Local certification of/on sparse graph classes

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joint works with
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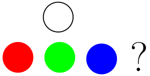
Disclaimer

All graphs are connected !

3-coloring and LOCAL model

All the vertices have a unique ID.

3-coloring and LOCAL model



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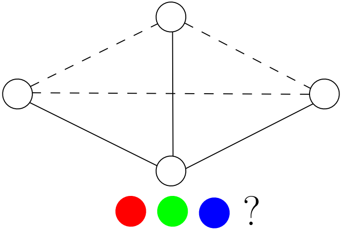

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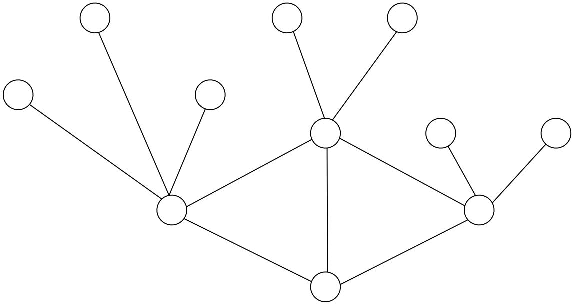

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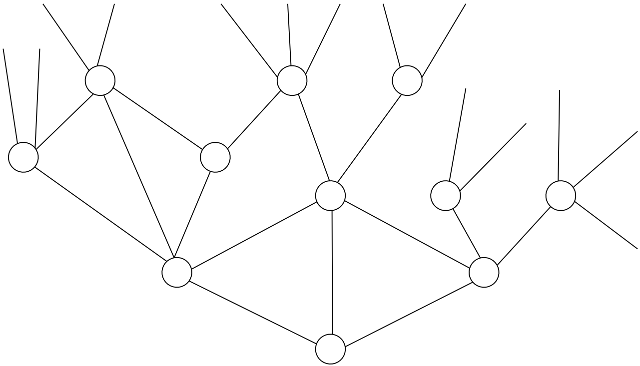

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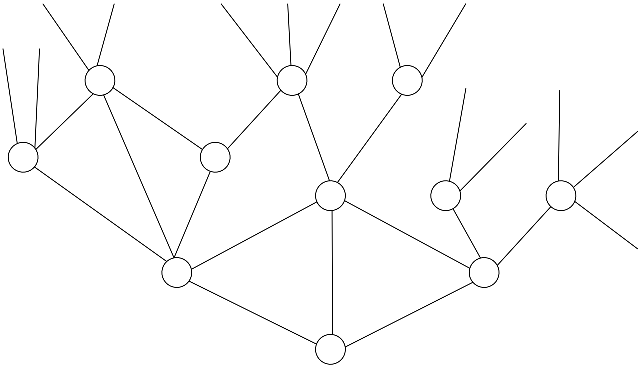

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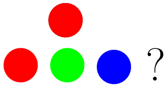
Question

Up to which distance do we have to look at to take a correct decision ?

Certify the 3-coloring



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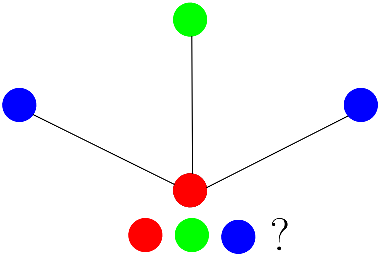

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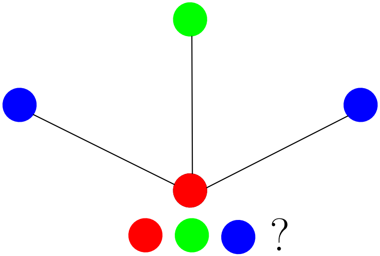

If vertices receive as labels their colors, we can check the coloring in the future by looking at vertices at distance 1 !

Local certification

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A property Π can be certified with $f(n)$ bits when :

- If Π is positive, there exists a certificate assignment to the nodes, each of size at most $f(n)$, such that all the nodes accept.
- If Π is negative, at least one node rejects for any possible certificate assignment.

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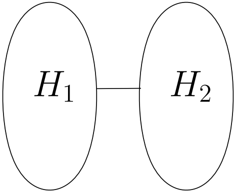
Idea :
Give the entire graph with IDs to each vertex.

Questions :

- Can we improve $\Omega(n^2)$ in general ?
- What is a decent lower bound ?
  $\rightarrow \Omega(\log n)$ (size of labels).

What can't be certified with small certificate ?

[Göös, Suomela '16] **Non trivial automorphism**



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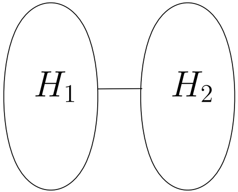

→ $\Omega(n^2)$ bits are needed.

[Censor-Hillel et al. '20] **Diameter 2.**
→ $\Omega(n)$ bits.

