E, \leq, \#)$ where

- E is a set of events
- $\leq$ is a partial order on E
- $\#$ is a (binary) conflict relation on E
- $\downarrow e := \{e' \in E : e' \leq e\}$ is finite for any $e \in E$
- $e \# e'$ and $e' \leq e'' \implies e \# e''$



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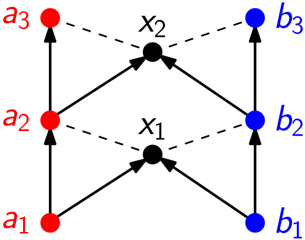
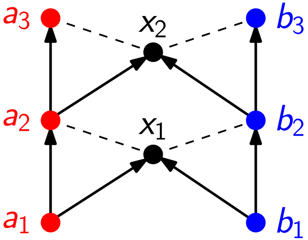


- e_1 and e_2 are in minimal conflict, $e_1 \#_\mu e_2$, if there is no event $e_1' \leq e_1$ such that $e_1' \# e_2$ (and vice versa)
- e_1 and e_2 are concurrent, $e_1 \| e_2$, if they are not comparable for $\leq$ and not in conflict

Configurations and Domains

A finite subset $c \subseteq E$ is a configuration if

- c is downward-closed: $e \in c$ and $e' \leq e \implies e' \in c$
- c is conflict-free: $e, e' \in c \implies (e, e') \notin \#$



- $\{a_1, a_2, b_1\}$ is a configuration
- $\{a_1, b_1, x_1\}$ is a configuration
- $\{a_1, a_2, b_2\}$ is not a configuration
- $\{a_1, a_2, b_1, x_1\}$ is not a configuration

Configurations and Domains

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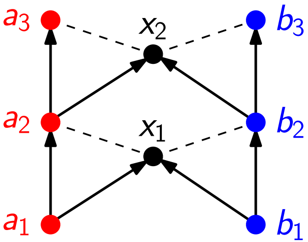
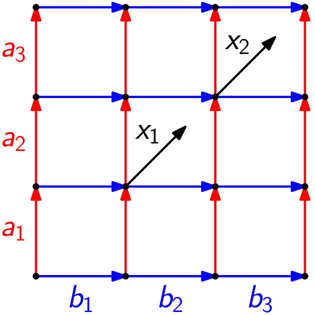
The domain $D(\mathcal{E})$ is a directed graph where

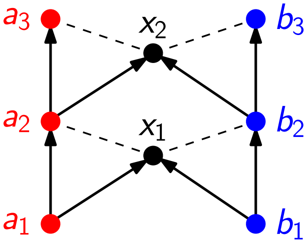
- the vertices of $D(\mathcal{E})$ are the configurations of $\mathcal{E}$
- $c \to c'$ if $c' = c \cup \{e\}$ for some event $e \notin c$

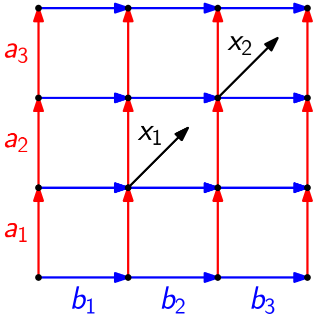




Labeled Event Structures

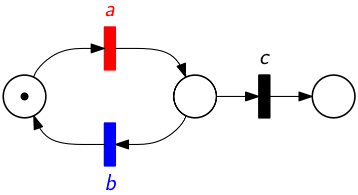
- A labeled event structure $(\mathcal{E}, \lambda)$ is an event structure $\mathcal{E}$ with a labeling $\lambda : E \to \Sigma$ (where Σ is a finite alphabet)
- λ is a nice labeling if $\lambda(e) \neq \lambda(e')$ when $e \| e'$ or $e \#_\mu e'$
- Equivalently, λ is a coloring of the edges of $D(\mathcal{E})$
  - Determinism: two edges with the same origin have distinct colors
  - Concurrency: two opposite edges of a square have the same color





Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



A 1-safe Petri Net is $N = (S, \Sigma, F, m_0)$

- S: places
- Σ: transitions
- $F \subseteq (S \times \Sigma) \cup (\Sigma \times S)$: flow relation
- $m_0 \subseteq S$: initial marking

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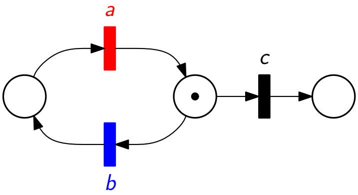

