Median in median graphs and their cube complexes in linear time

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Laurine Bénéteau, Jérémie Chalopin, Victor Chepoi, Yann Vaxès

Aix-Marseille Université, Laboratoire d'Informatique et Systèmes

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Preliminaries

Median set

Definition (Distance)

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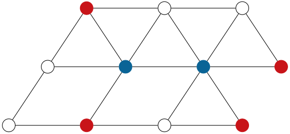
$$\mathrm{Med}_w(G) = \arg\min F_w$$



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$$\mathrm{Med}_w(G) = \arg\min F_w$$

Goal : Compute $\mathrm{Med}_w(G)$ faster than the distance matrix of G

Median graphs

Definition (Interval)

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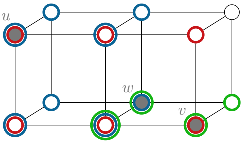

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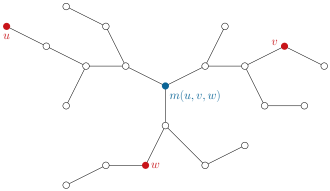
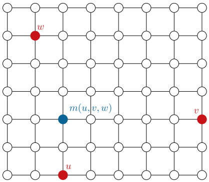
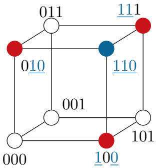


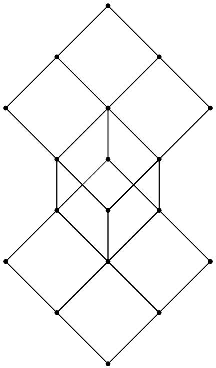




Median set in the ℓ_1-cube complexes of median graphs

Definition (ℓ_1-cube complex $\mathcal{G}$ van de Vel '93)

Each hypercube of a median graph G is replaced by a solid unity cube endowed with the ℓ_1-metric.



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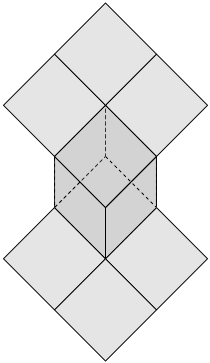

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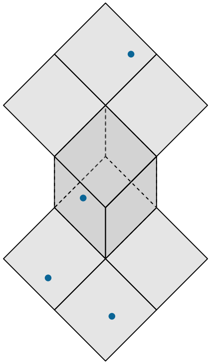
Definition (Median problem in $\mathcal{G}$)

$Med_w(\mathcal{G}) = \arg\min_{p \in \mathcal{G}} \sum_{q \in \mathcal{G}} w(q) d_1(p, q)$



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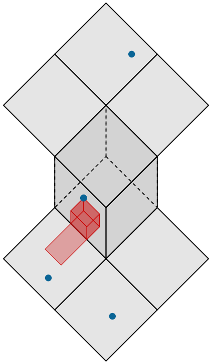

Main results

Theorem

For each median graph G and each weight function w, $\mathrm{Med}_w(G)$ can be computed in linear time.

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For each cube complex $\mathcal{G}$ of a median graph G, and each weighted set of points P in $\mathcal{G}$, $\mathrm{Med}_P(\mathcal{G})$ can be computed in linear time.

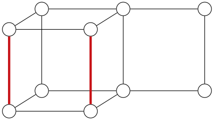
Ideas :

- Θ-classes
- Majority rule
- LexBFS
- Fellow-traveler property
- Peripheral halfspace peeling

Θ-classes

Definition (Oppositeness relation Θ_0)

$e\Theta_0 e'$ iff e and e' are two edges on the opposite sides of a square



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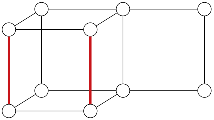
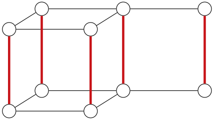


Definition (Parallelism relation Θ)

$\Theta = \Theta_0^*$ is the reflexive and transitive closure of Θ_0



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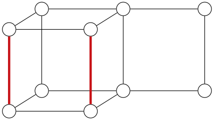

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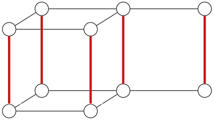
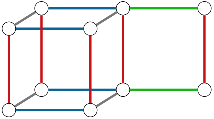


Definition (Θ-classes)

The Θ-classes denotes the equivalence classes of the relation Θ



Halfspaces (1/2)

Definition (Halfspaces)

Each Θ-classe split G in two convex and gated subgraphs called **halfspaces**

Definition (Convexity)

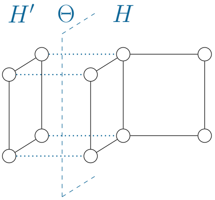
$S \subseteq V$ is convex if

$$\forall u, v \in S, I(u, v) \subseteq S$$

Definition (Gated set)

$S \subseteq V$ gated if

$$\forall x \in V \setminus S,\ \exists x' \in S,\ \forall y \in S,\ x' \in I(x, y)$$



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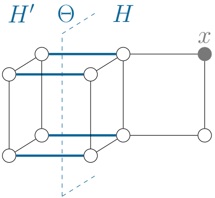

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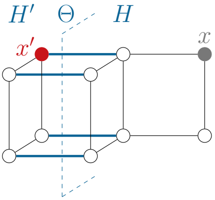

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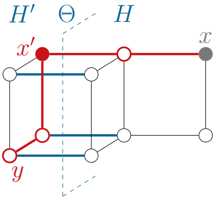
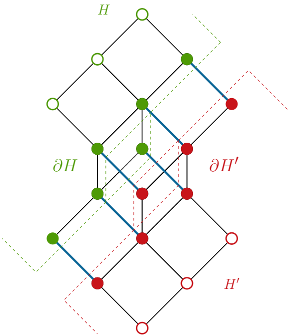


