Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion

Median in median graphs and their cube complexes in linear time MGT 2021, Marseille

Laurine Bénéteau, Jérémie Chalopin, Victor Chepoi, Yann Vaxès

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December 9, 2021

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Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion
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Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Definition (Distance)

For each $u, v \in V$, d(u, v) is the minimum number of edges in a (u, v)-path.

Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion
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Definition (Distance)

For each $u, v \in V$, d(u, v) is the minimum number of edges in a (u, v)-path.

Definition (Median function)

Let $w: V \to \mathbb{R}^+$ be a weight function $F_w(x) = \sum_{v \in V} w(v) d(x, v)$

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Definition (Median set)

 $\operatorname{Med}_w(G) = \operatorname{arg\,min} F_w$



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Definition (Median set)

 $\operatorname{Med}_w(G) = \operatorname{arg\,min} F_w$

Goal : Compute $Med_w(G)$ faster than the distance matrix of G

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Preliminaries	⊖-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V : d(u,v) = d(u,x) + d(x,v)\}$$

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Median graphs

Definition (Interval)

$$I(u,v) = \{x \in V : d(u,v) = d(u,x) + d(x,v)\}.$$

Definition (Median graph)

$$\forall u, v, w \in V$$
, $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$



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Θ-classes 00000 Median set in G 0000 Median set in the cube-complex ${\cal G}$ 00000000

Conclusion

Median set in the ℓ_1 -cube complexes of median graphs

Definition (ℓ_1 -cube complex \mathcal{G} van de Vel '93)

Each hypercube of a median graph G is replaced by a solid unity cube endowed with the ℓ_1 -metric.



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Θ-classes 00000 Median set in G

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Definition (Median problem in \mathcal{G})

 $Med_w(\mathcal{G}) = \arg\min_{p \in \mathcal{G}} \sum_{q \in \mathcal{G}} w(q) d_1(p,q)$

Θ-classes 00000 Median set in G 0000 Median set in the cube-complex ${\mathcal{G}}$ 00000000

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Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Main results

Theorem

For each median graph G and each weight function w, $Med_w(G)$ can be computed in linear time.

Theorem

For each cube complex \mathcal{G} of a median graph G, and each weighted set of points P in \mathcal{G} , $Med_P(\mathcal{G})$ can be computed in linear time.

Ideas :

- Θ-classes
- Majority rule
- LexBFS
- Fellow-traveler property
- Peripheral halfspace peeling

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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion 00
Θ-classes				

Definition (Oppositeness relation Θ_0)

 $e\Theta_0 e'$ iff e and e' are two edges on the opposite sides of a square



Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex ${\cal G}$ 00000000	Conclusion 00

 Θ -classes

Definition (Oppositeness relation Θ_0)

 $e\Theta_0 e'$ iff *e* and *e'* are two edges on the opposite sides of a square

Definition (Parallelism relation Θ)

 $\Theta=\Theta_0^*$ is the reflexive and transitive closure of Θ_0



Preliminaries 000000000	Θ-classes 00000	Median set in G	Median set in the cube-complex ${\mathcal G}$ 00000000	Conclusion
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Definition (Θ -classes)

The $\Theta\text{-classes}$ denotes the equivalence classes of the relation Θ







Preliminaries	⊖-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Definition (Halfspaces)

Each Θ -classe split *G* in two convex and gated subgraphs called halfspaces

Definition (Convexity)

 $S \subseteq V$ is convex if

$$\forall u, v \in S, I(u, v) \subseteq S$$

Definition (Gated set)

 $S \subseteq V$ gated if

$$\forall x \in V \setminus S, \ \exists x' \in S, \ \forall y \in S, \ x' \in I(x,y)$$

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Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex \mathcal{G}	Conclusion
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Preliminaries 0000000●0	Θ-classes 00000	Median set in G	Median set in the cube-complex ${\mathcal G}$ 00000000	Conclusion 00
Halfspace	$e_{s}(2/2)$			

Definition (Boundaries)

The boundary of a halfspace is :

$$\partial H' = \{ u \in H' : \exists v \in H, uv \in \Theta \}$$



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Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Peripheral halfspace

Definition (Peripheral halfspace)

A halfspace *H* is called peripheral if $H = \partial H$.

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Preliminaries 00000000●	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G}	Conclusion

Peripheral halfspace

Definition (Peripheral halfspace)

A halfspace *H* is called peripheral if $H = \partial H$.