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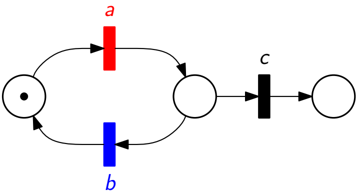

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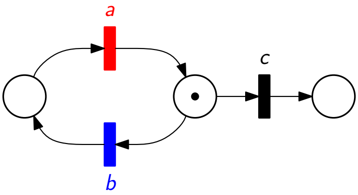

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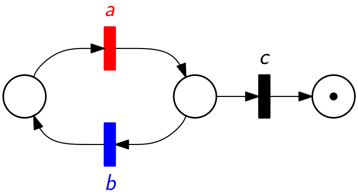

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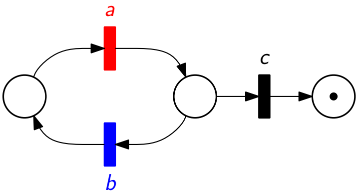

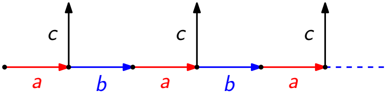

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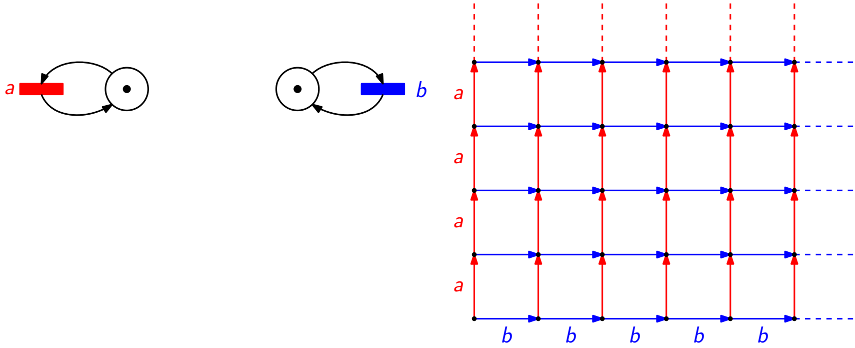

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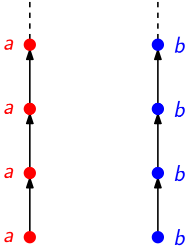

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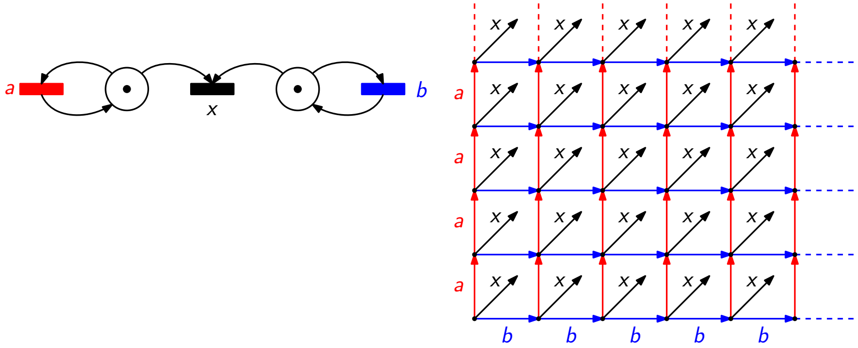

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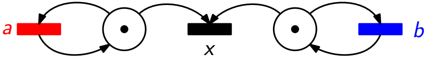
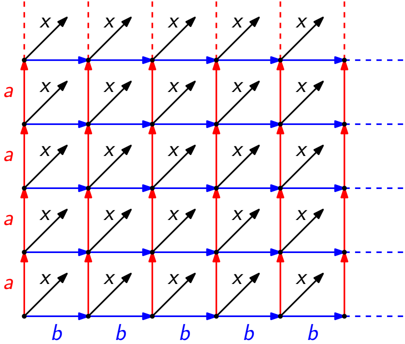
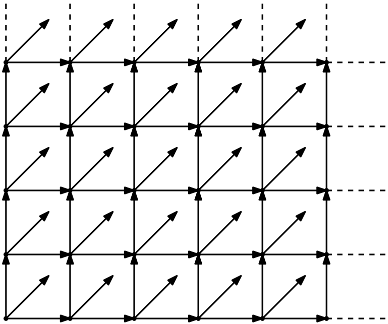