Halfspaces (2/2)

Definition (Boundaries)

The boundary of a halfspace is :

$$\partial H' = \{u \in H' \ : \ \exists v \in H, uv \in \Theta\}$$



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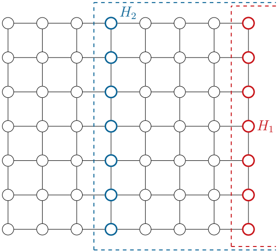

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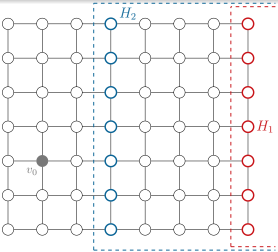
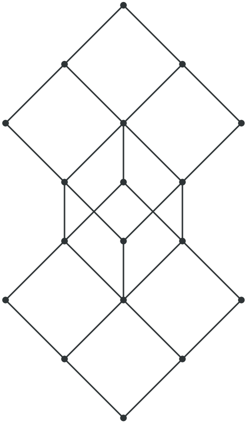
Property

If a halfspace H of G maximizes $d(v_0, H)$ for v_0 in V, then H is a peripheral halfspace



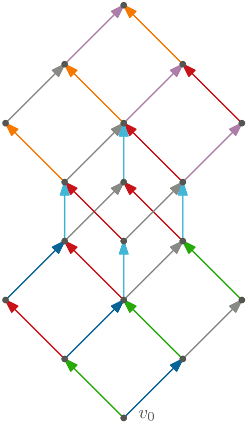
Θ-classes

BFS in median graphs



BFS in median graphs

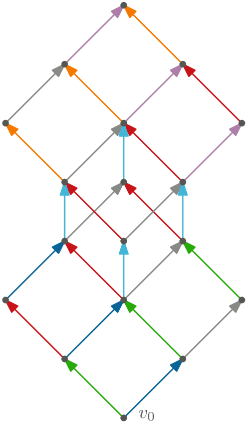
BFS orientation



BFS in median graphs

BFS orientation

For Each arc $\overrightarrow{uv}$:



BFS in median graphs

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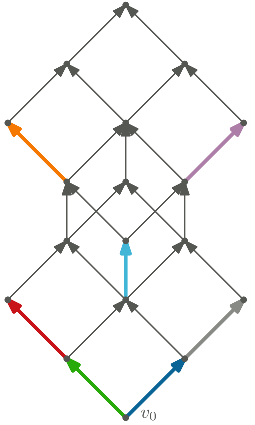
For Each arc $\overrightarrow{uv}$:

If $\overrightarrow{uv}$ is the only arc on v



BFS in median graphs

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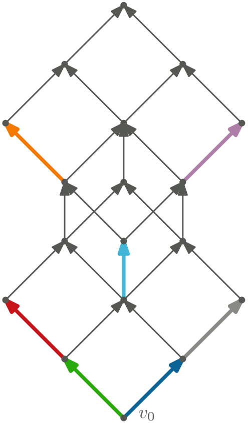
For Each arc $\overrightarrow{uv}$:

If $\overrightarrow{uv}$ is the only arc on v
uv is the closest to v_0 edge of its Θ-class.



BFS in median graphs

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For Each arc $\overrightarrow{uv}$:

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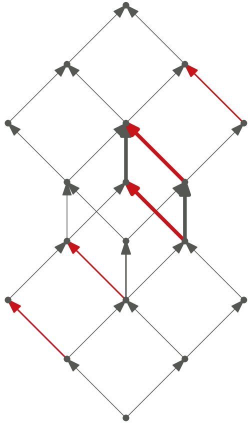
- **If** v is the target of two arcs $\overrightarrow{uv}$ and $\overrightarrow{xv}$,



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If v is the target of two arcs $\overrightarrow{uv}$ and $\overrightarrow{xv}$,
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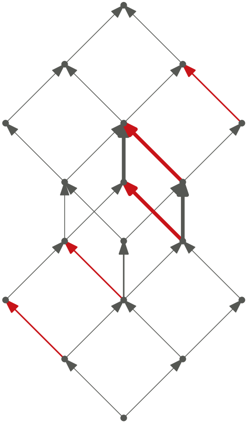

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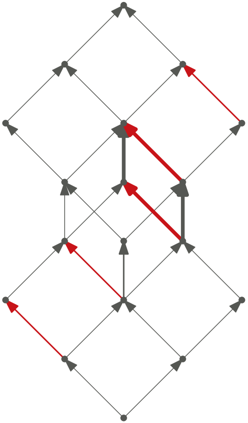
there is an edge yx closer to v_0 s.t. $uv\Theta yx$



LexBFS (Rose, Tarjan, Lueker, 1976)

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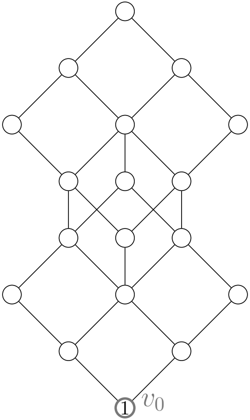

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The parent of $u \in V$, $f(u)$ is its smallest predecessor

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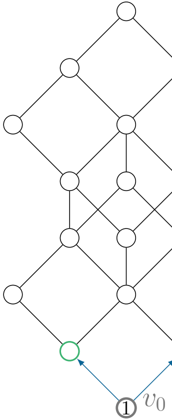

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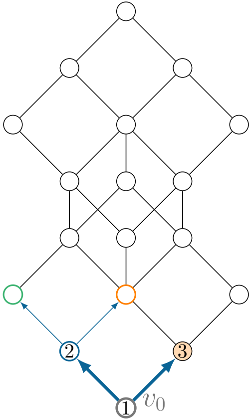

