All eccentricities on median graphs in subquadratic time

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Summary

I. Introduction to median graphs
   1. Definition and examples
   2. Θ-classes
   3. Diameter and eccentricities

II. Eccentricities in subquadratic time
   1. Splitting large Θ-classes
   2. Simplex graphs

III. Perspectives

Introduction to median graphs

1. Definition and examples

Graph $G = (V, E)$, undirected and unweighted

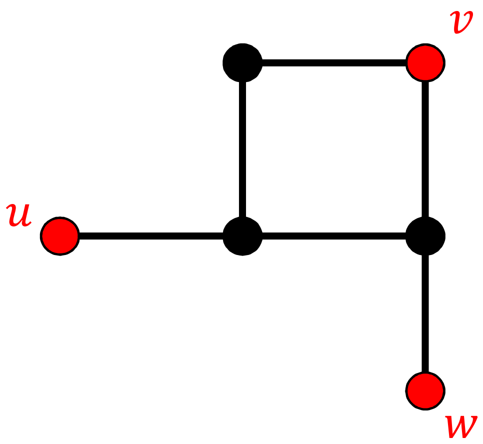
Interval $I(u, v) = \{x \in V: d(u, x) + d(x, v) = d(u, v)\}$

$= $ the set of vertices lying on a shortest (u, v)-path.

<u>Definition</u>: Median graph

Graph G is a *median graph* if, for any triplet of vertices u, v, w, set $I(u, v) \cap I(v, w) \cap I(w, u)$ is a singleton $\{m(u, v, w)\}$

1. Definition and examples



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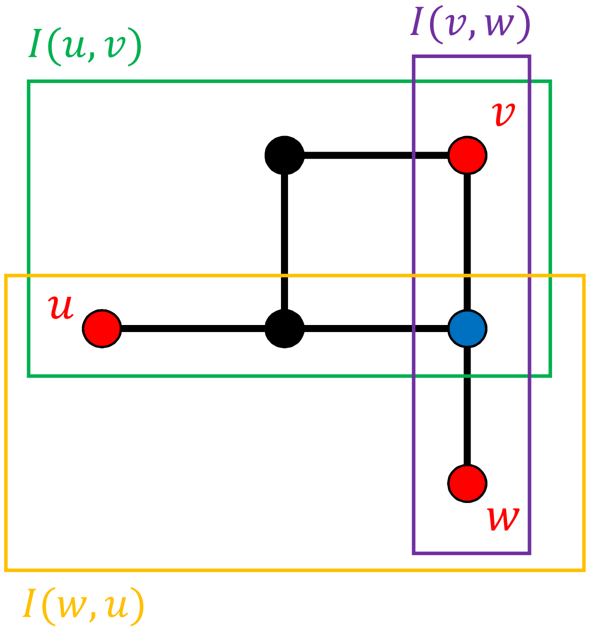

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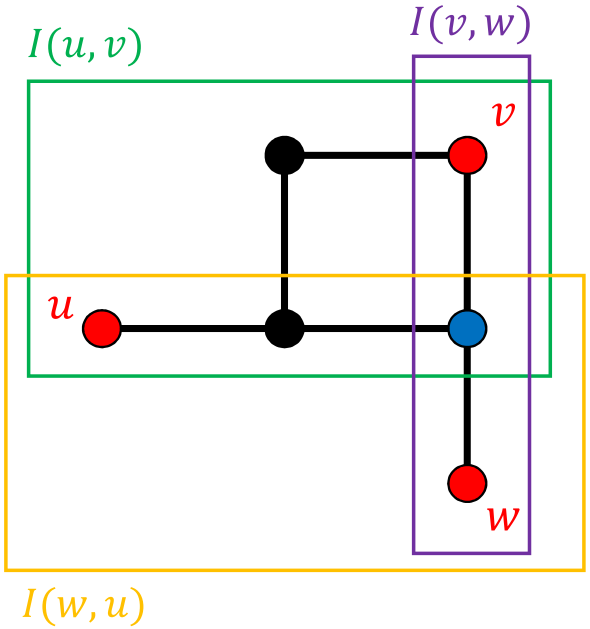

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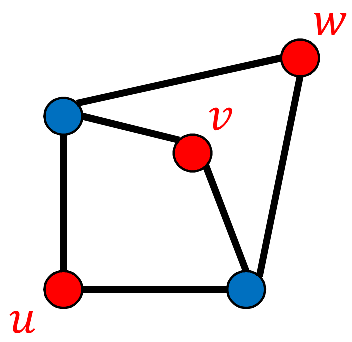

Triplet u, v, w has two medians: no induced $\boldsymbol{K_{2,3}}$ in median graphs

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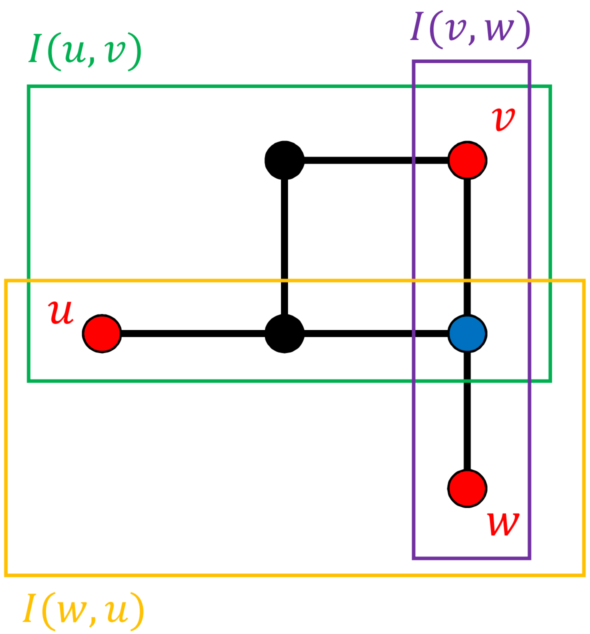