There is some regularity in the event structures arising from 1-safe Petri Nets

Regular Event Structures

- In $D(\mathcal{E})$, the future of a configuration c is the subgraph induced by the configurations reachable from c in $D(\mathcal{E})$
- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic futures



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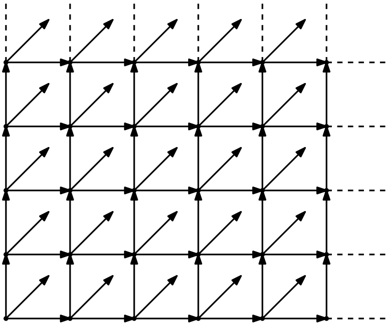
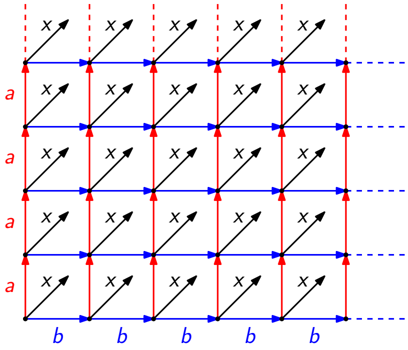
- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic futures
- A event structure $\mathcal{E}$ is regular if $D(\mathcal{E})$ has a finite degree and $R_{\mathcal{E}}$ has a finite number of equivalence classes



Regular Labeled Event Structures

If $(\mathcal{E}, \lambda)$ is a labeled event structure

- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic labeled futures
- $(\mathcal{E}, \lambda)$ is regular if λ is a nice labeling and $R_{\mathcal{E}}$ has a finite number of equivalence classes
- We say that λ is a regular nice labeling of $\mathcal{E}$



Event Structures and 1-safe Petri Nets

Any finite 1-safe Petri net gives a regular labeled event structure (and some extra properties)

Theorem [Thiagarajan '96] + [Morin '05]

Any regular labeled event structure $(\mathcal{E}, \lambda)$ is isomorphic to the event structure arising from a 1-safe Petri Net

Thiagarajan's regularity conjecture [Thiagarajan '96]

Any regular event structure $\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net

- True when $\mathcal{E}$ is conflict-free [Nielsen, Thiagarajan '02]
- True when the domain of $\mathcal{E}$ is context-free [Badouel, Darondeau, Raoult '99]

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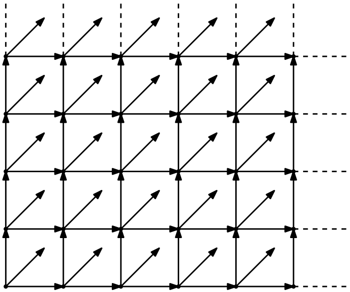
Any regular event structure $\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net

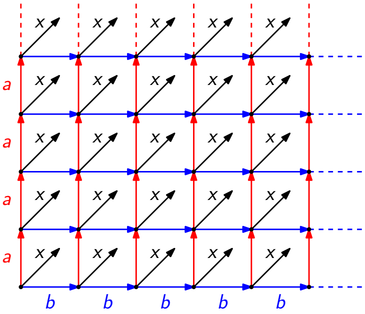
An equivalent condition

Any regular event structure $\mathcal{E}$ admits a regular nice labeling

Our Question

Given a regular event structure $\mathcal{E}$, can we always find a regular nice labeling of $\mathcal{E}$?





Our Results

Our Question

Given a regular event structure $\mathcal{E}$, can we always find a regular nice labeling of $\mathcal{E}$?

Theorem

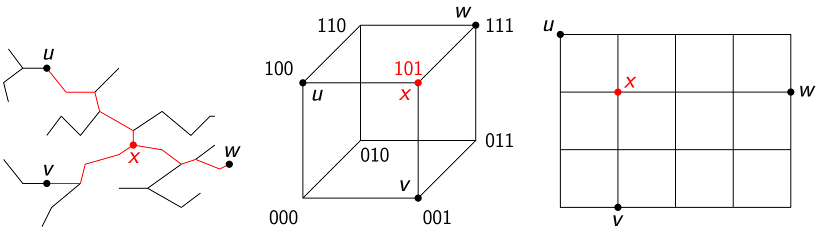
There exist regular event structures that do not *have any regular nice labeling.*

On the positive side, a correspondence between event structures that admit a regular nice labeling and the special cube complexes introduced by Haglund and Wise ('08)