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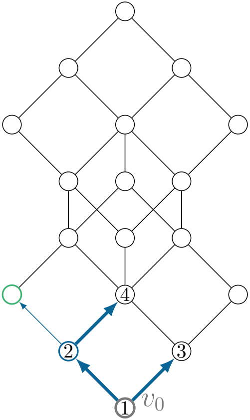

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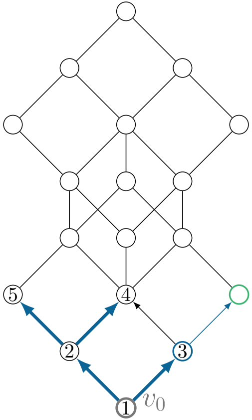

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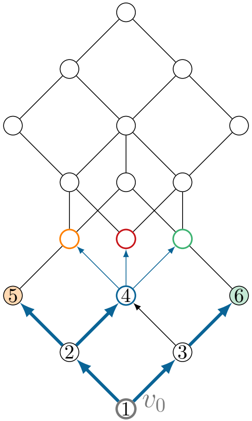

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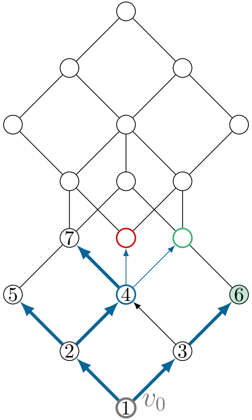

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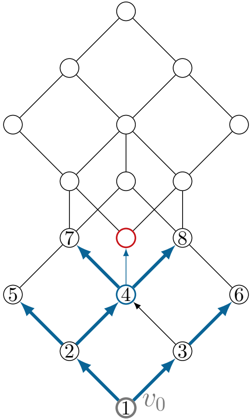

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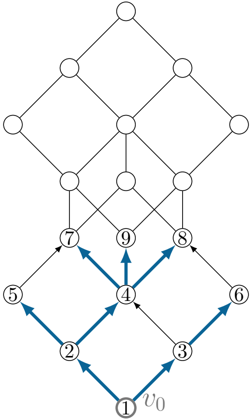

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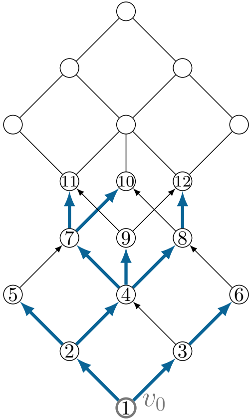

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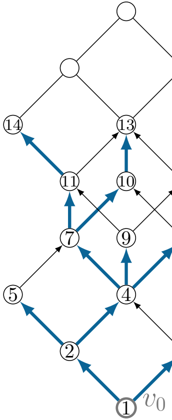

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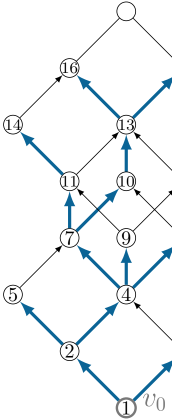

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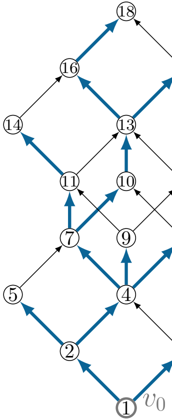

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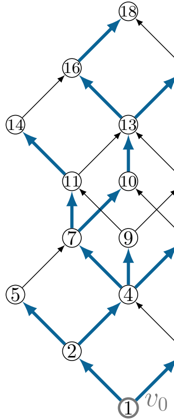

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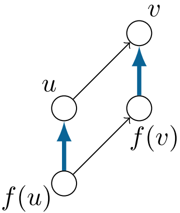
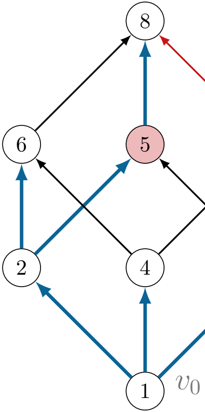
Theorem (Fellow-traveller Property)

For each arc $\overrightarrow{uv}$ of a LexBFS, if $u \neq f(v)$, then $f(u)$ and $f(v)$ are adjacent

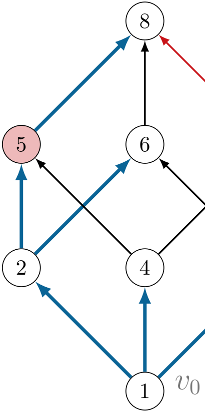


Difference between LexBFS and BFS

LexBFS



BFS



Computation of the Θ-classes in $O(m)$

Theorem

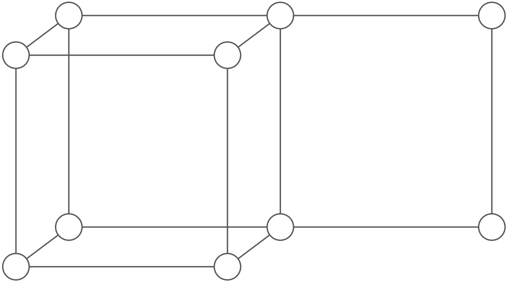
The Θ-classes of a median graph with m edges can be computed in $O(m)$ time

