A Cartan-Hadamard Theorem for median metric spaces

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8th December 2021

Let (M, σ) be a metric space.

Given $x, y, z \in M$, write $x.z.y$ to mean that

$$\sigma(x, y) = \sigma(x, z) + \sigma(z, y).$$

We say that z lies ***between*** x and y.
Let

$$[x, y] = \{z \in M \mid x.z.y\}$$

be the ***interval*** beween x and y.
Clearly, $[x, y] = [y, x]$ and $[x, x] = \{x\}$.

Given $x, y, z \in M$, let

$$\mathrm{Med}(x, y, z) = [x, y] \cap [y, z] \cap [z, x].$$

An element of $\mathrm{Med}(x, y, z)$ is a ***median*** of x, y, z.

Definition

(M, σ) is a ***median metric space*** if $\# \mathrm{Med}(x, y, z) = 1$ for all $x, y, z \in M$.

In this case, we write xyz for the unique median of x, y, z.
This gives us a symmetric ternary operation on M
(i.e. $xyz = yxz = yzx$).

Remark

With this structure M is a "median algebra": that is $xxy = x$ and $xy(xzw) = xz(xyw)$ for all $x, y, z, w \in M$.

Note that $x.z.y \Leftrightarrow z = xyz$.

Facts

(1) The completion of a median metric is a median metric.

(2) The l^1 direct product of median metrics is a median metric.

(3) A complete connected median metric space is geodesic and simply connected.

Definition

A ***geodesic*** from x to y in M is a path whose rectifiable length is equal to $\sigma(x, y)$.
A metric space is ***geodesic*** if any pair of points are connected by a geodesic.

Examples

of median metrics.

(1) $\mathbb{R}$ with the standard metric.

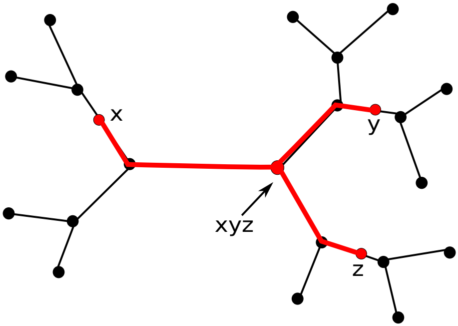
x.z.y

(2) $\mathbb{R}^n$ with the l^1 metric.

Infinite dimensional l^1-spaces.

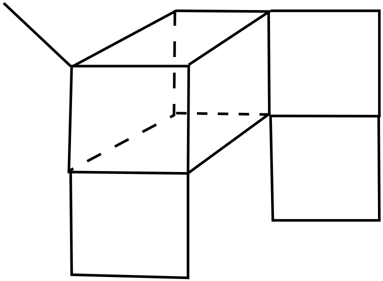
Examples

(3) Trees (simpicial trees, $\mathbb{R}$-trees...).
xyz is the centre of the "tripod" spanned by x, y, z.



Examples

(4) Cominatorial CAT(0) cube complexes with the l^1 metric on each cube.



(In the picture all the cubes have unit side-lengths, but we can allow sides to have different lengths.)
Note the usual "CAT(0)" metric is different: for that, we put the eucidean metric on each cube. (Hence "combinatorial".)

Examples

(5) median graphs.

A ***median graph*** is a connected graph such that the vertex set is a median metric space in the combinatorial metric.

These have been studied for a long time by numerous authors, and there are many equivalent formulations.

For example (Chepoi 2000): a graph is median if and only if it can be realised as the 1-skeleton of a CAT(0) cube complex.

Median metrics arise in various contexts, and have many applications.
For example:

Spaces with measured walls (Cherix, Martin, Valette)
These arise from group actions on various kind of spaces (eg. real or complex hyperbolic spaces...)
Applications include the Haagerup property (Chatterji, Druțu, Haglund...).