Median graphs

Definition

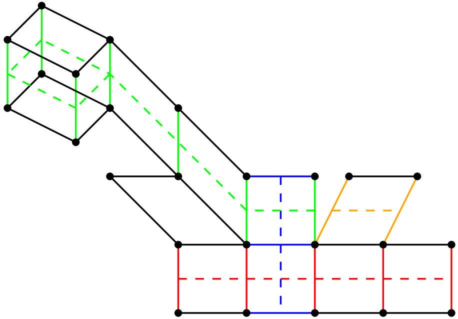
A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



Hyperplanes [Sageev]

In a median graph G, the Djoković-Winkler relation Θ is defined as follows:

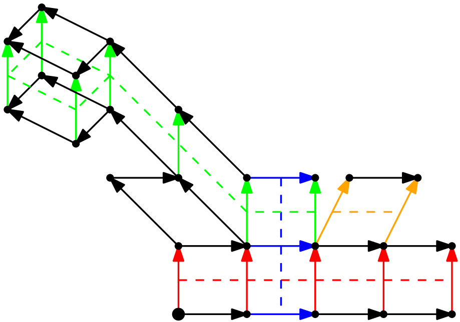
- $e_1 \Theta_1 e_2$ if e_1 and e_2 are two two opposite edges of a square
- $\Theta = \Theta_1^*$
- an hyperplane of G is an equivalence class of Θ



Median Graphs and Event Structures

Theorem [Barthélémy and Constantin '93]

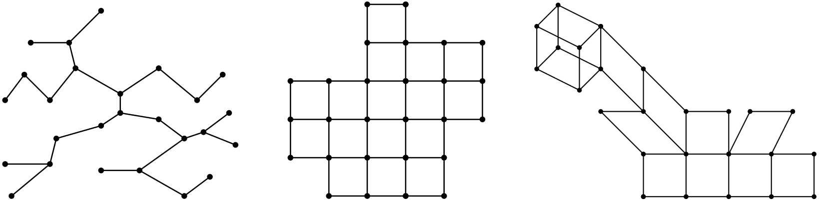
- $D(\mathcal{E})$ is a median graph (forgetting the orientation)
- Any pointed median graph is the domain of an event structure



CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

The 1-skeleton of X is the underlying graph $(V(X), E(X))$

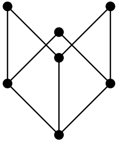


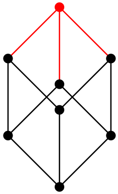
CAT(0) cube complexes

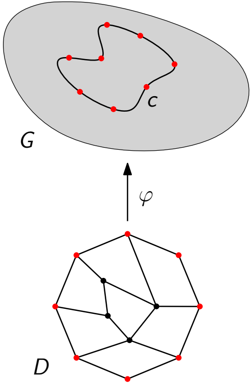
A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

A cube complex X is CAT(0) if

- X is nonpositively curved (NPC) [Gromov]
- X is simply connected







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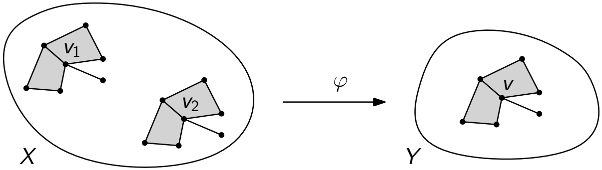
Theorem [Chepoi '00]

Median graphs are exactly the 1-skeletons of CAT(0) cube complexes

Covers of cube complexes

A cube complex X is a cover of the cube complex Y if there is a simplicial map $\varphi : V(X) \to V(Y)$ that is locally bijective



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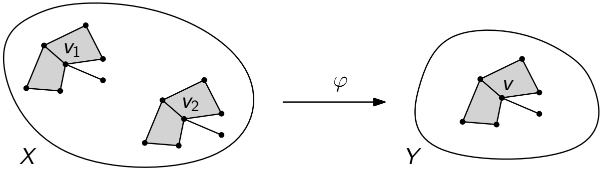

- Any complex X has a universal cover $\widetilde{X}$ such that if Y is a cover of X then $\widetilde{X}$ is a cover of Y
- X is simply connected if and only if $\widetilde{X} = X$

Constructing Event Structures from NPC complexes

Recall that a cube complex is Non Positively Curved (NPC) if it satisfies Gromov's cube condition