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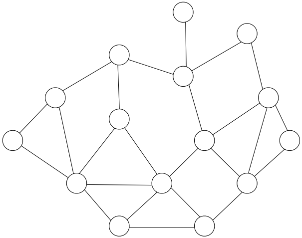

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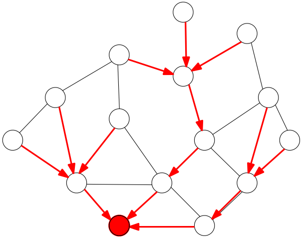

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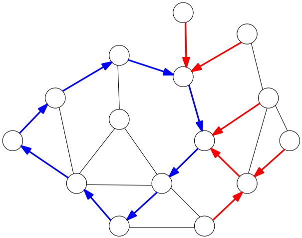

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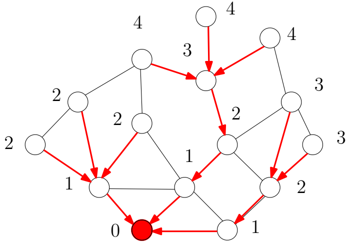

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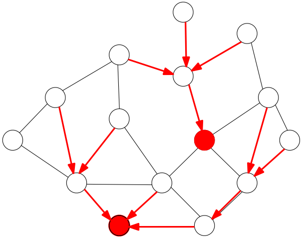

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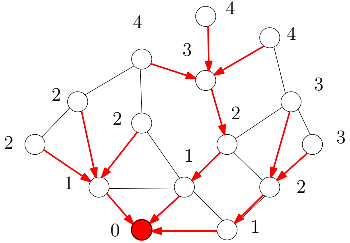

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On the way :
Certification of 2 and 3-connectivity, block cut trees, development of new tools...

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Theorem (B, Feuilloley, Pierron '21+)

- $\mathrm{td}(G) \leq k$ can be certified with $O(k \log n)$ bits.
- Every MSO formula can be certified with $O(\log n)$ bits on bounded treedepth graphs.

Certifying K_4-minor free graphs (I)

Lemma (B., Feuilloley, Pierron)

If H is 2-connected :
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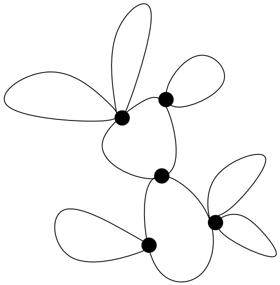

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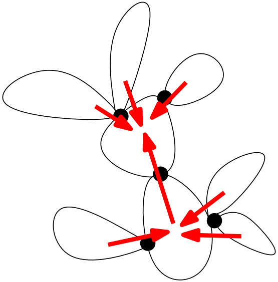

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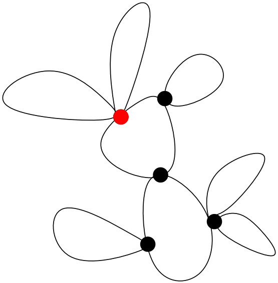

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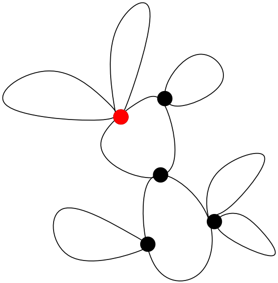

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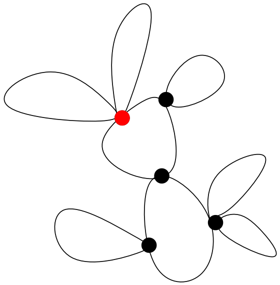
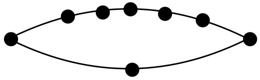
- Certify that 2-connected components are 2-connected.
- Give the H-certificate to each 2-connected component.



Ear decompositions $\Leftrightarrow$ 2-connected.

- Start from a cycle



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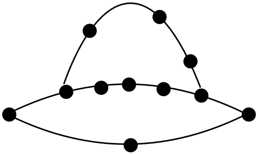

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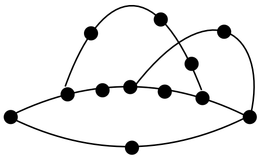

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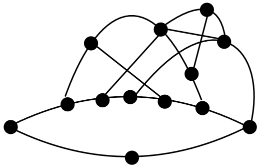

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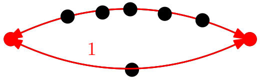

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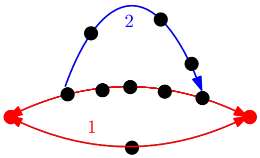

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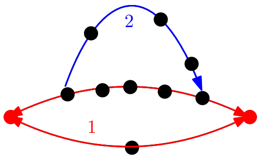

Theorem (Eppstein)

The following are equivalent :

- G is a 2-connected K_4-minor-free graph,
- G is a 2-connected series-parallel graphs,
- G has a nested ear decomposition.