Asymptotic cones of various space (Gromov, van den Dries, Wilkie)
We rescale the metric and pass to a limit.
In various cases, this limit has a median metric structure.
Eg. mapping class groups of surfaces, Teichmüller space... (Behrstock, Druțu, Sapir...)

Quasimedian graphs (Mulder)
These are generalisations of median graphs.
They arise in group theory and phylogenetics.
One can canonically associate a median graph to a quasimedian graph (Genevois).

Guirardel cores
These arise in the theory of group splittings.

Etc...

Main result:

Definition

Given some $\epsilon > 0$, we say that a metric space is ϵ-***locally median*** if $\#\,\mathrm{Med}(x, y, z) = 1$ for all $x, y, z \in M$ with $\mathrm{diam}\{x, y, z\} \leq \epsilon$.
We say M is ***uniformly locally median*** if it is ϵ-locally median for some $\epsilon > 0$.

Theorem

Complete & geodesic (*) & simply connected & uniformly locally median
$\Rightarrow$ *median.*

(*): only really need that M is a path-metric space.

Question

Is uniformity really necessary?: maybe ***locally median*** is enough.
That is every point has a neighbourhood U such that $\#\,\mathrm{Med}(x, y, z) = 1$ for all $x, y, z \in U$.

In some cases, this is true. We'll say more at the end, if there's time...

Being (uniformly) locally median can be thought of as a kind of non-positive curvature condition.
In this way, the Theorem is analogous to the "Cartan-Hadamard Theorem" in riemannian geometry.

Remark

There are analogous results for other "non-positive conditions". For example:

(1) CAT(0) spaces (Busemann, Aleksandrov, Toponogov, Gromov)

(2) Injective metric space (Miesch).

(3) Weakly modular graphs (Chalopin, Chepoi, Hirai, Osajda).

The broad structure of our argument broadly follows that of (3), though the details a bit different.

Remark

A version of the main Theorem is given in the Thesis of Miesch, under the additional assumption that M admits a "conical geodesic bicombing": a global conditon which implies simply connected.

We will need:

Definition

(M, σ) is a ***modular metric space*** if $\mathrm{Med}(x, y, z) \neq \emptyset$ for all $x, y, z \in M$.

That is, medians always exist, but are not necesarily unique.

We quote the following:

Proposition

Complete & geodesic & modular & uniformly locally median $\Rightarrow$ *median.*

So we can focus on showing that M is modular.

Easy lemma:

Lemma

Let (M, σ) be a metric space.
Let $a, b, c, d \in M$ with $a.b.c$ and $b.c.d$.
Then $\sigma(b, c) \leq \sigma(a, d)$.

Proof.

We just add the inequalities:

$$\sigma(a, b) + \sigma(b, c) = \sigma(a, c) \leq \sigma(a, d) + \sigma(c, d)$$
$$\sigma(b, c) + \sigma(c, d) = \sigma(b, d) \leq \sigma(a, b) + \sigma(a, d)$$

and cancel $\sigma(a, b)$, $\sigma(c, d)$. □

Let (M, σ) be a metric space with some "basepoint", $p \in M$.

Consider the following properties:

$\nabla(p)$: for all $x, y \in M$ we have $\mathrm{Med}(p, x, y) \neq \varnothing$.

$\nabla(p, r)$: if $x, y \in M$ and $\sigma(x, y) \leq r$, then $\mathrm{Med}(p, x, y) \neq \varnothing$.

Thus "modular" means $(\forall p \in M)\, \nabla(p)$.

Lemma

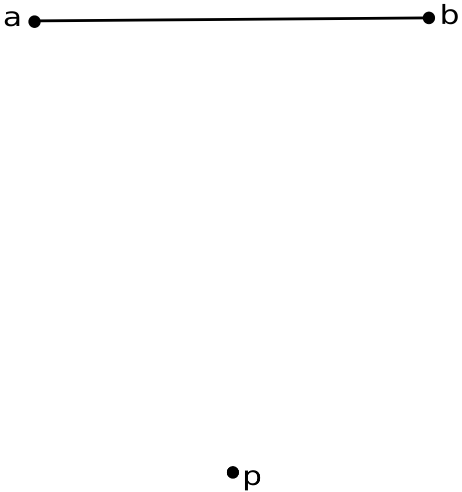
Geodesic & $(\exists r > 0)\, \nabla(p, r) \Rightarrow \nabla(p)$.