- Starting from a finite NPC cube complex X, its universal cover $\widetilde{X}$ is a CAT(0) cube complex
- We have a finite number of equivalence classes of vertices in $\widetilde{X}$ up to isomorphism

Problem

We need to have some orientations on the edges to get the domain of an event structure

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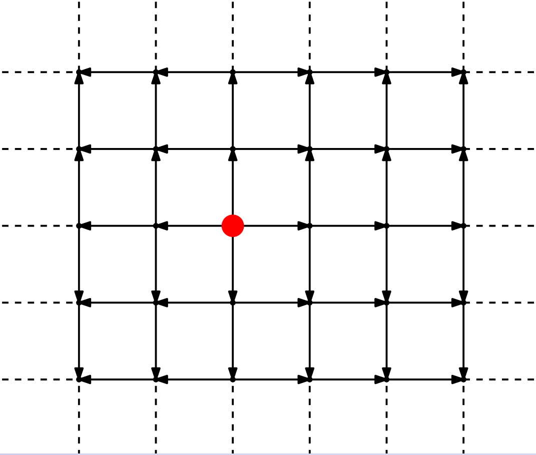
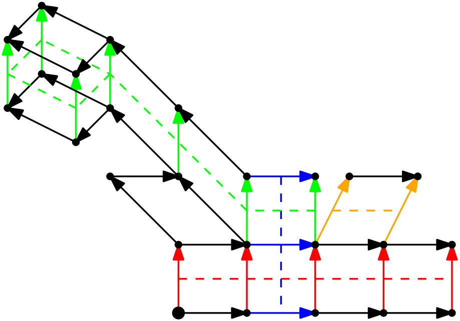


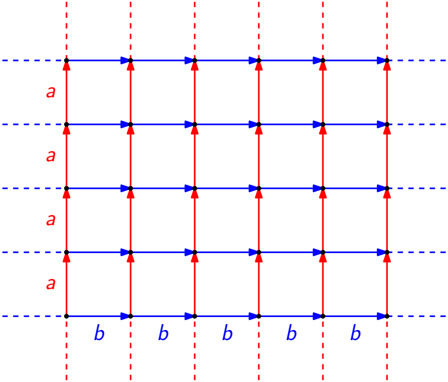
Directed NPC complexes

A directed NPC complex is a complex such that each edge is directed in such a way that two opposite edges of a square have the same direction



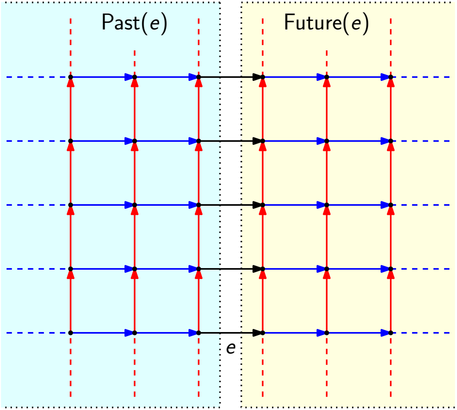
From Directed NPC complexes to Event Structures

- Starting from a finite directed NPC complex X, we construct its universal cover $\widetilde{X}$
- We have a finite number of classes of futures
- But vertices can have an infinite past ...



Cutting along Hyperplanes

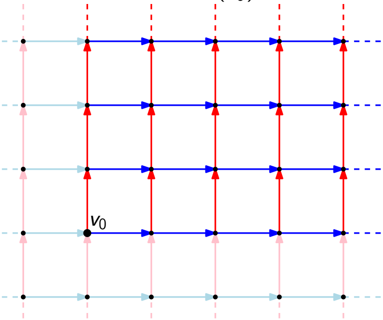
- In $\widetilde{X}$, edges belonging to the same hyperplane have the same orientation
- In a CAT(0) cube complex, hyperplanes are separators
  - For each hyperplane e, we define $\mathsf{Past}(e)$ and $\mathsf{Future}(e)$