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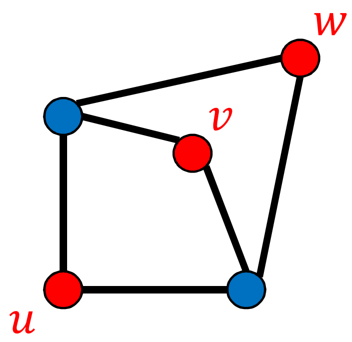

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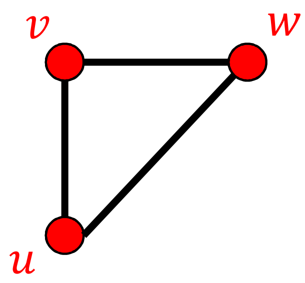
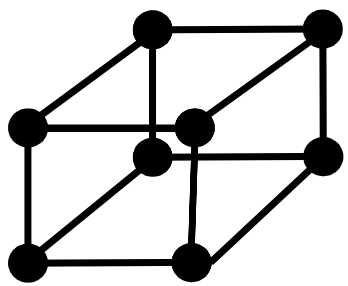
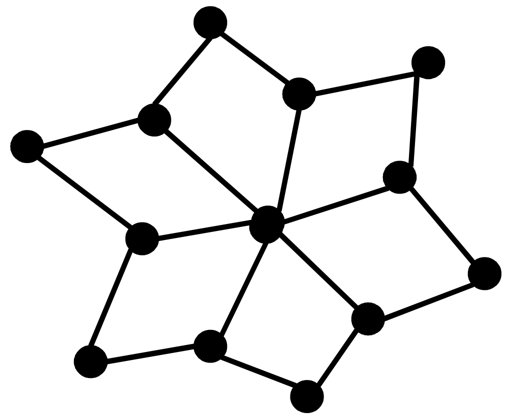


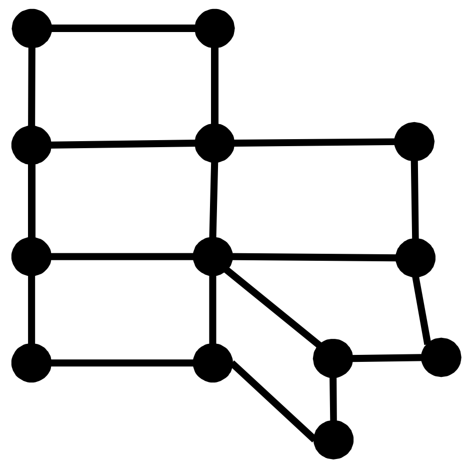
Triplet u, v, w has no median: median graphs are triangle-free (in fact, bipartite)

1. Definition and examples

Examples: trees, hypercubes, grids, cogwheels, squaregraphs







2. Θ-classes

We say edges uv and xy are in relation Θ_0 if there is an induced square $uvyx$ in the median graph G

Definition: Θ-classes

Relation Θ is the transitive closure of Θ_0. Equivalence classes of Θ are called *Θ-classes* and are denoted by $E_1, \ldots, E_q$.

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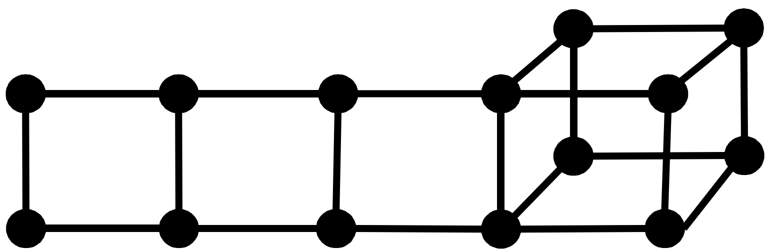

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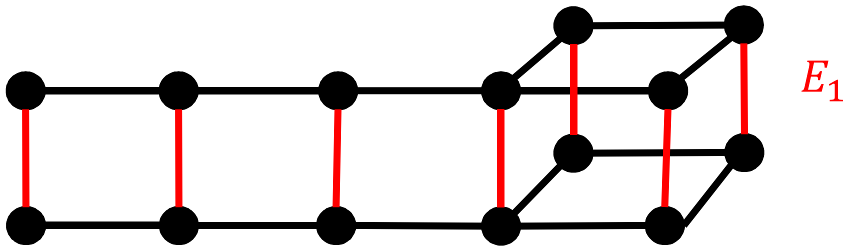

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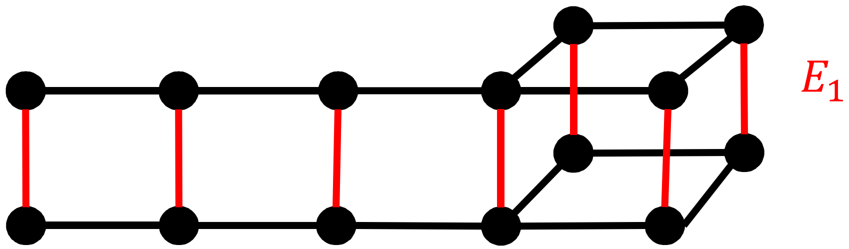
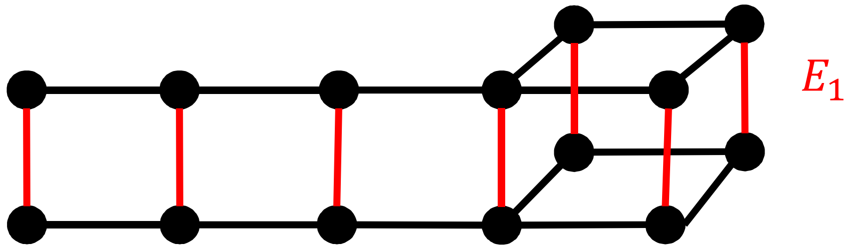


Theorem: Bénéteau *et al.*, 2020

Θ-classes can be determined in linear time $O(|E|) = O(n \log n)$.