Property

If a halfspace H of G maximizes $d(v_0, H)$ for v_0 in V, then H is a peripheral halfspace



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Preliminaries	⊖-classes ●0000	Median set in G	Median set in the cube-complex \mathcal{G}	Conclusion

$\Theta ext{-classes}$

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Preliminaries	Θ -classes	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion
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BFS in median graphs



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 Preliminaries
 Θ-class

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Θ-classes Ν ο●οοο α

Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

BFS in median graphs

BFS orientation



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Θ-classes 0●000 Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

BFS in median graphs

BFS orientation

For Each arc \overrightarrow{uv} :



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⊖-classes 0●000 Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

BFS in median graphs

BFS orientation

For Each arc \overrightarrow{uv} :

If \overrightarrow{uv} is the only arc on v



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Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

BFS in median graphs

BFS orientation

For Each arc \overrightarrow{uv} :

If \overrightarrow{uv} is the only arc on vuv is the closest to v_0 edge of its Θ -class.



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⊖-classes o●ooo Median set in G

Median set in the cube-complex \mathcal{G} 00000000

Conclusion

BFS in median graphs

BFS orientation

For Each arc \overrightarrow{uv} :

If \overrightarrow{uv} is the only arc on vuv is the closest to v_0 edge of its Θ -class.

If v is the target of two arcs \overrightarrow{uv} and \overrightarrow{xv} ,



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⊖-classes 0●000 Median set in G

Median set in the cube-complex \mathcal{G} 00000000

Conclusion

BFS in median graphs

BFS orientation

For Each arc \overrightarrow{uv} :

- If \overrightarrow{uv} is the only arc on vuv is the closest to v_0 edge of its Θ -class.
- If v is the target of two arcs \overrightarrow{uv} and \overrightarrow{xv} , they belong to a single square



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Median set in the cube-complex \mathcal{G} 00000000

Conclusion

BFS in median graphs

BFS orientation

- For Each arc \overrightarrow{uv} :
 - If \vec{uv} is the only arc on vuv is the closest to v_0 edge of its Θ -class.
 - If v is the target of two arcs \overrightarrow{uv} and \overrightarrow{xv} , they belong to a single square there is an edge yx closer to v_0 s.t. $uv\Theta yx$



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⊖-classes 00●00 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

LexBFS (Rose, Tarjan, Lueker, 1976)

Ordering in LexBFS :

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⊖-classes 00●00 Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 00000000

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LexBFS (Rose, Tarjan, Lueker, 1976)

Ordering in LexBFS :



Definition (Parent)

The parent of $u \in V$, f(u) is its smallest predecessor

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Median set in the cube-complex \mathcal{G} 00000000

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Theorem (Fellow-traveller Property) For each arc \overrightarrow{uv} of a LexBFS, if $u \neq f(v)$, then f(u) and f(v) are adjacent



Θ-classes 000●0 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

Difference between LexBFS and BFS

LexBFS





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<mark>Θ-classes</mark> 0000● Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 00000000

Conclusion

Computation of the Θ -classes in O(m)

Theorem

The Θ -classes of a median graph with m edges can be computed in O(m) time

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Preliminaries	Θ-classes 00000	Median set in G ●000	Median set in the cube-complex \mathcal{G} 00000000	Conclusion

Median set in ${\it G}$

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Preliminaries	Θ -classes 00000	Median set in G 0●00	Median set in the cube-complex ${\mathcal{G}}$	Conclusion

Property (Majority rule, Bandelt and Barthélémy '84, Soltan and Chepoi '87)

$$\mathsf{Med}_w(G) = \cap \{H \mid w(H) \ge \frac{1}{2}w(G)\}$$



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Preliminaries	Θ-classes 00000	Median set in G 00●0	Median set in the cube-complex ${\mathcal G}$ 00000000	Conclusion
Algorithm				

Median (G = (V, E)):

Compute and order the Θ -classes

For Each Θ -class Θ :

Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H



Preliminaries	Θ -classes 00000	Median set in G 00●0	Median set in the cube-complex ${\mathcal G}$ 00000000	Conclusion
Algorithm	ı			

Median (G = (V, E)):

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For Each $\Theta\text{-class}\ \Theta$:

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Preliminaries	Θ-classes 00000	Median set in G 00●0	Median set in the cube-complex ${\cal G}$	Conclusion 00

Median (G = (V, E)):

Algorithm

Compute and order the Θ -classes

For Each Θ -class Θ :

Compute $w(H) = w(\partial H)$

Deduce the majoritary and minoritary halfspace

Direct each edge in Θ from the minoritary halfspace to the majoritary one

Report the weights of H



Preliminaries	Θ-classes 00000	Median set in G 00●0	Median set in the cube-complex ${\cal G}$	Conclusion 00

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Θ-classes 00000 Median set in G

Median set in the cube-complex \mathcal{G} 00000000

Conclusion

Median set computation in O(m) time

Theorem

For each median graph G with m edges and the weighted function w, $Med_w(G)$ can be computed in linear time O(m)

L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} $\bullet \circ \circ \circ \circ \circ \circ \circ \circ$	Conclusion 00