Median set in G

Majority Rule

Property (Majority rule, Bandelt and Barthélémy '84, Soltan and Chepoi '87)

$$\mathrm{Med}_w(G) = \cap\{H \mid w(H) \geq \tfrac{1}{2}w(G)\}$$



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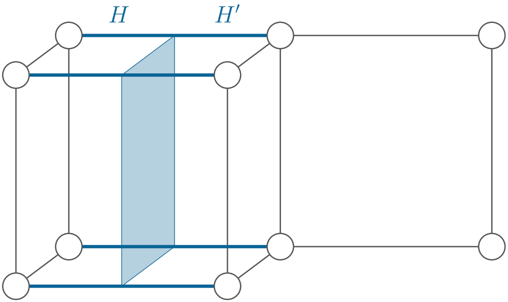

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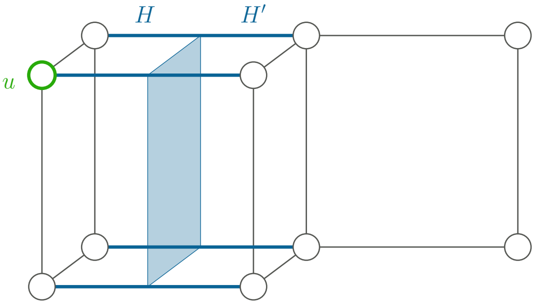

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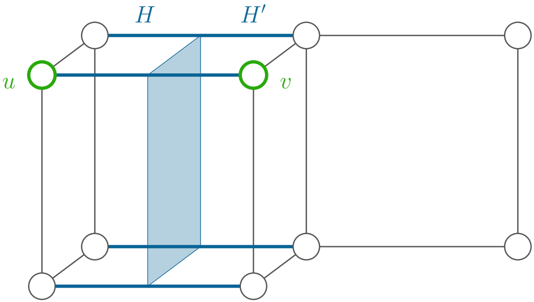

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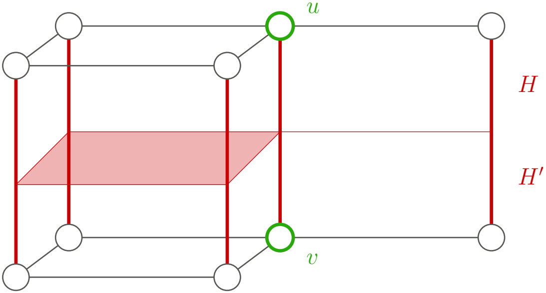


Algorithm

Median $(G = (V, E))$:

Compute and order the Θ-classes

For Each Θ-class Θ :

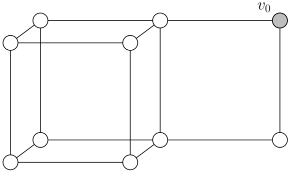
Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H

Return the sinks



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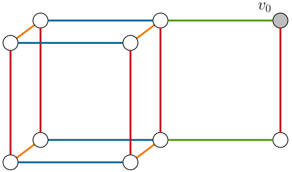

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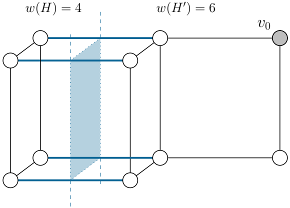

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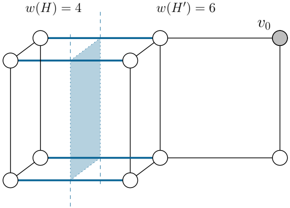

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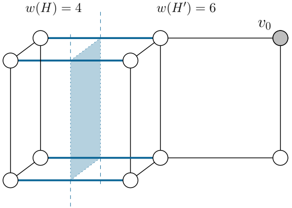

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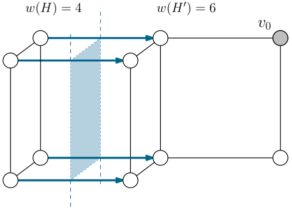

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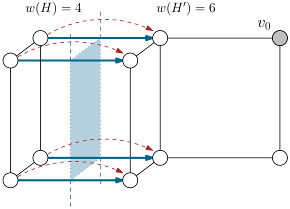

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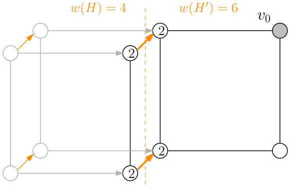

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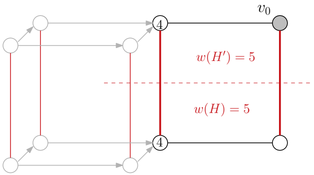

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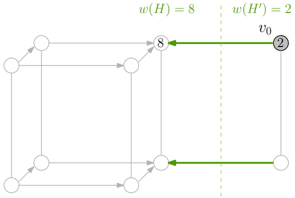

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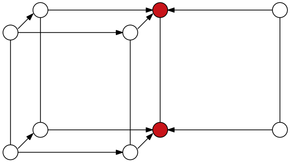

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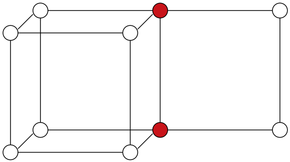

Median set computation in $O(m)$ time

Theorem

For each median graph G with m edges and the weighted function w, $\mathrm{Med}_w(G)$ can be computed in linear time $O(m)$

Median set in the cube-complex $\mathcal{G}$

Input

- The complex

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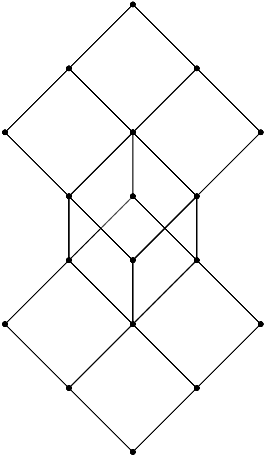

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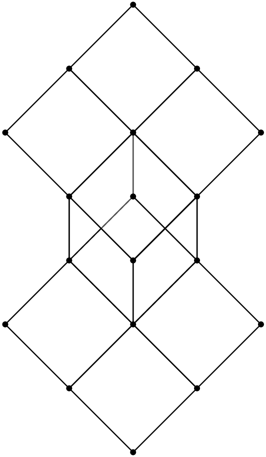