Question 1 - Next step ?

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$\Rightarrow$ Certification of $(H + K_1)$-minor free graphs with $O(\log n)$ bits ?

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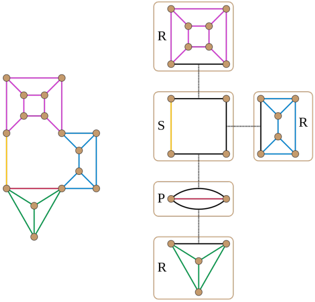
[Fraigniaud et al. '21] Certificate of size $O(\log^2 n)$.

Question 2 - SPQR trees partition



Source : wikipedia

- S node - Cycle of length at least 3.
- P node - Multigraph with 2 vertices and ≥ 3 edges.
- Q node - Single real edge.
- R node - 3-connected graph that is not S or P.

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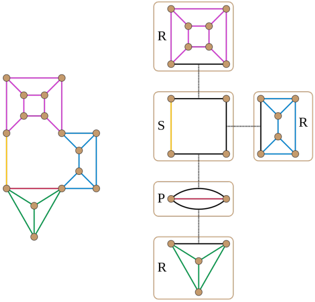

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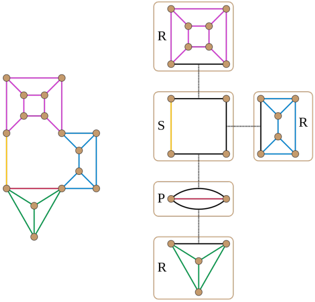

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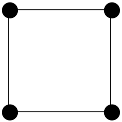

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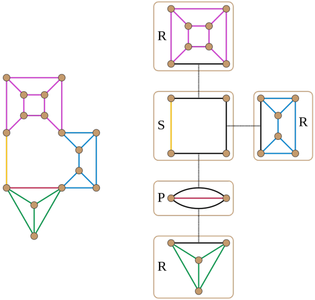

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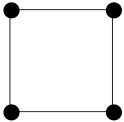

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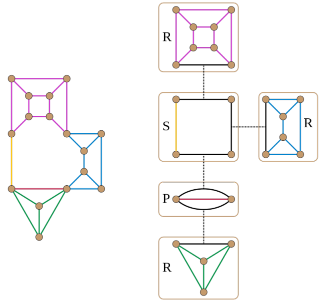

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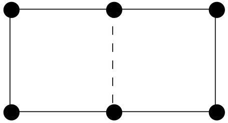
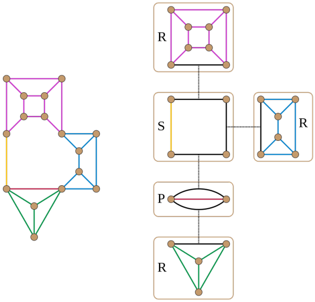


Question 2 - SPQR trees partition



Source : wikipedia

- S node - Cycle of length at least 3.
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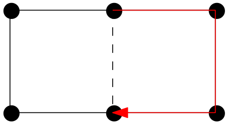

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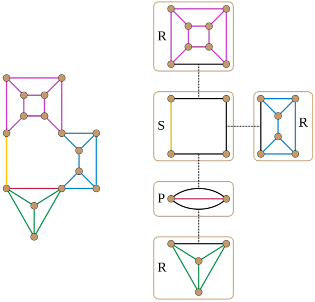

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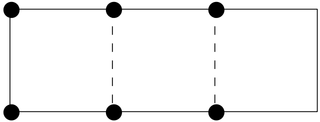

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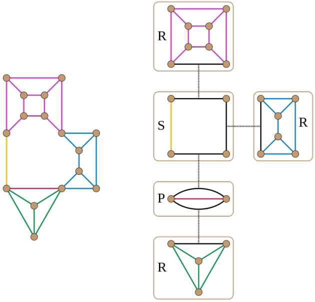

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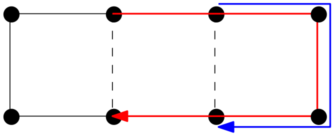

Question 3 - K_5-minor free graphs

Theorem (Wagner ‘’37)

K_5-minor free graphs $\Leftrightarrow$ 3-sums of planar graphs and Wagner graph.

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Remark : A similar result for $K_{3,3}$-graphs exist.