We sketch a proof in pictures.

Suppose $a, b \in M$.

Connect a, b by a geodesic, and cut it into segments of length at most $r/2$.

Apply $\nabla(p, r)$ repeatedly...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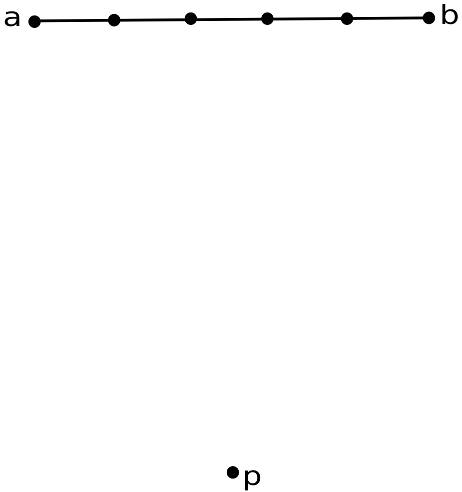
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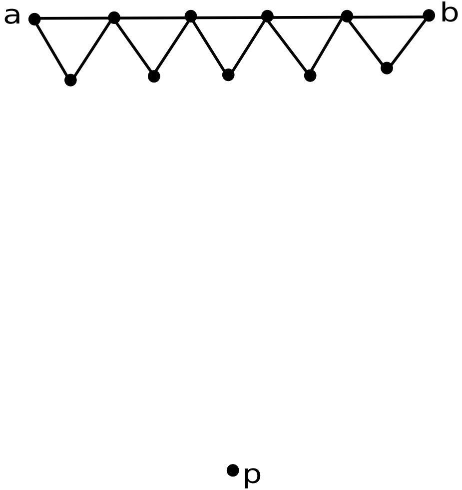
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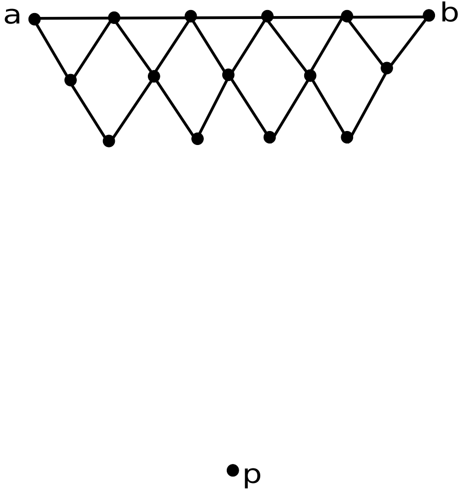
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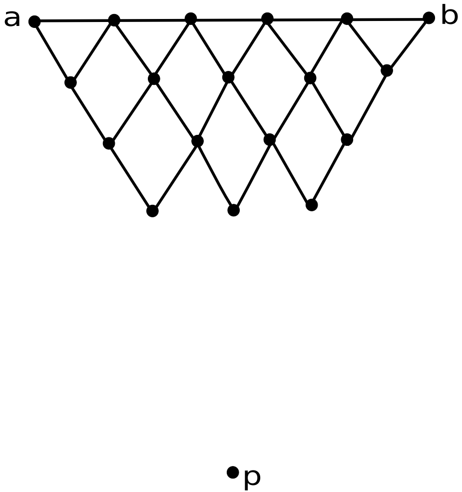
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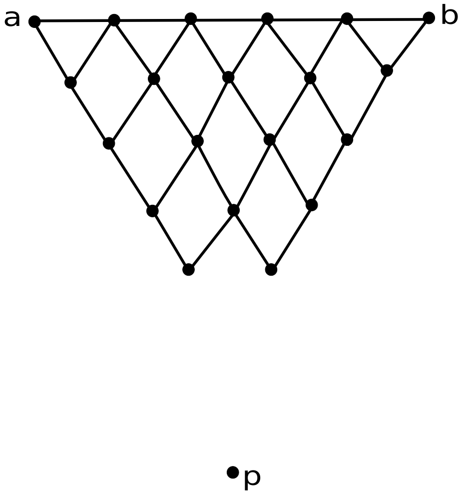
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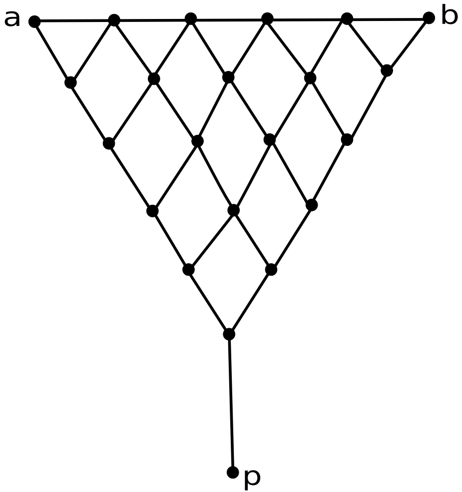
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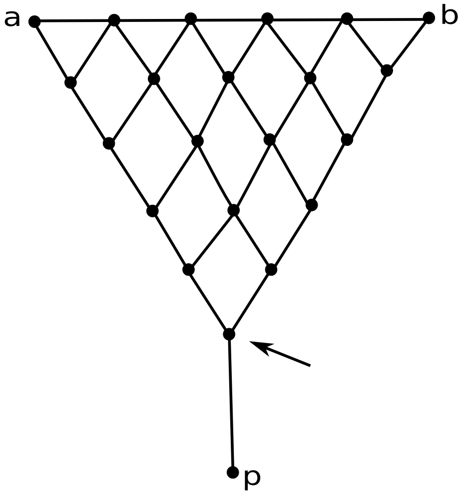
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Idea of proof of main Theorem.

