# Construction of Bolzmann and Mc Kean Vlasov flows

## (the sewing argument)

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*In honor of Denis Talay*

**1. Bolzmann type equations**

$$\int_{\mathbb{R}^d} \varphi(x) f_{s,t}(dx) \tag{1}$$
$$= \int_{\mathbb{R}^d} \varphi(x) d\rho(x) + \int_s^t \int_{\mathbb{R}^d} \langle \nabla \varphi(x), b(x, f_{s,r}) \rangle f_{s,r}(dx) dr$$
$$+ \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}^d} f_{s,t}(dx) f_{s,r}(dv) \int_E \Big( \varphi(x + c(v,z,x,f_{s,t})) - \varphi(x) \Big) \gamma(v,z,x) \mu(dz) dr.$$

We look for a family of probability measures $f_{s,t}(dv), s \leq t$ which verify the above weak integro-differential equation.

a. If the coefficients does not depend on $f_{s,t}$ this is a Bolzmann type equation.

b. If $f_{s,r}(dv)$ is a given (known) $g_{s,r}(dv)$ and the coeffieints depend on $f_{s,t}$ this is a McKean Vlaso type equation

## 2. Stochastic equation

We consider the stochastc equation

$$X_{s,t} = X + \int_s^t b(X_{s,r}, \mathcal{L}(X_{s,r}))dr \tag{2}$$

$$+ \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X_{s,r-}, \mathcal{L}(X_{s,r}))1_{\{u \leq \gamma(v,z,X_{s,r-})\}} N_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr). \tag{3}$$

With $N_{\mathcal{L}(X_{s,r})}$ a Poisson point measure with compensator

$$\widehat{N}_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr) = \mathcal{L}(X_{s,r})(dv)\mu(dz)dudr$$

This represents a "probabilistic representation" (Tanaka) for the solution of the weak eqaution :

$$f_{s,r}(dx) = \mathcal{L}(X_{s,r})(dx)$$

## 3. Flow solution (a new formulation of the problem)

$$\mathcal{P}_1 = \{\mu \text{ probailiy, } \int_{R^d} |x|\, \mu(dx) < \infty\}$$

$$W_1(\mu,\nu) = \sup_{|\nabla f|\leq 1} \left| \int f d\mu - \int f d\nu \right|$$

**Folws of maps**

$$\mathcal{E} = \{\theta : \mathcal{P}_1 \to \mathcal{P}_1\},$$

$$d_*(\theta,\theta') = \sup_{\rho\in\mathcal{P}_1} \frac{W_1(\theta(\rho),\theta'(\rho))}{1+\int_{R^d} |x|\, \rho(dx)}$$

Then $(\mathcal{P}_1, W_1)$ and $(\mathcal{E}, d_*)$ are complete metric spaces.

**One step Euler scheme :** Given $\rho \in \mathcal{P}_1$ we take $X \sim \rho$ and construct

$$Y_{s,t}(\rho) = X + b(X, \rho)(t-s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) 1_{\{u \leq \gamma(v,z,X)\}} N_\rho(dv, dz, du, dr)$$

Then define $\Theta_{s,t} : \mathcal{P}_1 \to \mathcal{P}_1$ by

$$\rho \to \Theta_{s,t}(\rho) = \mathcal{L}(Y_{s,t}(\rho)).$$

**Theorem (flow solution)** There exists a unique flow $\theta_{s,t} \in \mathcal{E}$, $s < t$, such that

$$\theta_{s,t} = \theta_{r,t} \circ \theta_{s,r} \qquad s < r < t \qquad \text{(flow property)}$$

and

$$d_*(\theta_{s,t}, \Theta_{s,t}) \leq C(t-s)^2.$$

Moreover, $\theta_{s,t}(\rho)$ solves the weak equation (1) and admits the stochastic representation (2). We call $\theta_{s,t}$ the "**flow solution**"

**Remark 1** A simlar point of view has been introduced by Devie in the framework of rough path equations.

**Remark 2** The construction of the flow solution is done as limit of the Euler schemes :

$$\Theta^{\mathcal{P}}_{s,t} = \Theta_{s_{n-1},s_n} \circ .... \circ \Theta_{s_0,s_1} \quad \mathcal{P} = \{s = s_0 < .... < s_n = t\}$$

Then

$$d_*(\theta_{s,t}, \Theta^{\mathcal{P}}_{s,t}) \leq C(t-s) \times \max_{i=1,n}(s_{i+1} - s_i).$$

**Remark 3 : Uniqueness** of the solution of the weak equation is a difficult problem - for the flow solution we have uniquenss easely : there exists a unique solution constructed as **limit of Euler schemes** (but this produces just "one possible solution" of the weak equation).