Monadic Second Order logic

First order (FO) :

- Quantifies on vertices
- Predicate $\rightarrow$ adjacency

$$\forall x \forall y, (x = y) \vee (x - y) \vee \exists z (x - z \wedge z - y)$$

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Monadic second order :

- Quantifies on sets of vertices (MSO_1) and edges (MSO_2)
- Predicates $\rightarrow$ adjacency + membership

$$\begin{aligned} &\exists V_1 \exists V_2, \\ &\quad \forall v, (v \in V_1) \Leftrightarrow \neg(v \in V_2) \\ &\wedge \forall v \forall w, v - w \Rightarrow (v \in V_1 \Leftrightarrow w \in V_2) \end{aligned}$$

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- Excluding a minor is MSO_1
- Hamiltonian cycle is MSO_2

Certifying MSO formulas

Theorem (Courcelle, informal)

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[Fraigniaud et al. '21]

- 3-approximation of tw in $O(\log^2 n)$ bits.
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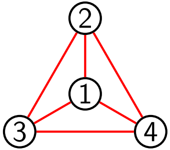
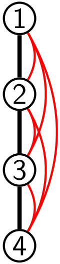
One step back : Treedepth

Find a (rooted) tree T on $|V(G)|$ vertices such that each edge of G links a vertex with an ancestor.
$\mathrm{td}(G)$ = minimum depth of such a T





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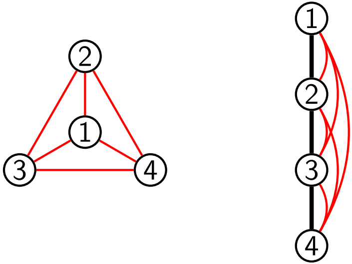

Bounded treedepth $\Rightarrow$ No long path

Zoology of depths

$$\mathrm{tw}(G) \leq \mathrm{pw}(G) \leq \mathrm{td}(G)$$

Advantages of treedepth :

- Diameter is bounded.
- [Gajarský, Hlinený '16] For every graph of bounded schrubdepth we can construct a kernel that satisfies the same FO formulas.

  And bounded schrubdepth $\Rightarrow$ Bounded treedepth.
- For bounded schrubdepth graphs, (informally) FO = MSO.

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Theorem (B, Feuilloley, Pierron '21+)

$\mathrm{td}(G) \leq k$ can be certified with $O(k \log n)$ bits.

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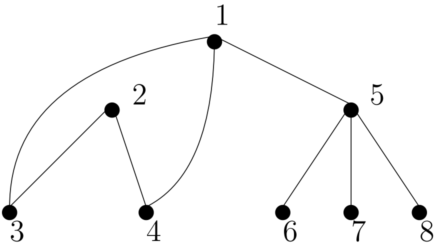

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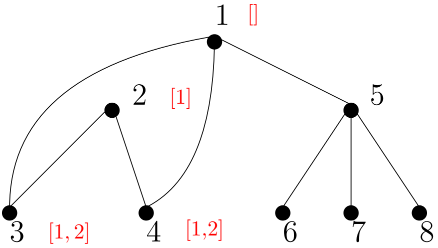

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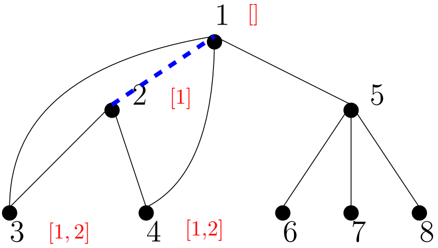

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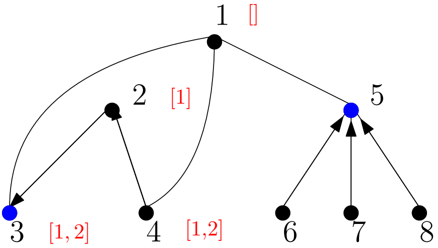

Certificate :

- List of ancestors
- Subtree rooted in a node connected to an ancestor for the graph below v for every v.

MSO certification

Theorem

Every MSO formula can be certified with $O(\log n)$ bits on bounded treedepth graphs.

- Existence of a (constructive and easy to certify) kernel for FO formulas on bounded treedepth graphs.

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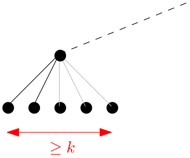

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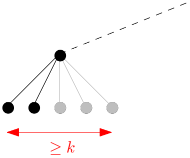

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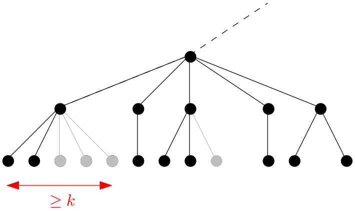

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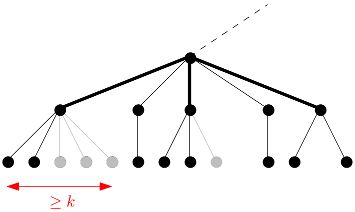

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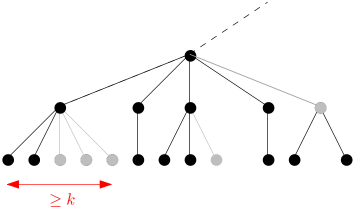

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The proof technique for MSO seem a bit different.

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Thanks for your attention !