2. Θ-classes

- Θ-classes are matching cutsets



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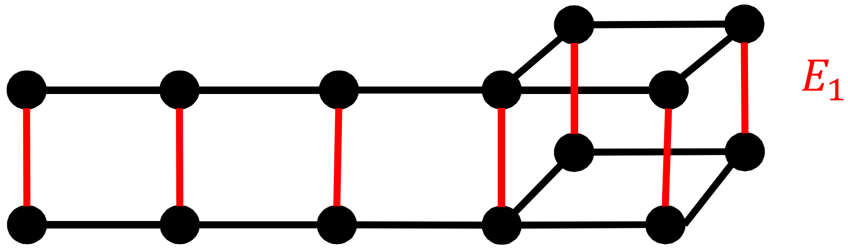
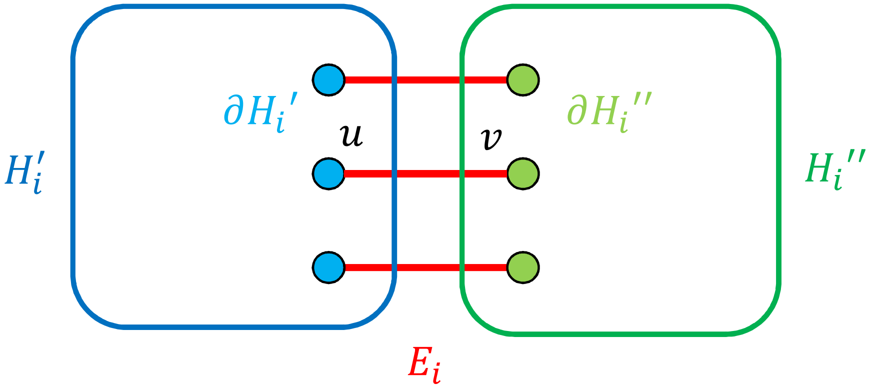


- Halfspaces and boundaries are convex



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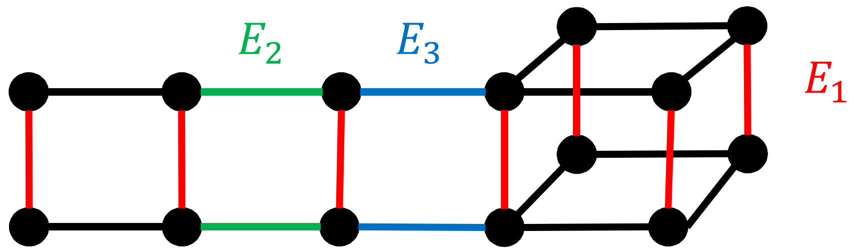
Definition: Orthogonal Θ-classes

Classes E_i and E_j are *orthogonal* if there is an induced square of G with two edges of E_i and two edges of E_j.



- All Θ-classes in an hypercube are pairwise orthogonal

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A set X of Θ-classes is a *POF* if for any $E_i, E_j \in X$, $E_i \perp E_j$

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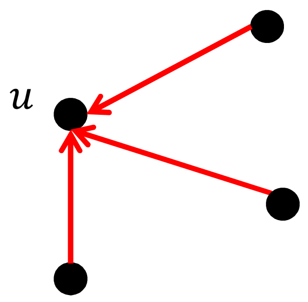
BFS from an arbitrary v_0: for any vertex u, the Θ-classes of all edges incoming to u form a POF.



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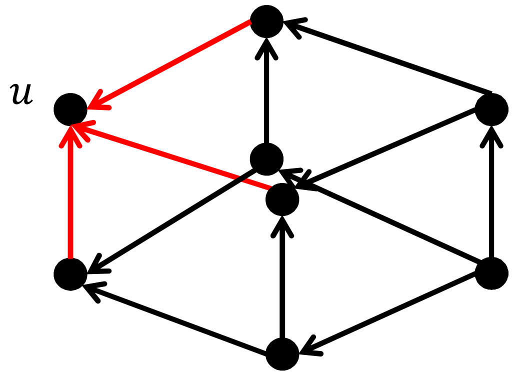

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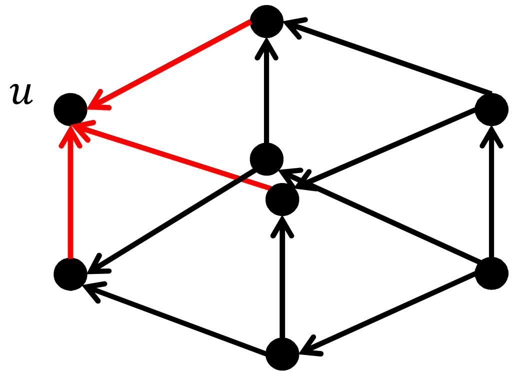

Definition: Dimension

The dimension d of a median graph is the dimension of the largest induced hypercube of G: $d \leq \log n$.