Let $q \in M$.

We construct a geodesic metric space Y, with basepoint $p \in Y$, satisfying $\nabla(p)$, and a map $f : Y \longrightarrow M$, with a local isometry, with $f(p) = q$.

This is a covering space, hence a homeomorphism (since M is simply connected).

Since Y and M are both geodesic spaces, f is an isometry, so M satisfies $\nabla(q)$.

Let $\lambda = \epsilon/100$.

We construct Y as an increasing union,

$$Y = \bigcup_{n=1}^{\infty} X_n$$

with

$$p \in X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$$

with $X_n = N(p, n\lambda) \subseteq Y$, and with maps

$$f_n : X_n \longrightarrow M,$$

such that $f_n | X_m = f_m$ for $m \leq n$.

We construct X_n inductively, starting with $X_1 = N(p, \lambda)$, and $f_1 : X_1 \longrightarrow M$ the inclusion map.
Here is the inductive step.
To simplify notation, we write $X = X_n$, $X' = X_{n+1}$ and $R = n\lambda$.

Suppose we have constructed a metric space (X, ρ) with basepoint p, and $f : X \longrightarrow M$.

We suppose that there is a constant $R > 0$ such that the following hold.

(1): X is a geodesic space.

(2): $\rho(p, x) < R$ for all $x \in X$.

(3): X satisfies $\nabla(p)$.

(4): f is a local isometry.

(5): Any path β of length less than R in M starting at q lifts to a path α in X starting at p. (That is $f(\alpha) = \beta$.)