- The terminals

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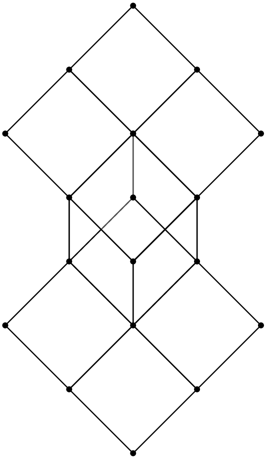

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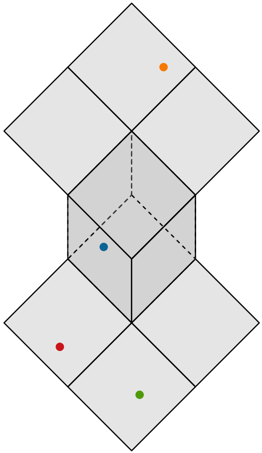

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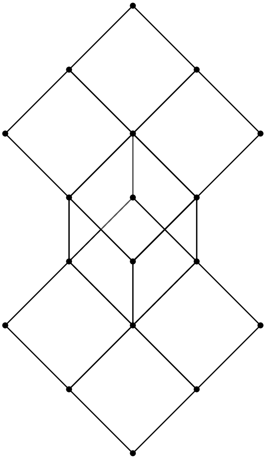

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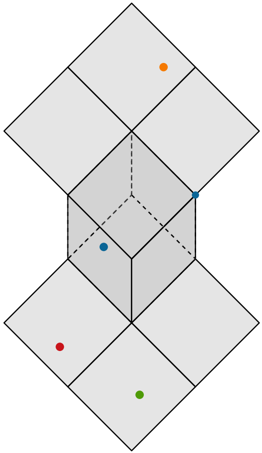

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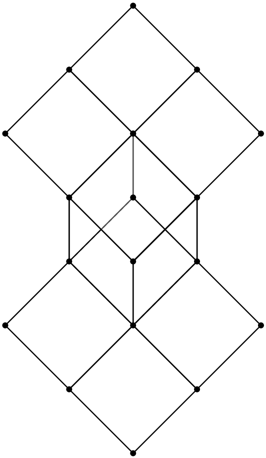

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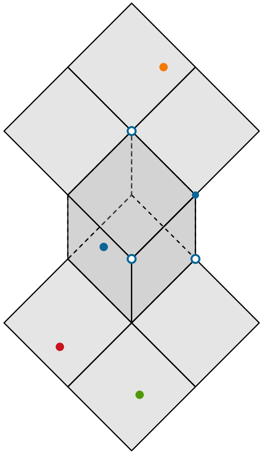

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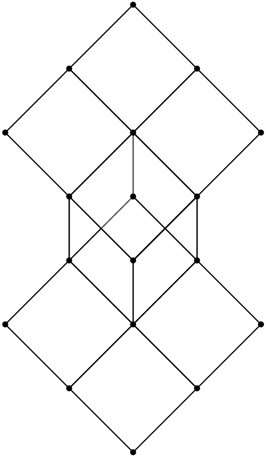

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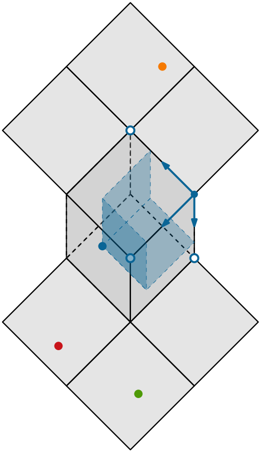

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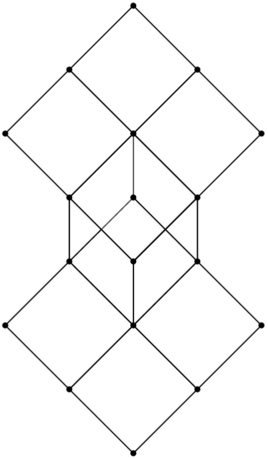

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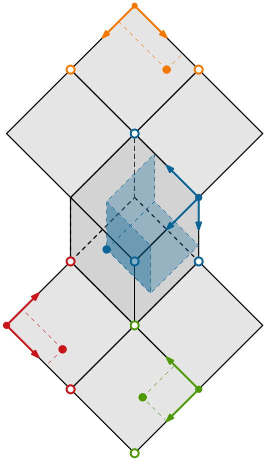

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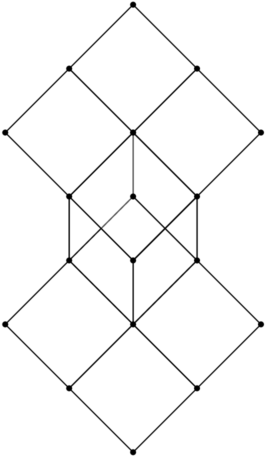

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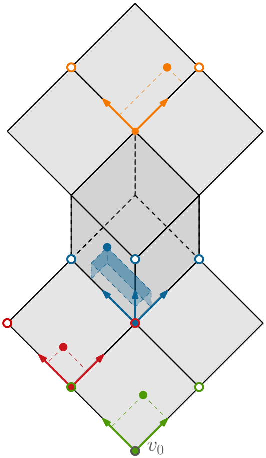
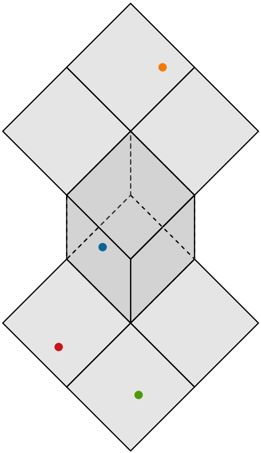