Cutting along Hyperplanes

- In $\widetilde{X}$, edges belonging to the same hyperplane have the same orientation
- In a CAT(0) cube complex, hyperplanes are separators
- Pick $v_0 \in \widetilde{X}$, let $\mathsf{Past}(v_0) = \{e \mid v_0 \in \mathsf{Future}(e)\}$ and

$$\widetilde{X}_{v_0} = \bigcap_{e \in \mathsf{Past}(v_0)} \mathsf{Future}(e)$$



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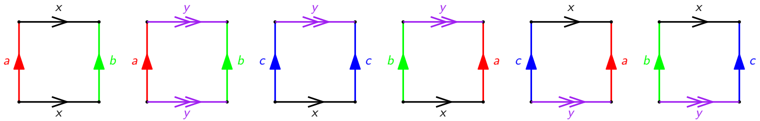
- Starting from a finite directed NPC complex X, we have constructed a pointed CAT(0) cube complex $\widetilde{X}_{v_0}$, i.e., the domain of an event structure
- The number of classes of futures is bounded by $|V(X)|$
- $\widetilde{X}_{v_0}$ is the domain of a regular event structure

Wise's directed NPC complex X

A colored directed NPC complex with 1 vertex, 2 "horizontal" edges (x and y), 3 "vertical" edges (a, b, and c), 6 squares



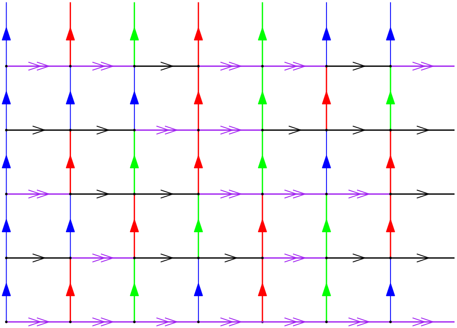
- it defines a square complex
- it is directed non positively curved

Warning!!

Colors have nothing to do with the labels of an event structure

An aperiodic tiling in the universal cover $\widetilde{X}$ of X

In the universal cover $\widetilde{X}$ of X, the quarter of plane defined by y^ω and c^ω is aperiodic

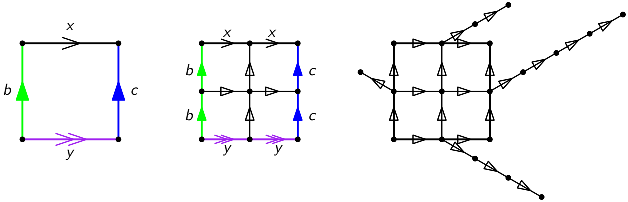


Proposition [Wise '96]

All horizontal words starting on the side of the quarter of plane are distinct

From $\widetilde{X}$ to a colorless domain $\widetilde{W_v}$

We encode the colors of the edges of X and we get a colorless directed cube complex W



Consider the universal cover $\widetilde{W}$ of W and the domain $\widetilde{W_v}$

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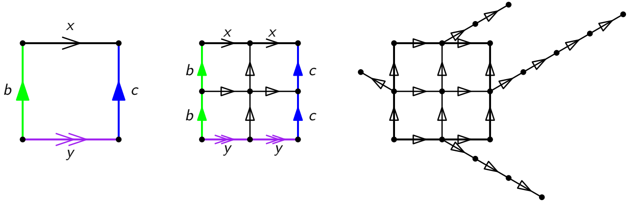
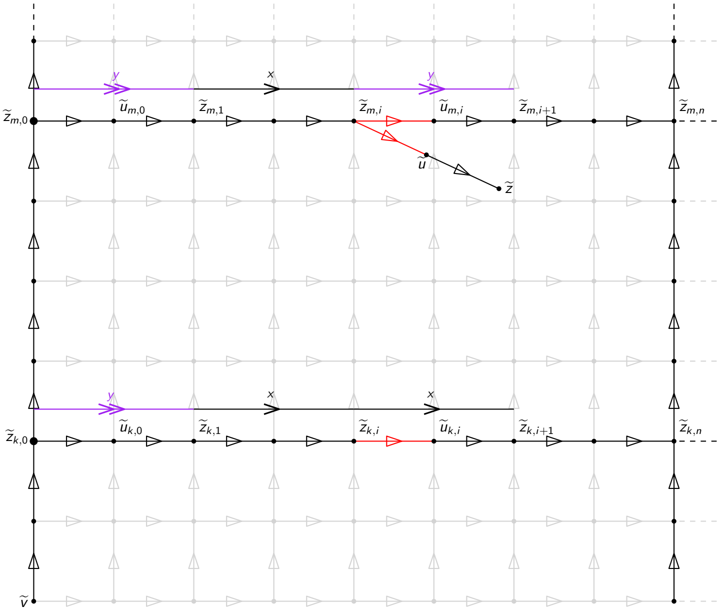