**Remark 4 : Stability**

$$W_1(\theta_{s,t}(\rho), \theta_{s,t}(\xi)) \leq C(t-s)W_1(\rho, \xi).$$

**Sewing Lemma** (Feyel De la Pradele, et independement Gubinelli 2004/Ismael Bailleul 2015) Take $V$ abstract set and consider the maps

$$\mathcal{E}(V) = \{\Theta : V \to V\}$$
$$d_* = \quad \text{metric on } \mathcal{E}(V) \text{ such that } (\mathcal{E}(V), d_*) \text{ is complete.}$$

In our case : $V = \mathcal{P}_1$ and $\Theta_{s,t}$ is the one step Euler scheme. We construct Euler schemes

$$\Theta^{\mathcal{P}}_{s,t} = \Theta_{s_{n-1},s_n} \circ .... \circ \Theta_{s_0,s_1} \quad \mathcal{P} = \{s = s_0 < .... < s_n = t\}$$

**General** Let $\Theta_{s,t} \in \mathcal{E}(V), s < t$ which has the following two properties :

**Lipschitz** $\quad d_*(\Theta^{\mathcal{P}}_{s,t} \circ U, \Theta^{\mathcal{P}}_{s,t} \circ U') \leq C d_*(U, U') \quad \forall U, U' \in \mathcal{E}(V).$

And **SEWING property**

$$d_*(\Theta_{s,t}, \Theta_{r,t} \circ \Theta_{s,r}) \leq C(t-s)^{1+\varepsilon} \qquad \forall s < r < t.$$

**Sewing Lemma** Under the above hypothesis there exists a unique $\theta_{s,t} \in \mathcal{E}(V), s < t$, which has the flow property $\theta_{s,t} = \theta_{r,t} \circ \theta_{s,r}$ and such that $d(\Theta_{s,t}, \theta_{s,t}) \leq C(t-s)^{1+\varepsilon}$.

**Stability :** Given $X$ and $\rho$ define

$$Y_{s,t}(\rho, X) = X + b(X, \rho)(t-s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) 1_{\{u \leq \gamma(v,z,X)\}} N_\rho(dv, dz, du, dr)$$

Then define $\Theta_{s,t}(\rho, X) = \mathcal{L}(Y_{s,t}(\rho, X))$.

Take also $\overline{X}$ and $\overline{\rho}$ and construct $Y_{s,t}(\overline{\rho}, \overline{X})$ and $\Theta_{s,t}(\overline{\rho}, \overline{X}) = \mathcal{L}(Y_{s,t}(\overline{\rho}, \overline{X}))$ as above. Then

$$W_1(\Theta_{s,t}(\rho, X), \Theta_{s,t}(\overline{\rho}, \overline{X})) \leq (E\left|X - \overline{X}\right| + W_1(\rho, \overline{\rho})) \times (1 + C(t-s)).$$

We concatenate

$$W_1(\Theta^{\mathcal{P}}_{s,t}(\rho, X), \Theta^{\mathcal{P}}_{s,t}(\overline{\rho}, \overline{X})) \leq (E\left|X - \overline{X}\right| + W_1(\rho, \overline{\rho})) \times e^{C(t-s)}$$

**Proof** Take $\Pi(\rho, \overline{\rho})$ optimal coupling of $\rho$ and $\overline{\rho}$ and construct $\tau = (\tau^1, \tau^2) : [0,1] \to R^d \times R^d$ such that

$$\int_{R^d \times R^d} f(x, x') \Pi(\rho, \overline{\rho})(dx, dx') = \int_0^1 f(\tau(w)) dw$$

Consider also $\Pi(X, \overline{X})$ an optimal coupling of the laws of $X$ and $\overline{X}$

and construct $(X', \overline{X}') \sim \Pi(X, \overline{X})$. Then we take

$$\begin{aligned} Y'_{s,t}(\rho, X) &= X' + b(X', \rho)(t-s) \\ &\quad + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(\tau^1(w), z, X', \rho) 1_{\{u \leq \gamma(v,z,X')\}} N(dw, dz, du, dr) \end{aligned}$$

with

$$\widehat{N}(dw, dz, du, dr) = 1_{(0,1)}(w) dw \mu(dz) du dr).$$

In a similar way

$$\overline{Y}'_{s,t}(\overline{\rho},\overline{X}) = \overline{X}' + b(\overline{X}',\overline{\rho})(t-s) \\ + \int_s^t \int_{\mathbb{R}^d\times E\times\mathbb{R}_+} c(\tau^2(w), z, \overline{X}', \overline{\rho}) 1_{\{u\leq\gamma(v,z,\overline{X}')\}} N(dw, dz, du, dr)$$

Now we are on the same probability space and so

$$\begin{aligned} & W_1(\Theta_{s,t}(\rho,X), \Theta_{s,t}(\overline{\rho},\overline{X})) \\ & \leq E\left|Y'_{s,t}(\rho, ) - Y'_{s,t}(\overline{\rho},\overline{X})\right| \\ & \leq E\left|X'-\overline{X}'\right| + C\int_s^t E\left|X'-\overline{X}'\right| + \int_0^1 f(\tau^1(w)-\tau^2(w))dw + W_1(\rho,\overline{\rho})dr \\ & \leq (E\left|X'-\overline{X}'\right| + W_1(\rho,\overline{\rho})) \times (1 + C(t-s)). \end{aligned}$$