Box complex (Van de Vel '93)



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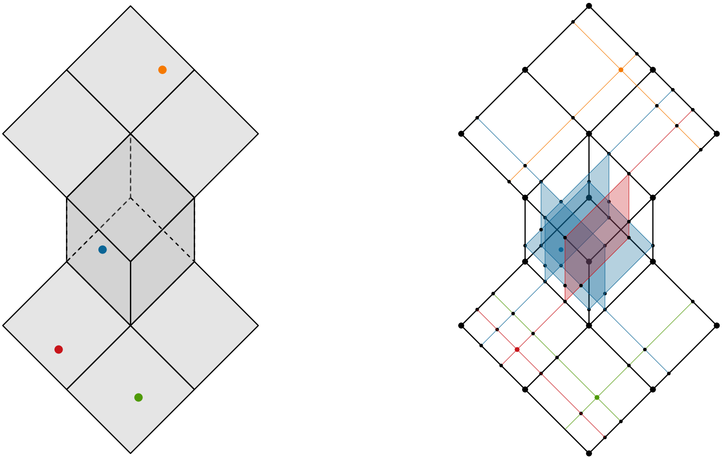
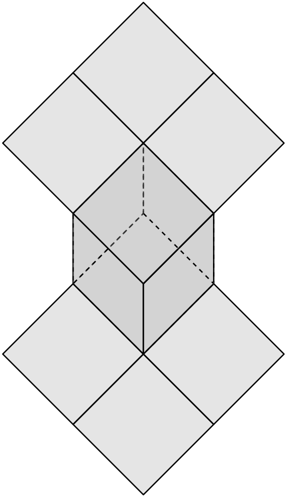


Halfspaces and Carriers



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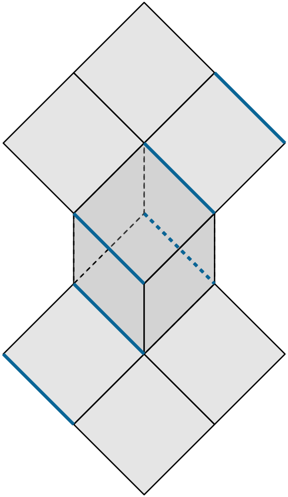

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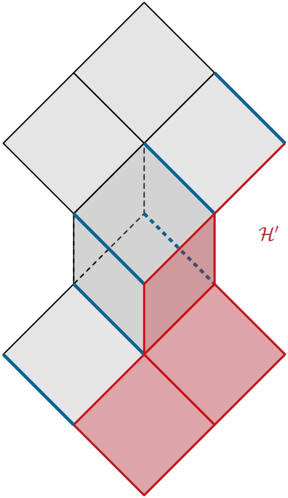

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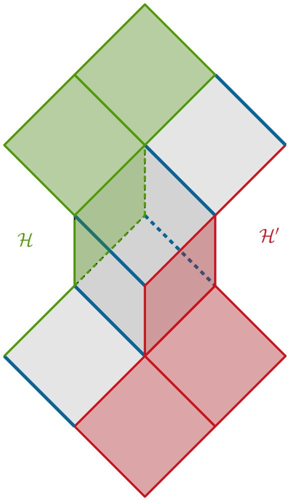

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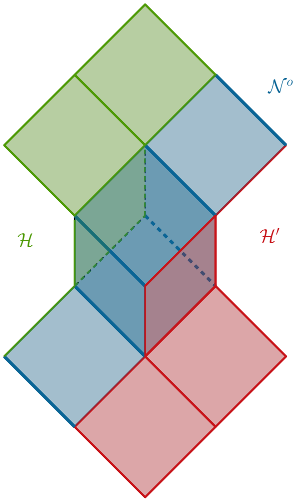
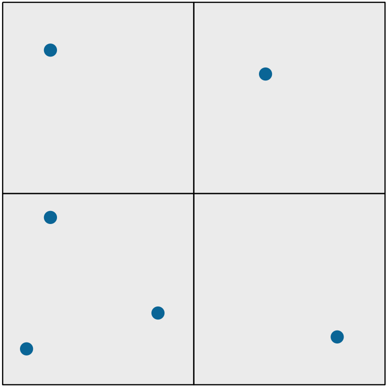


E_i-median



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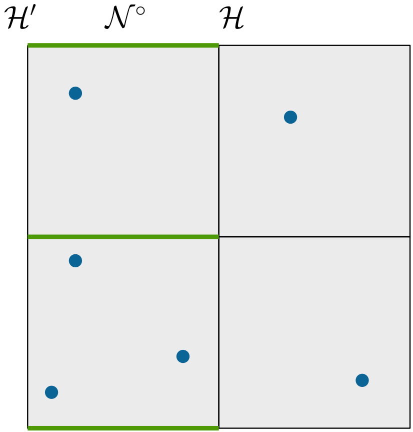

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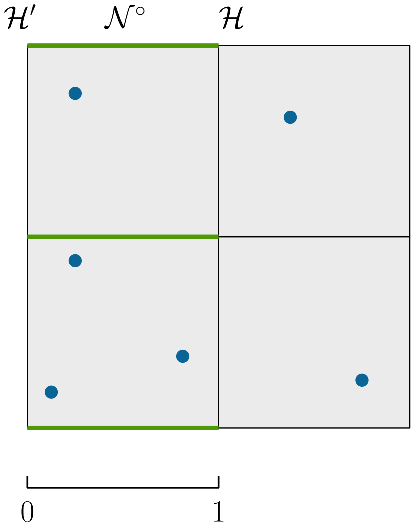