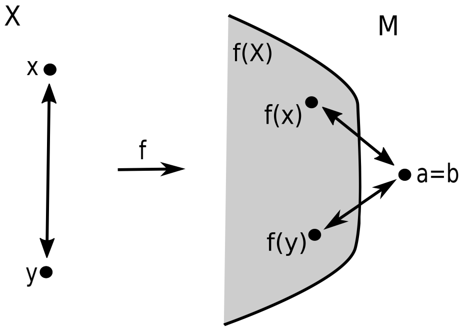
We aim to construct, a metric space (X', ρ') with basepoint p' satisfying the above, with R replaced by $R' := R + \lambda$, together with maps $\iota : X \longrightarrow X'$ and $f' : X' \longrightarrow M$, with $\iota(p) = p'$ and $f' \circ \iota = f$.

Let

$$P = \{(x, a) \in X \times M \mid \sigma(a, fx) < \lambda\}.$$

Define a relation, $\sim$, on P by

$$(x, a) \sim (y, b) \Leftrightarrow a = b \ \& \ \rho(x, y) < 2\lambda.$$



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Define a relation, $\sim$, on P by

$$(x, a) \sim (y, b) \Leftrightarrow a = b \;\&\; \rho(x, y) < 2\lambda.$$

Lemma

$\sim$ is an equivalence relation on P.

Let

$$X' = P/\sim.$$

Write $[[x, a]] \in X'$ for the $\sim$-class of (x, a).
Define

$$\iota : X \longrightarrow X'$$

by

$$\iota(x) = [[x, fx]].$$

Define

$$f' : X' \longrightarrow M$$

by

$$f'([[x, a]]) = a.$$

We now have a series of lemmas:

(1) ι is injective.

(2) $f : (X', \rho') \longrightarrow (M, \sigma)$ is a local isometry.

(3) ρ' is a geodesic metric on X'.

(4) $\iota X = N(p', R) \subseteq X'$.

(5) X' satisfies $\nabla(p')$.

(6) $\iota : (X, \rho) \longrightarrow (X', \rho')$ is isometric embedding.

Idea to show $\nabla(p)$.
Suppose we have

$$X \hookrightarrow X'$$

an isometric embedding (so ι is just inclusion), and

$$f : X' \longrightarrow M$$

with $f(p) = q$.

Suppose we have already achieved the following:

(1) $X = N(p, R) \subseteq X'$.

(2) X, X' are geodesic spaces.

(3) X satisfies $\nabla(p)$.

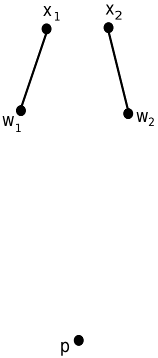
(4) f restricted to each ϵ-ball is an isometry.

We want to show that X' satisfies $\nabla(p)$.

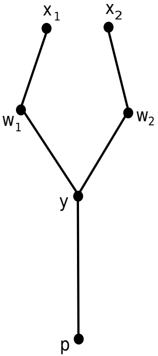
By earlier Proposition, it's enough to verify $\nabla(p, \lambda)$ in X'.

Let $x_1, x_2 \in X'$ with $\rho'(x_1, x_2) \leq \lambda$.

Let $x_1, x_2 \in X'$ with $\rho'(x_1, x_2) \leq \lambda$.
Since X' is geodesic, we can find $w_i \in X$ with $p.w_i.x_i$ and with $\rho'(w_i, x_i) < \lambda$.



Let $x_1, x_2 \in X'$ with $\rho'(x_1, x_2) \leq \lambda$.
Since X' is geodesic, we can find $w_i \in X$ with $\rho(w_i, x_i) < \lambda$.
By $\nabla(p)$ in X, there is some $y \in \mathrm{Med}(p, w_1, w_2)$.
Note that $p.w_i.x_i$ and $p.y.w_i$, so $p.y.x_i$.