**Lipschitz**

$$W_1(\Theta^{\mathcal{P}}_{s,t}(\rho), \Theta^{\mathcal{P}}_{s,t}(\overline{\rho})) \leq W_1(\rho,\overline{\rho}) \times e^{C(t-s)}$$

**Sewing Property** One takes $s < r < t$ and $X \sim \rho$ and writes

$$\begin{aligned} Y_{s,t}(X) &= X + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) 1_{\{u \le \gamma(v,z,X)\}} N_\rho(dv, dz, du, dr) \\ &= Z + \int_r^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) 1_{\{u \le \gamma(v,z,X)\}} N_\rho(dv, dz, du, dr) \end{aligned}$$

with

$$Z = X + \int_s^r \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) 1_{\{u \le \gamma(v,z,X)\}} N_\rho(dv, dz, du, dr)$$

Notice that

$$Z \sim \Theta_{s,r}(\rho).$$

Then write

$$\begin{aligned} &Y_{r,t}(Z) \\ &= Z + \int_r^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, Z, \Theta_{s,r}(\rho)) 1_{\{u \le \gamma(v,z,Z)\}} N_{\Theta_{s,r}(\rho)}(dv, dz, du, dr). \end{aligned}$$

Then, the stability property give

$$E\left|Y_{s,t}(X)-Y_{r,t}(Z)\right| \le C\int_r^t E\left|X-Z\right| + W_1(\rho,\Theta_{s,r}(\rho))dr$$
$$\le C(t-r)E\left|X-Z\right| \le C(t-r)(r-s).$$

Since

$$Y_{s,t}(X) \sim \Theta_{s,t}(\rho) \quad and \quad Y_{r,t}(Z) \sim \Theta_{r,t}\circ\Theta_{s,r}(\rho)$$

we get

$$W_1(\Theta_{s,t}(\rho),\Theta_{r,t}\circ\Theta_{s,r}(\rho)) \le E\left|Y_{s,t}(X)-Y_{r,t}(Z)\right|$$
$$\le C(t-r)(r-s) \le C(t-s)^2.$$

**Remark** Finite variation :

$$E\left|Y_{s,t}(X)-Y_{s,t+h}(X)\right| \sim h \quad \to \quad h\times h = h^2$$

If a maritgale term appears then

$$E\left|Y_{s,t}(X)-Y_{s,t+h}(Z)\right| \sim h^{1/2} \quad \to \quad h^{1/2}\times h^{1/2} = h$$

**Particle systhem approximation.** For $i = 1, ..., N$

$$X^i_{k+1} = X^i_k + b(X^i_k, \frac{1}{N}\sum_{i=1}^N \delta_{X^i_k})(s_{k+1} - s_k)$$
$$+ \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X^i_k, \frac{1}{N}\sum_{i=1}^N \delta_{X^i_k}) 1_{\{u \leq \gamma(v,z,X^i_k)\}} N^i_k(dv, dz, du, dr)$$

with $N^i_k, i = 1, ..., N$ independent $PPM$ with intensity

$$\widehat{N}^i_k(dv, dz, du, dr) = (\frac{1}{N}\sum_{i=1}^N \delta_{X^i_k}(dv)) \times \mu(dz) du dr$$

**Theorem** For $f$ Lipscitz

$$\left| E(\frac{1}{N}\sum_{i=1}^N f(X^i_{nk})) - \int f(x)\theta_{0,t}(\rho)(dx) \right| \leq \frac{C}{N^{1/d}} + \frac{C}{n}$$

## Homogenous Bolzmann equation

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v)\mathbf{1}_{\{u \le |V_k^i - v|^\alpha)\}} N_k^i(dv, dz, du, dr)$$

with

$$\widehat{N}_k^i(dv, dz, du, dr) = \frac{1}{N} \sum_{j=1}^N \delta_{V_k^j}(dv) \times \mu(dz) du dr$$

## Inhomogenous Bolzmann equation

$$X_{k+1}^i = X_k^i + V_k^i (s_{k+1} - s_k)$$

and

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v)\mathbf{1}_{\{u \le |V_k^i - v|^\alpha)\}} \mathbf{1}_{\{|X_k^i - x| \le R\}} N_k^i(d(v, x), dz, du,$$

with

$$\widehat{N}_k^i(d(x, v), dz, du, dr) = \frac{1}{N} \sum_{j=1}^N \delta_{(V_k^j, X_k^j)}(dv, dx) \times \mu(dz) du dr$$

## Inhomogenuous equation with min field interaction

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v) 1_{\{u \leq |V_k^i - v|^\alpha)\}} 1_{\{u \leq \frac{1}{N}\sum_{j=1}^N n_\varepsilon(X_k^i - X_k^j)\}}$$
$$N_k^i(d(v,x), dz, du, dr)$$