Consider the universal cover $\widetilde{W}$ of W and the domain $\widetilde{W}_v$

Theorem

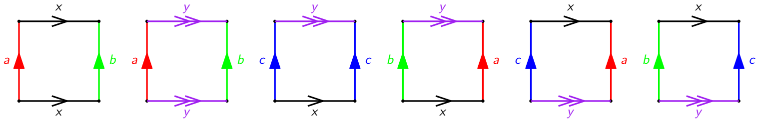
$\widetilde{W}_v$ is the domain of a regular event structure that does not admit a regular nice labeling

$\widetilde{W}_v$ has no regular nice labeling



Counterexamples arise from aperiodic tilesets

Wise's complex is obtained from a 4-way deterministic tileset



Counterexamples arise from aperiodic tilesets

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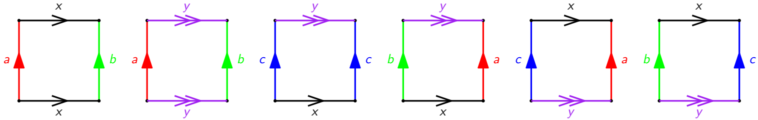


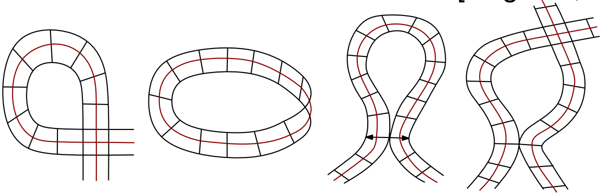
Any aperiodic 4-way deterministic tileset gives a counterexample to Thiagarajan's conjecture

Theorem

- There exists a 4-way deterministic aperiodic tileset [Kari, Papasoglu '99]
- Deciding if a 4-way deterministic tileset tiles the plane is undecidable [Lukkarila '09]

On the positive side: special cube complexes

A NPC complex is special if its hyperplanes behave nicely
[Haglund, Wise '08]



(a) no self-intersection
(b) no 1-sided hyperplane
(c) no direct self-osculation
(d) no interosculation

A finite NPC complex is virtually special if it has a finite cover that is special

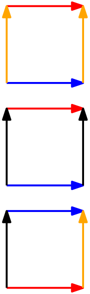
1-safe Petri nets and special cube complexes

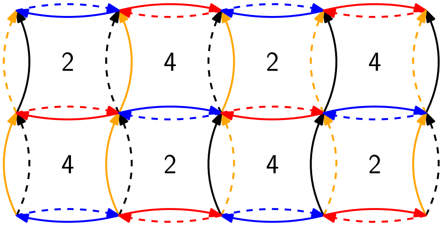
Theorem

An event structure $\mathcal{E}$ admits a regular nice labeling

$\Leftrightarrow$ *$\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net* *[Thiagarajan '96]*

$\Leftrightarrow$ *there exists a finite directed (virtually) special cube complex X such that $D(\mathcal{E}) \simeq \widetilde{X}_v$*





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Theorem [Agol'13]

If the universal cover $\widetilde{X}$ of a finite NPC complex X is hyperbolic, then X is virtually special

$\widetilde{X}$ is hyperbolic $\Leftrightarrow$ isometric square grids in $\widetilde{X}$ are bounded

Conclusion

- Counterexamples to Thiagarajan's regularity conjecture

- On the positive side, the regularity conjecture is true for particular ("antinomic") cases
  - conflict-free event structures [Nielsen, Thiagarajan '02]
  - context-free event domains [Badouel, Darondeau, Raoult '99]
  - domains obtained from finite NPC complexes with an hyperbolic universal cover

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  - domains obtained from finite NPC complexes with an hyperbolic universal cover

- Questions:
  - Is Thiagarajan's regularity conjecture true for hyperbolic domains?
  - Can we decide if a regular event structure admits a regular nice labeling?

Conclusion

- Nice connections between event structures and NPC complexes
  - CAT(0) cube complexes correspond to event structures
  - finite (virtually) special cube complexes correspond to regular event structures that admit a regular nice labeling
  - Question: Do finite NPC complexes correspond to regular event structures?

- Using these connections, we provided a counterexample to another conjecture of Thiagarajan ('14):
  - Given a labeled event structure $\mathcal{E} = (E, \leq, \#)$ that admits a regular nice labeling, MSO($\mathcal{E}$) is decidable iff $\mathcal{E}$ is grid-free

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Thank you!