Median set in the cube-complex ${\mathcal G}$

L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Preliminaries O	-classes 1 0000 0	Median set in G	Median set in the cube-complex <i>G</i> ⊙●○○○○○○	Conclusion
Input				

• The complex

L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès
Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion 00
Input				



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Preliminaries	Θ-classes 00000	Median set in G 0000	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
Input				



• The terminals

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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
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• The terminals



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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
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L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion 00
Input				



• The terminals



L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
Input				



• The terminals



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Θ**-classes** 00000 Median set in G

Median set in the cube-complex ${\mathcal{G}}$ 0000000

Conclusion

Box complex (Van de Vel '93)



Θ**-classes** 00000 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Median set in the cube-complex ${\mathcal{G}}$ 0000000

Conclusion

Box complex (Van de Vel '93)





L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

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Θ-classes 00000 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Conclusion

Halfspaces and Carriers



L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

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Halfspaces and Carriers



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Θ-classes 00000 Median set in G

Conclusion

Halfspaces and Carriers



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Θ-classes 00000 Median set in G

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Halfspaces and Carriers



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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000 \bullet 000	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 0000000	Conclusion
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L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
<i>E_i-mediar</i>	ı			



Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Θ**-classes** 00000 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Median set in the cube-complex \mathcal{G} 00000000

Conclusion

Majority rule in the ℓ_1 -cube complex

$$egin{aligned} &
ho'=
ho=1\ & Med_w(\mathcal{G})\subseteq\mathcal{H}' \end{aligned}$$



Θ**-classes** 00000 $\begin{array}{c} \text{Median set in } G \\ \text{0000} \end{array}$

Median set in the cube-complex \mathcal{G} 0000000

Conclusion

Majority rule in the ℓ_1 -cube complex

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- $egin{aligned} & 0 <
 ho \leq
 ho' < 1 \ & \mathcal{M}ed_w(\mathcal{G}) \subseteq \mathcal{N}^\circ \end{aligned}$



L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Θ**-classes** 00000 Median set in G

Median set in the cube-complex \mathcal{G} 0000000

Conclusion

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 $\begin{aligned} \bullet \qquad 0 < \rho < \rho' = 1 \\ \textit{Med}_w(\mathcal{G}) \subseteq \mathcal{H}' \cup \mathcal{N}^\circ \\ \textit{Med}_w(\mathcal{G}) \text{ intersects } \mathcal{H}' \text{ and } \mathcal{N}^\circ \end{aligned}$



L. Bénéteau, J.Chalopin, V.Chepoi, Y.Vaxès

Θ**-classes** 00000 Median set in G

Median set in the cube-complex \mathcal{G} 0000000

Conclusion

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• $0 = \rho \leq \rho' = 1$ $Med_w(\mathcal{G})$ intersects $\mathcal{H}, \mathcal{H}'$ and \mathcal{N}°







Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 000000 \bullet 0	Conclusion 00
Algorithm	ı			



Preliminaries	<mark>Θ-classes</mark> 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion
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Preliminaries	<mark>Θ-classes</mark> 00000	Median set in G	Median set in the cube-complex \mathcal{G} 000000 \bullet 0	Conclusion
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 000000 \bullet 0	Conclusion 00
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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex \mathcal{G} 00000000	Conclusion 00
Algorithm	ı			



Θ**-classes** 00000 Median set in G

Median set in the cube-complex \mathcal{G} 0000000

Conclusion

Median set computation in linear time

Theorem

For each cube complex \mathcal{G} of a median graph G, and for each weighted set of points in \mathcal{G} , $Med_w(\mathcal{G})$ can be computed in linear time.

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Preliminaries	Θ-classes 00000	Median set in G	Median set in the cube-complex ${\cal G}$ 00000000	Conclusion ●○

Conclusion

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Preliminaries	Θ -classes 00000	Median set in G	Median set in the cube-complex ${\mathcal G}$ 00000000	Conclusion ○●
Perspecti	ves			

• Computation of the center and diameter in median graphs in subquadratic time.

Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion
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Perspectiv	ves			

• Computation of the center and diameter in median graphs in subquadratic time.

• Computation of the barycenter in linear time

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Median in median graphs and their cube complexes in linear time

Preliminaries	Θ-classes	Median set in G	Median set in the cube-complex ${\cal G}$	Conclusion
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Perspecti	ves			

• Computation of the center and diameter in median graphs in subquadratic time.

• Computation of the barycenter in linear time

• Computation of the median set in other graph classes with strong metric properties

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Median in median graphs and their cube complexes in linear time

Preliminaries	⊖-classes	Median set in G	Median set in the cube-complex ${\mathcal G}$	Conclusion
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Thank you!

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Median in median graphs and their cube complexes in linear time