More generally, for any POF X, there is an hypercube of G whose edges belong to the Θ-classes in X, so $|X| \leq d$.

3. Diameter and eccentricities

Theorem: B., Habib, 2021

All eccentricities of median graphs can be determined in time $O\left(2^{O(d\log d)}n\right)$, linear for constant dimension.

Best exact algorithm for the general case: Multiple BFS in $O(dn^2)$

Objective: Find an exact algorithm in $O(n^c \log^{O(1)} n)$, $c < 2$, without restriction on d

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Partition refinement techniques

Intermediate result

All eccentricities of median graphs can be determined in time $O\left(2^{2d} n \log^3 n\right)$.

Splitting large Θ-classes

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Eccentricities in subquadratic time

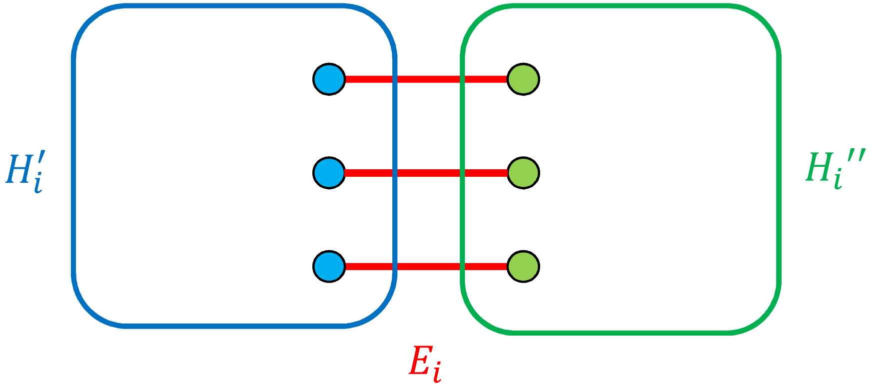
1. Splitting large Θ-classes

<u>Intermediate result</u> (assumption)

All eccentricities of median graphs can be determined in time $\tilde{O}(2^{2d}n)$.

<u>Objective</u>: design of a subquadratic-time algorithm

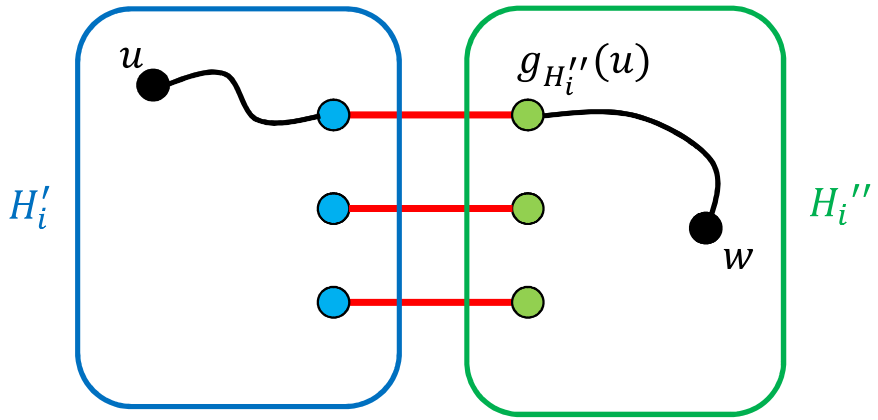
<u>Idea</u>: Recursive calls to the halfspaces of large Θ-classes



Deduce all eccentricities of G given all eccentricities of both H_i' and H_i''

1. Splitting large Θ-classes

In median graphs, convex = gated



A set H is *gated* if for any vertex $u \in V - H$, there exists a unique vertex $g_H(u) \in H$ such that, for any $w \in H$, a shortest (u, w)-path passes through $g_H(u)$.

$$\forall w \in H, d(u,w) = d\big(u, g_H(u)\big) + d(g_H(u), w)$$

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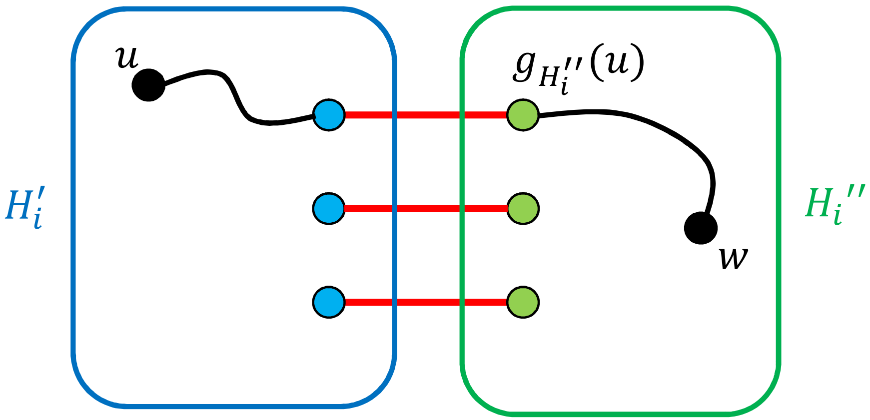

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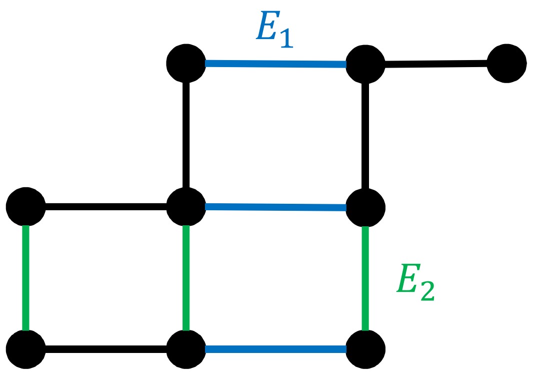
Consequence: $\mathrm{ecc}_G(u) = \max\left\{\mathrm{ecc}_{H_i'}(u), d\left(u, g_{H_i''}(u)\right) + \mathrm{ecc}_{H_i''}(g_{H_i''}(u))\right\}$