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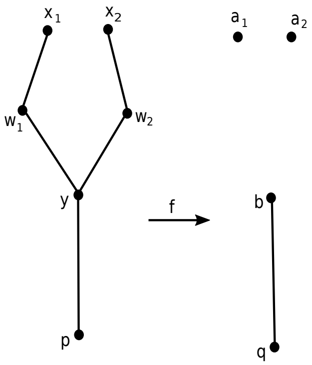
Let $a_i = f(x_i)$ and $b = f(y)$ in M.



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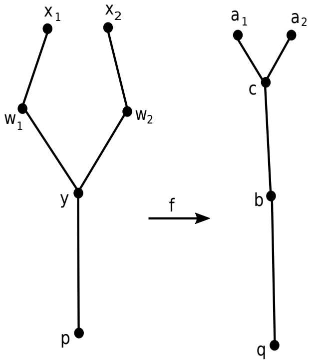
Note that $p.w_i.x_i$ and $p.y.w_i$, so $p.y.x_i$.
Let $a_i = f(x_i)$ and $b = f(y)$ in M.
Now $\mathrm{diam}\{b, a_1, a_2\} \leq 6\lambda < \epsilon$, so there is some $c \in \mathrm{Med}(b, a_1, a_2)$.



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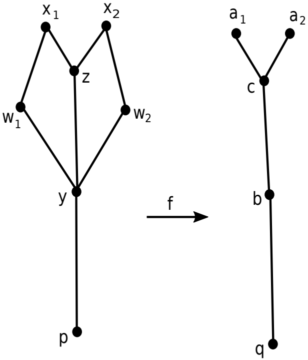
Note that $p.w_i.x_i$ and $p.y.w_i$, so $p.y.x_i$.
Let $a_i = f(x_i)$ and $b = f(y)$ in M.
Now $\mathrm{diam}\{b, a_1, a_2\} \leq 6\lambda < \epsilon$, so there is some $c \in \mathrm{Med}(b, a_1, a_2)$.
Connect q, c, a_1, a_2 by geodesics, and lift back to X' (since f' is a local isometry).
Let $z \in X'$ be be the preimage of c.
Then $z \in \mathrm{Med}(y, x_1, x_2)$, so $p.z.x_i$, so $z \in \mathrm{Med}(p, x_1, x_2)$.



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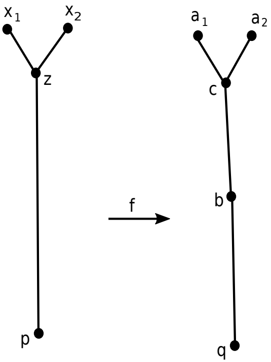

This doesn't quite work as stated...

We need to use "approximate medians", where betweenness is defined up to an arbitrarily small positive constant, etc.

The details get a bit technical.

We return to:

Question

Does the main result hold if we just assume M is locally median?
That is every point has a neighbourhood U such that $\# \operatorname{Med}(x, y, z) = 1$ for all $x, y, z \in U$.

In some cases this is true:

(1) If M is locally compact.

Fix some basepoint for all R there is some $\epsilon > 0$ so that M is $\epsilon(R)$-locally median on $N(p, R)$.

There is a sequence $R_n \to \infty$ such that $R_{n+1} \leq R_n + \epsilon(R_n)/100$ for all n.

Given $q \in M$, use the above construction to give a sequence

$$p \in X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$$

to get

$$Y = \bigcup_{n=1}^{\infty} X_n$$

with $X_n = N(p, R_n) \subseteq Y$ for all n.

We end up with an isometry $f : Y \longrightarrow M$ with $f(p) = q$ as before, so M satisfies $\nabla(q)$.

(2) It is also true if M satisfies the following "uniform simple connectedness" property:

For all $\epsilon > 0$, there is some $\delta = \delta(\epsilon) > 0$, such that any two paths of length less than δ connecting a pair of points, $a, b \in M$, are homotopic relative to a, b through paths of length at most ϵ.

(This certainly holds in a complete connected median metric space, with $\delta(\epsilon) = \epsilon$.)

Question

How can this fail?