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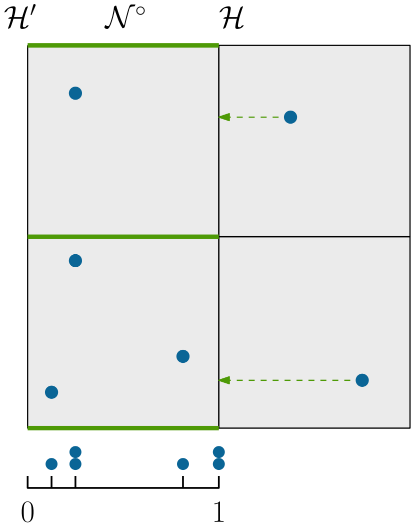

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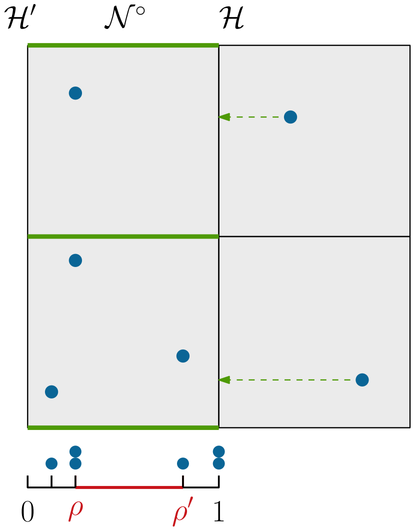

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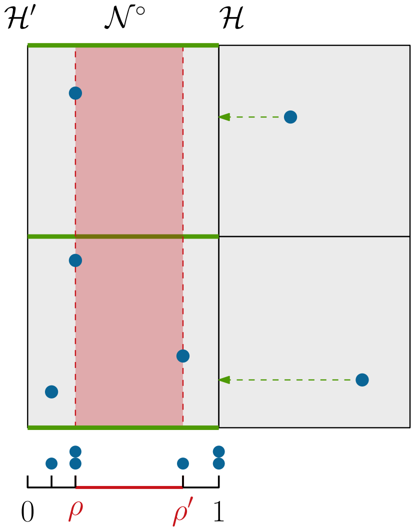

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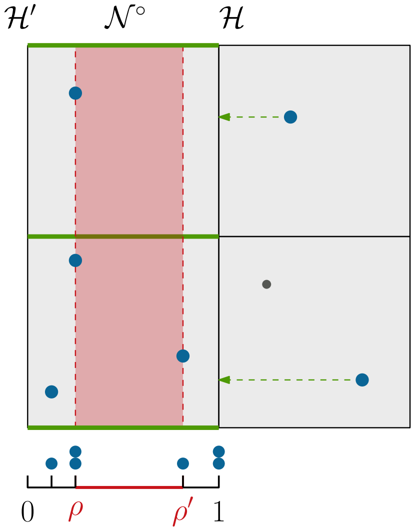

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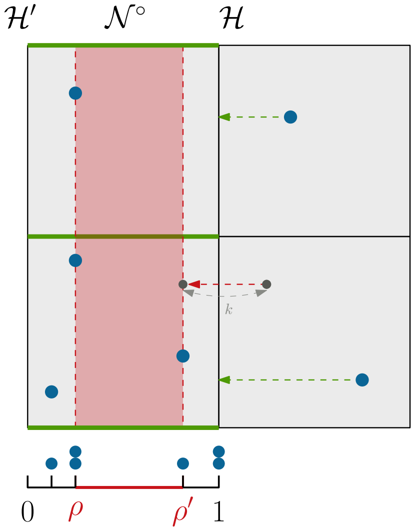

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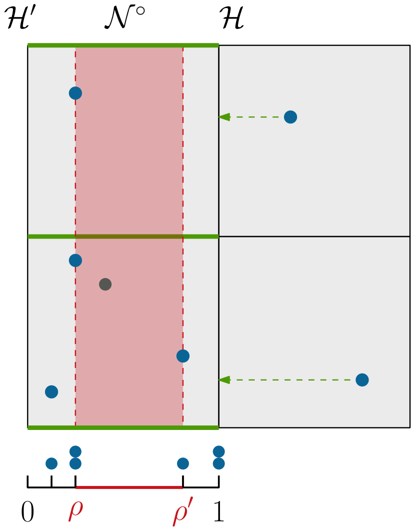

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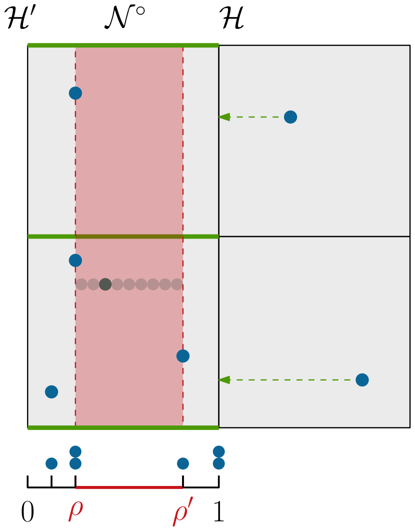
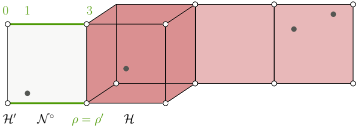


Majority rule in the ℓ_1-cube complex

- $\rho' = \rho = 1$

  $Med_w(\mathcal{G}) \subseteq \mathcal{H}'$



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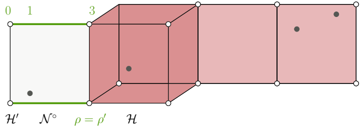

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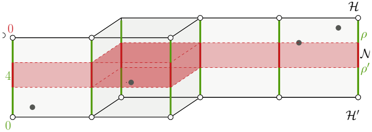