1. Splitting large Θ-classes

Let $E_1, \ldots, E_p$ be the Θ-classes with cardinality $\geq D$ (to be determined)



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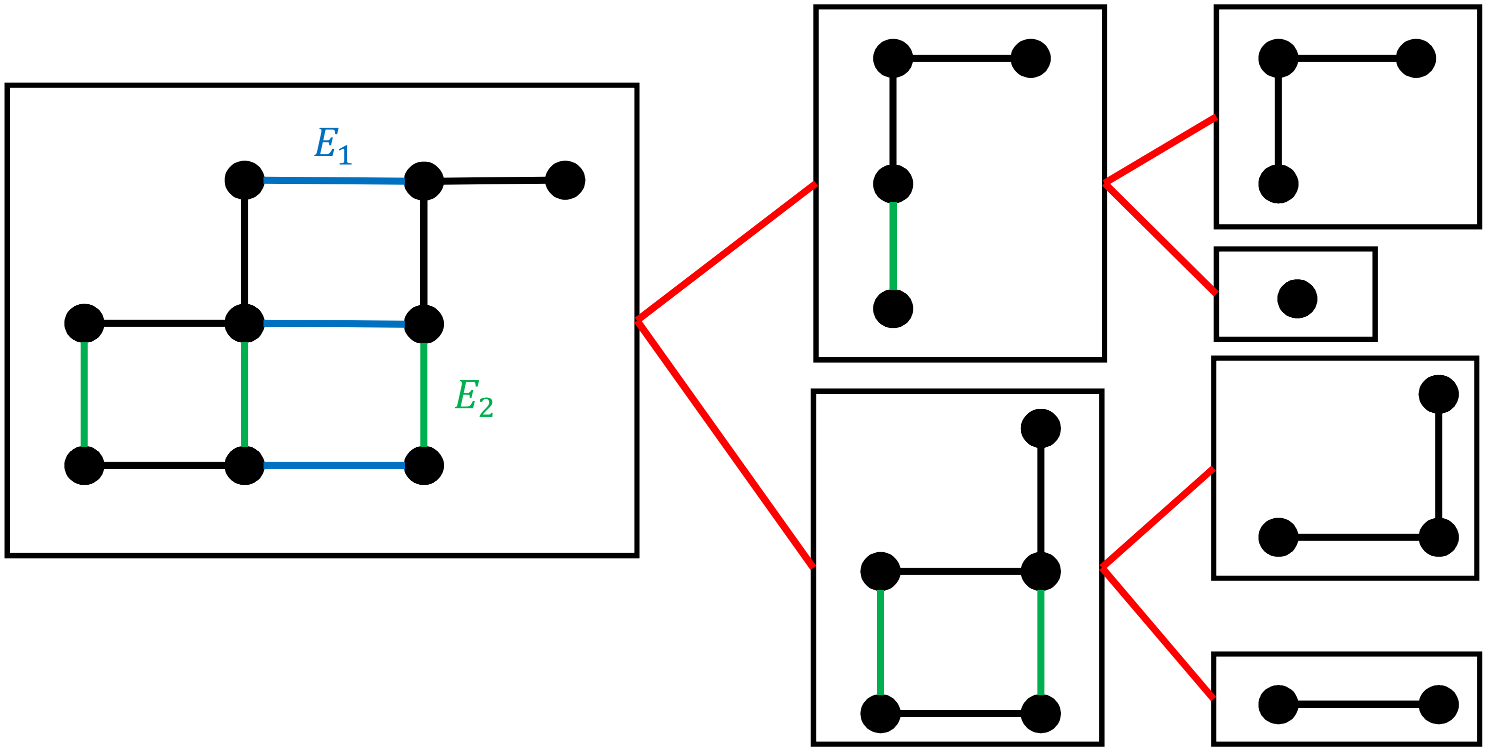

Depth $p \leq \frac{|E|}{D} \leq \frac{n\log n}{D}$

Hypercube of dimension d
$\Rightarrow$ Θ-class of size 2^{d-1}

The leaves of the tree have dimension at most $1 + \log D$, their eccentricities are obtained in $\tilde{O}\left(2^{2(1+\log D)}n\right) = \tilde{O}(D^2 n)$

