Relational Models for the Lambek calculus with Intersection and Unit

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The Lambek Calculus

The Lambek calculus is a version of non-commutative intuitionistic linear logic, formulated as the following sequent calculus:

$$\frac{}{A \to A}\ Id \qquad \frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ Cut$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \backslash B, \Delta \to C}\ \backslash L \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ \backslash R \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ \cdot L$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B / A, \Pi, \Delta \to C}\ / L \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ / R \qquad \frac{\Pi \to A \quad \Delta \to B}{\Pi, \Delta \to A \cdot B}\ \cdot R$$

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- In $\backslash R$ and $/ R$, the antecedent Π should be non-empty.
- This restriction is motivated by linguistic applications: otherwise, having "extremely interesting book" validated as $(N / N) /(N / N), N / N, N \to N$, we would also validate "extremely book" as $(N / N) /(N / N), N \to N$.

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- The two systems, $\mathbf{L}$ and $\mathbf{L}^{\Lambda}$, are not directly reducible to one another, so theory here goes in parallel.
- In this talk, we show one example of different behaviour of $\mathbf{L}^{\Lambda}$ and $\mathbf{L}$.

Relational Models

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An R-model is a triple $\mathcal{M} = (W, U, v)$, where W is a non-empty set, $U \subseteq W \times W$ is transitive and $v \colon \mathrm{Fm} \to \mathcal{P}(U)$ obeys the following:

$$v(A \cdot B) = v(A) \circ v(B) = \{(x,z) \mid \exists y \in W \, (x,y) \in v(A) \text{ and } (y,z) \in v(B)\};$$

$$v(A \setminus B) = v(A) \setminus_U v(B) = \{(y,z) \in U \mid \forall x \in W \, (x,y) \in v(A) \Rightarrow (x,z) \in v(B)\};$$

$$v(B \,/\, A) = v(B) \,/_U\, v(A) = \{(x,y) \in U \mid \forall z \in W \, (y,z) \in v(A) \Rightarrow (x,z) \in v(B)\}.$$

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A sequent $A_1, \ldots, A_n \to B$, where $n > 0$, is true in $\mathcal{M}$ if $v(A_1) \circ \ldots \circ v(A_n) \subseteq v(B)$.

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- Square R-models are natural models for $\mathbf{L}^{\Lambda}$, while arbitrary ones are for $\mathbf{L}$.

Completeness Results

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- We shall now explore what happens with these proofs when we extend the Lambek calculi with extra operations.

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- No loops are allowed: $(x, x) \notin E$
- The graph is constructed iteratively, and in the limit we get a universal model for a given set of hypotheses $\mathcal{H}$.

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- In particular, there should be 'labels' $p\,/\,p$ and $q\,/\,q$, which are **incomparable** if p and q are different variables.
- Andréka and Mikulás consider **sets** of formulae as labels: now $v(A) = \{(x, y) \in W \times W \mid \vdash_{\mathbf{L}^{\Lambda}} A' \to A \text{ for some } A' \in \mathcal{L}(x, y)\}$.

Intersection

- Let us extend **L** and $\mathbf{L}^{\Lambda}$ with **intersection** (additive conjunction):

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \wedge L_1 \qquad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \wedge L_2 \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \wedge B} \wedge R$$

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- The corresponding calculi will be denoted by $\mathbf{L}\wedge$ and $\mathbf{L}^{\Lambda}\wedge$, depending on whether Lambek's restriction is imposed.

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Theorem
$\vdash_{\mathbf{L}^{\Lambda}\wedge} \Pi \to B \iff \mathcal{M} \vDash \Pi \to B$ *for each square R-model* $\mathcal{M}$.

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- We propose another approach, which uses an explicit unit constant.

Unit in Relational Models

- The unit $\mathbf{1}$ is axiomatized as follows:

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- Examples: $(\mathbf{1} \wedge F \wedge G) \to (\mathbf{1} \wedge F) \cdot (\mathbf{1} \wedge G)$ (Andréka and Mikulás), $\mathbf{1} /(F / F) \to (\mathbf{1} /(F / F)) \cdot (\mathbf{1} /(F / F))$ (Buszkowski).

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- Examples: $(\mathbf{1} \wedge F \wedge G) \to (\mathbf{1} \wedge F) \cdot (\mathbf{1} \wedge G)$ (Andréka and Mikulás), $\mathbf{1} /(F / F) \to (\mathbf{1} /(F / F)) \cdot (\mathbf{1} /(F / F))$ (Buszkowski).
- With such extra principles we conjecture undecidability (cf. Kanovich et al. 2020).

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Definition
An $\mathfrak{A}$-unit $\mathbf{1}_{\mathfrak{A}}$ is such an element of $\mathfrak{A}$ that for $R = \mathbf{1}_{\mathfrak{A}} \circ R = R \circ \mathbf{1}_{\mathfrak{A}}$ for each $R \in \mathfrak{A}$.
A non-standard model is $\mathcal{M}^{\mathfrak{A}} = (W, \mathfrak{A}, \mathbf{1}_{\mathfrak{A}}, v)$.

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- This, however, will not work with hypotheses, since a crucial part here is $\vdash \mathbf{1} \to B \cdot C \Rightarrow \vdash \mathbf{1} \to B$ and $\vdash \mathbf{1} \to C$.
- Reducts of non-standard models to the language without **1** are **standard R-models.**
- Thus, we get another, more straightforward proof of Mikulás 2015 theorem.

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- $A^* \setminus B \equiv \bigwedge_{n=0}^{\infty} (A^n \setminus B)$, same for $B \,/\, A^*$.
- We get (weak) R-completeness with the extension of $\mathbf{L}^{\Lambda}\wedge\mathbf{1}$ with such operations.

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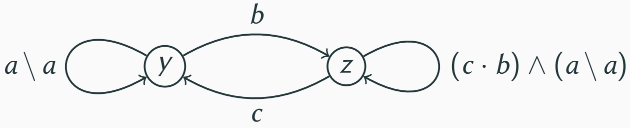

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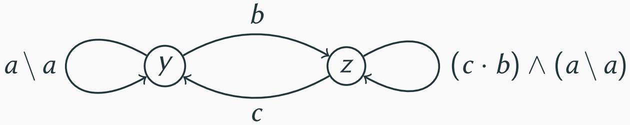

- ... but Mikulás didn't prove that it is not derivable in $\mathbf{L}^{\Lambda}\wedge$.

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- This rule admits cut elimination, so we establish non-derivability of $\Lambda \to b \cdot \big((c \cdot b) \wedge (a \setminus a)\big) \cdot c$ by exhaustive proof search.

Further Questions

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- Finite axiomatizability of semantic entailment on square R-models (Mikulás 2015).

Further Questions

- Algorithmic complexity: with **1** and extra axioms and with iterated divisions.
- Finite axiomatizability of semantic entailment on square R-models (Mikulás 2015).
- Strong completeness without product.

Thanks! Merci! Köszönöm!