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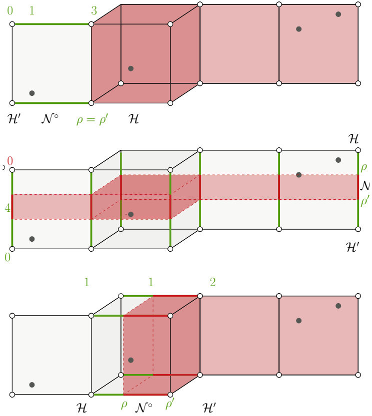
  $Med_w(\mathcal{G})$ intersects $\mathcal{H}'$ and $\mathcal{N}^\circ$



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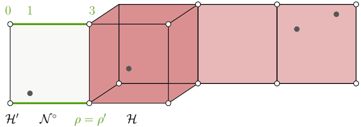

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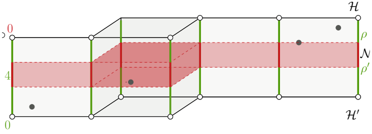

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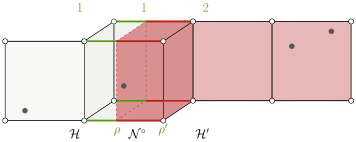
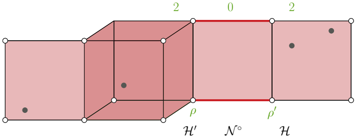
  $Med_w(\mathcal{G})$ intersects $\mathcal{H}'$ and $\mathcal{N}^\circ$

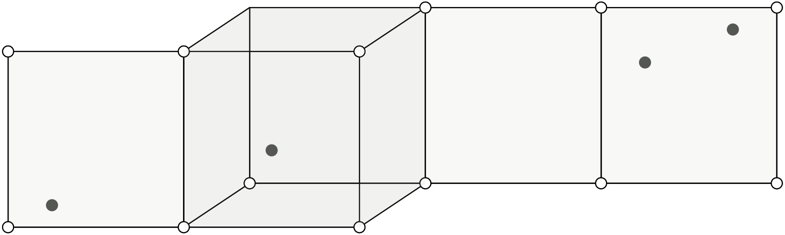


- $0 = \rho \leq \rho' = 1$

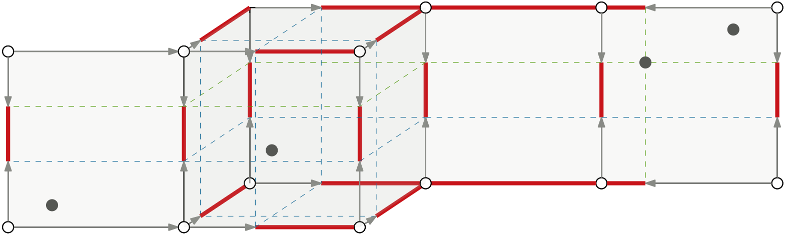
  $Med_w(\mathcal{G})$ intersects $\mathcal{H}, \mathcal{H}'$ and $\mathcal{N}^\circ$



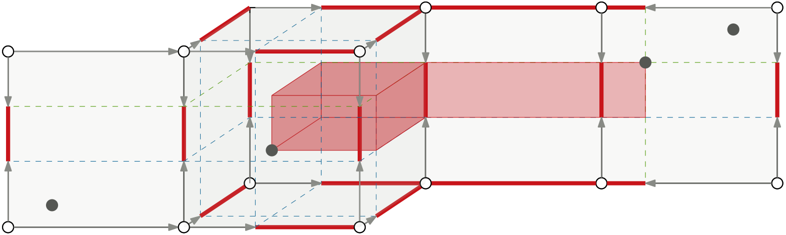
Algorithm



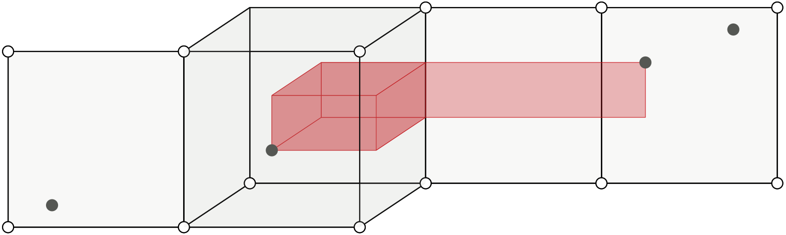
Algorithm



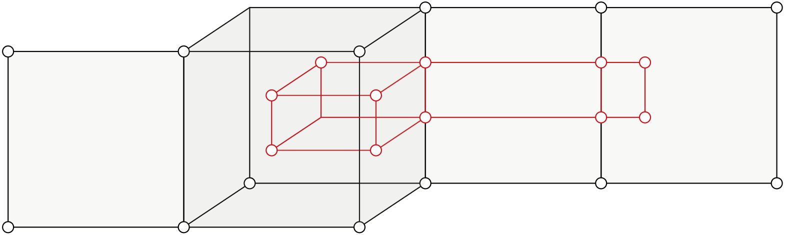
Algorithm



Algorithm



Algorithm



Median set computation in linear time

Theorem

For each cube complex $\mathcal{G}$ of a median graph G, and for each weighted set of points in $\mathcal{G}$, $\mathrm{Med}_w(\mathcal{G})$ can be computed in linear time.

Conclusion

Perspectives

- Computation of the center and diameter in median graphs in subquadratic time.

Perspectives

- Computation of the center and diameter in median graphs in subquadratic time.

- Computation of the barycenter in linear time

Perspectives

- Computation of the center and diameter in median graphs in subquadratic time.
- Computation of the barycenter in linear time
- Computation of the median set in other graph classes with strong metric properties

Thank you !