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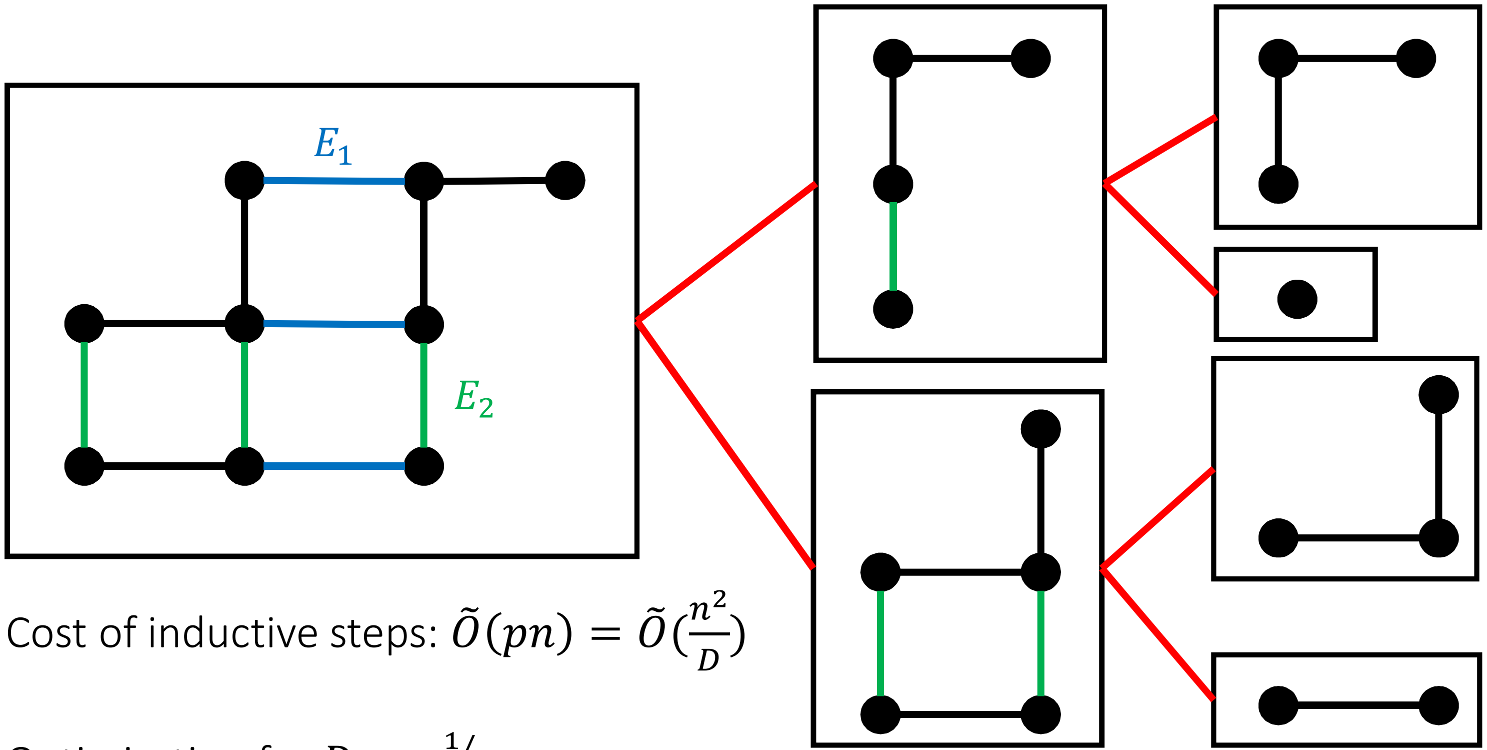

Cost of inductive steps: $\tilde{O}(pn) = \tilde{O}(\frac{n^2}{D})$

Optimization for $D = n^{1/3}$

Depth $p \leq \frac{|E|}{D} \leq \frac{n\log n}{D}$

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2. Simplex graphs

Result A

Result B

Result C

Result

All eccentricities of *simplex graphs* $K(G)$ can be determined in $\tilde{O}(n)$.

Intermediate result

All eccentricities of median graphs can be determined in time $\tilde{O}\left(2^{2d}n\right)$.

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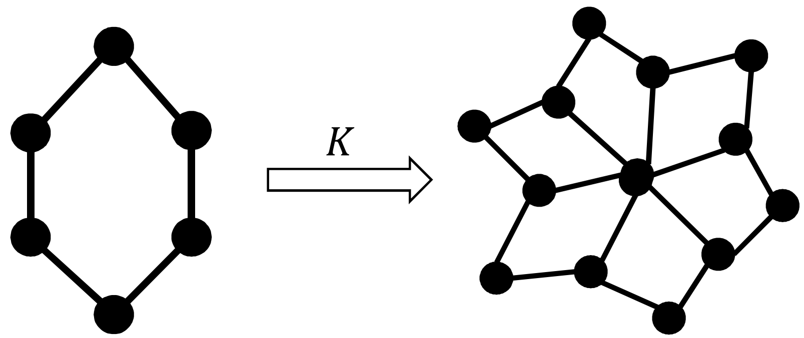
All eccentricities of median graphs can be determined in time $\tilde{O}\left(2^{2d} n\right)$.

Given G, the simplex graph $K(G)$ is defined as:

- $V(G)$: set of induced cliques of G (even not maximal)
- $E(G)$: C and C' adjacent if $C = C' \cup \{u\}$

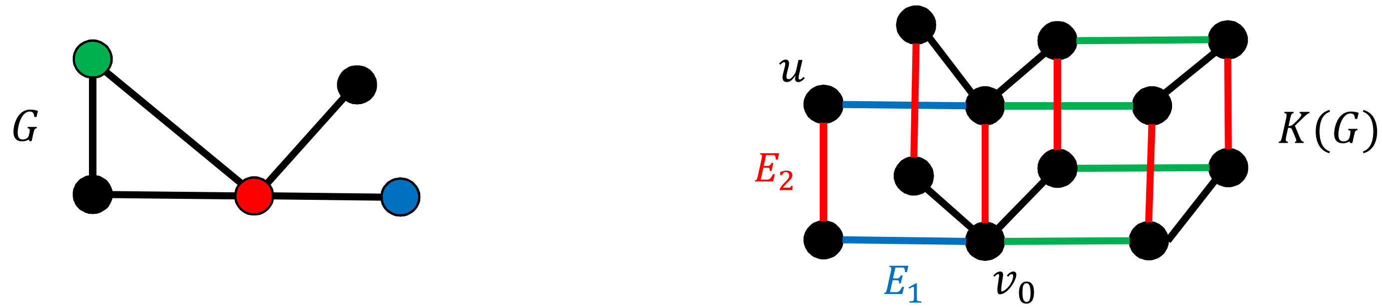


2. Simplex graphs

<u>Characterization</u>: Simplex graphs are the median graphs s.t. all Θ-classes are incident to the same vertex v_0

- 1-to-1 corresp.: vertices of G $\Leftrightarrow$ Θ-classes of $K(G)$

  cliques of G $\Leftrightarrow$ vertices of $K(G)$ $\Leftrightarrow$ hypercubes of $K(G)$



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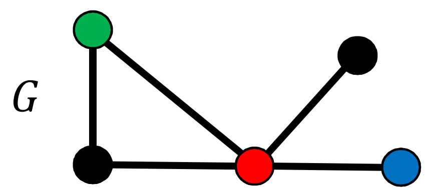
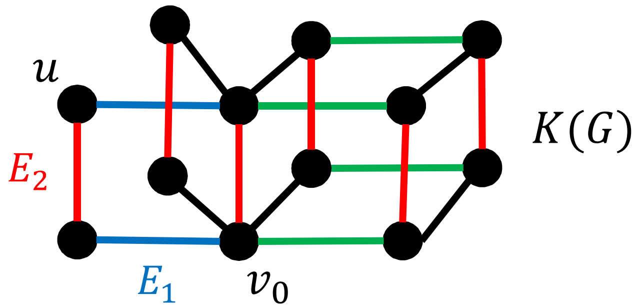




$f(u) = \{E_1, E_2\}$

$f(u)$: set of Θ-classes of $K(G)$ incoming into u with the v_0-orientation
$\Leftrightarrow$ set of vertices of G contained into the clique represented by u

$$\boxed{\mathrm{ecc}(u) = |f(u)| + \max_{f(v)\cap f(u)=\emptyset} |f(v)|}$$

2. Simplex graphs

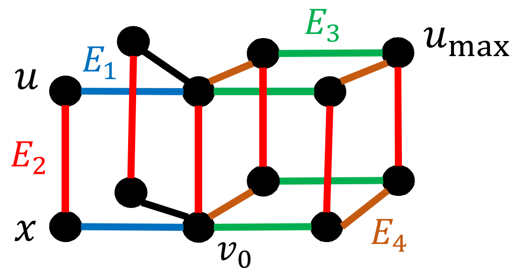
Theorem

All eccentricities of *simplex graphs* $K(G)$ can be determined in $\tilde{O}(n)$.

$$E_2E_3E_4\ ,E_1E_2\ ,E_3E_4\ ,\ldots$$

Idea: - sort all vertices of $K(G)$ in function of $|f(u)|$

- for each Θ-class E_i of $f(u_{\max})$, split the set and preserves the order (partition refinement)
- for each set obtained, restart the process
- stop at depth d^2



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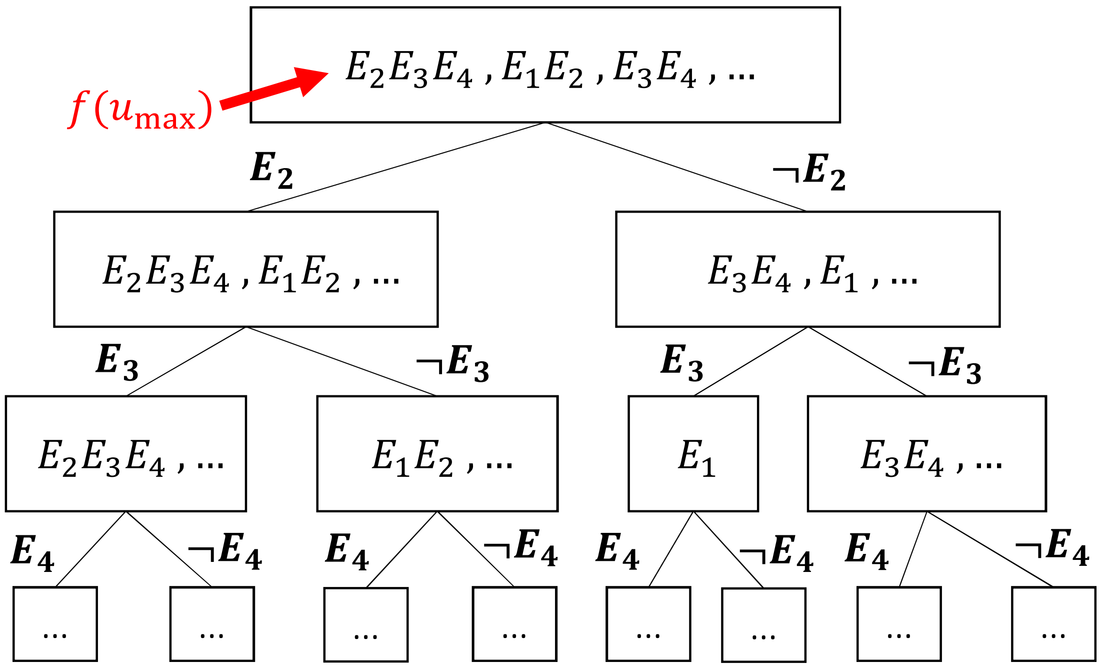

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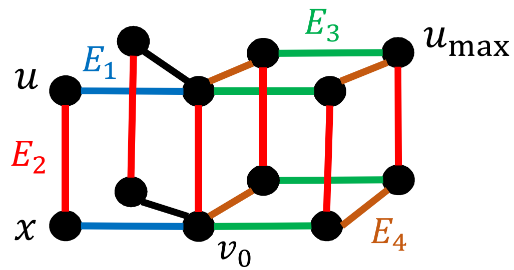
Examples:

$f(x) = E_1$, then $\mathrm{ecc}(u) = |\{E_1\}| + |u_{\max}| = 4$

$f(u) = E_1E_2$, then go to $\neg E_2$



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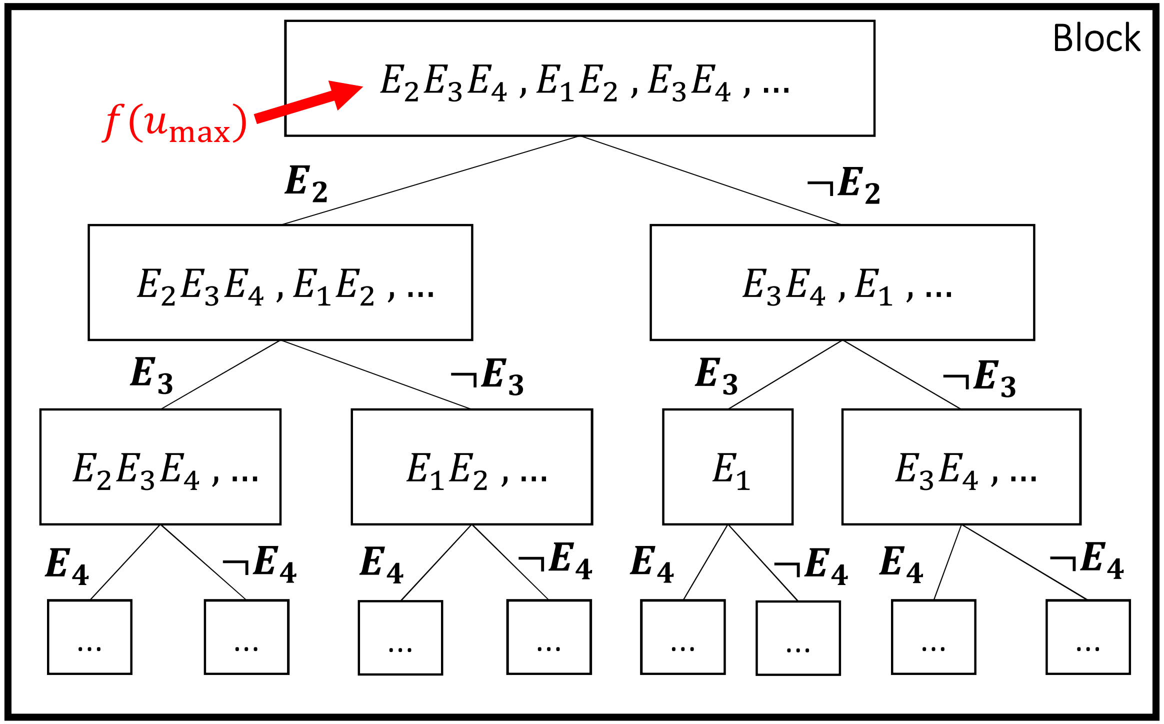

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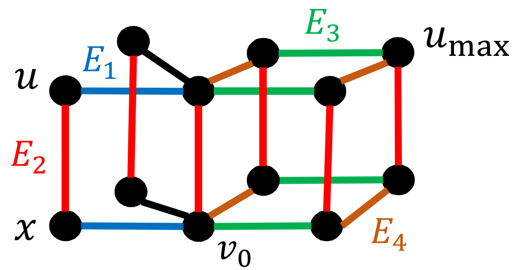

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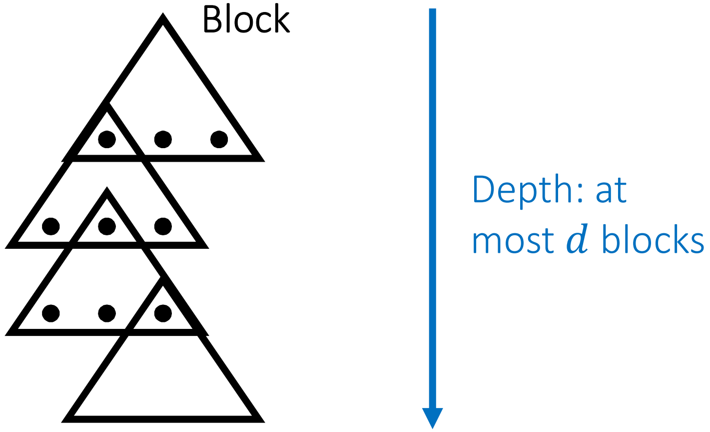

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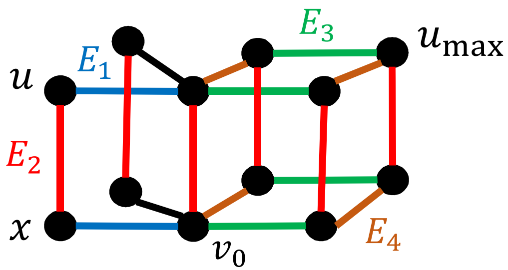

Perspectives

Open questions

Diameter and eccentricities : Can we obtain a quasilinear time algorithm ?

- Already true for simplex graphs
- With our current technique, not possible to overpass $\tilde{O}(n^{1.5})$

Idea : characterize median graphs without balanced Θ-classes

- Perhaps their structure is not far from simplex graphs

Long-term : focus on larger families of graphs ? Reach/betweeness centrality ?