Cox Constructions

Une marche aléatoire dans l'analyse stochastique et les probabilités numériques
A Random Walk in the Land of Stochastic Analysis and Numerical Probability

Philip Protter, Statistics Department, Columbia University
Based on joint work with **Andrés Riveros**

October, 2023

- Suppose we have a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ and a stopping time T defined on the space
- Assume $T < \infty$ a.s. and let

$$S_t = 1_{\{t \geq T\}}$$

- Since S is zero until T and then is 1, it is a submartingale (it is adapted because T is a stopping time)
- By the Doob-Meyer Decomposition Theorem, there exists a unique, increasing, predictable process A with $A_0 = 0$ such that

$$M_t = 1_{\{t \geq T\}} - A_t \text{ is a martingale} \tag{1}$$

- A stopping time T is **predictable** if there exists a sequence of stopping times $(S_n)_{n=1,2,\ldots}$, each $S_n < T$ a.s., and increasing to T such that $\lim_{n\to\infty} S_n = T$ a.s.
- A stopping time T is **accessible** if there exists a sequence of predictable times $(S_n)_{n=1,2,\ldots}$ such that

$$P(\cup_{n=1}^{\infty}\{S_n = T < \infty\}) = P(T < \infty)$$

- A stopping time is **totally inaccessible** if for every predictable stopping time S we have

$$P(\{T = S < \infty\}) = 0$$

- If the underlying filtration comes from a Hunt process, for example, then stopping times can be classified as either predictable or totally inaccessible, or a combination of the two. No need for accessible times.
- If T is predictable, then so too is the process $1_{\{t \geq T\}}$ hence the Doob-Meyer decomposition gives $1_{\{t \geq T\}} - 1_{\{t \geq T\}} = 0$ which is a martingale, and this case is uninteresting.
- **Therefore the interesting case is when T is totally inaccessible.** Such times T arise in the study of **Credit Risk.**

- A common assumption made in the literature is that the process A in (3) has absolutely continuous paths. That is,

$$A_t = \int_0^t \alpha_s ds \text{ a.s.} \tag{2}$$

- The Ethier-Kurtz Criterion says that in the decomposition $M_t = 1_{\{t \geq T\}} - A_t$ of (3) if

$$E(A_t - A_s | \mathcal{F}_s) \leq K(t-s) \text{ a.s. for } 0 \leq s \leq t < \infty$$

  Then A has the form $A_t = \int_0^t \alpha_s ds$ for almost all paths.
- Yan Zeng extended the Ethier-Kurtz Criterion to give necessary and sufficient conditions, but they're less easy to verify in practice
- This can be clarified in the case of a strong Markov Hunt semimartingale X

- E. Çinlar and J. Jacod showed back in 1981 that any $\mathbb{R}^d$br valued strong Markov process which is a Hunt process, and which is also a semimartingale, up to a change of time via an additive functional "clock," can be represented as the solution of a stochastic differential equation driven by dt, dW_t, and $n(ds, dz)$; where W is a standard multidimensional Brownian motion, and n is a standard Poisson random measure with mean measure given by $ds\nu(dz)$.
- Assume as given a strong Markov Hunt process semimartingale which can be represented on a space $(\Omega, \mathcal{F}, \mathbb{F}, P^x)$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, as follows:

$$
\begin{aligned}
X_t = X_0 \quad &+ \quad \int_0^t b(X_s)ds + \int_0^t c(X_s)dW_s \\
&+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)| \leq 1\}} [n(ds, dz) - ds\nu(dz)] \\
&+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)| > 1\}} n(ds, dz) \qquad (3)
\end{aligned}
$$

- For this situation with a strong Markov Hunt Process Semimartingale with the representation on the previous slide, we have the following result:
- For any totally inaccessible stopping time R on the space $(\Omega, \mathcal{F}, \mathbb{F}^\mu, P^\mu)$ the predictable increasing process A, with $A_0 = 0$, such that $1_{\{t \geq R\}} - A_t = M_t$ is a martingale, has the form $A_t = \int_0^t \lambda_s ds$ for some adapted process λ.
- This is nice, because the expression $\int_0^t \lambda_s ds$ lends itself to the interpretation of being a **hazard rate**:

$$\lambda_t = \lim_{h \to 0} \frac{1}{h} P(t \leq T < t + h | T \geq t) \quad (4)$$

- One more detail regarding stopping times: For a stopping time T and an event $\Lambda \in \mathcal{F}_T$, we define

$$T_\Lambda(\omega) = T(\omega) \text{ if } \omega \in \Lambda \text{ and } \infty \text{ if } \omega \notin \Lambda$$

- In most cases, a given stopping time T can be decomposed into $T = T_\Lambda \wedge T_{\Lambda^c}$, where T_Λ is totally inaccessible and T_{Λ^c} is predictable
- For a Hunt Markov process X, if T is totally inaccessible, let $\Lambda = \{X_{T-} \neq X_T\}$. Then $T = T_\Lambda$.
- For an arbitrary time R, $R_{\{X_{T-} \neq X_T\}}$ is the totally inaccessible part of R (Old result of P.A. Meyer)

Cox Constructions of Totally Inaccessible Times

- David Lando with Rick Durrett (circa 1998)
- Suppose we want to construct a totally inaccessible stopping time T with a given compensator $\int_0^t \alpha_s ds$
- Let Z be an exponential random variable independent of the process α_s and define

$$T = \inf_{t \geq 0} \{ \int_0^t \alpha_s ds > Z \}$$

- Note however that the jump times of our underlying Hunt process are totally inaccessible without the need of an independent exponential random variable Z

- This raises the question: **Are Cox Constructions intrinsic to Markov processes (and hence jump times of Markov processes)?**
- **Yes, they are**
- The exponential time used in a Cox Construction is always there in a Hunt process, but it's not independent. It turns out within the framework of Markov processes one does not need the independence and the Cox Construction still works
- The idea is to use a change of time argument with the process A_t that it continuous and increasing, to arrive at a compensator $\tilde{A}_t = t \wedge T$ which gives us that A_T is an exponential time

- One can then ask as a converse: **Can we find, for any given totally inaccessible stopping time, a Hunt process such that that stopping time is a jump time for the Hunt process?**
- **Yes, we can** – Barack Obama, 2008

- Let T be a totally inaccessible time on a filtered, complete probability space, with $P(T > 0) = 1$. Let

$$X_t = 1_{\{t \geq T\}}$$

- Then X is a Feller process, and we have the converse
- **Now, what about predictable times?**
- For this case we have the question of **Monique Jeanblanc:**
- Given a predictable time τ, can we express it as the hitting time of 0 of a continuous process?
- Since τ is predictable there exists a sequence $(S_n)_{n=1,2,3\ldots}$ of stopping times increasing to τ with $S_n < \tau$ a.s.
- We can use this sequence to construct the sought-after process

- We simply connect the successive stopping times with line segments. This creates a process which is anticipating, however.
- Our construction is typically not adapted to the underlying filtration, but we can correct this with a simple filtration enlargement.
- We need to mention G. Lowther who treated these issues in a blog, in 2009 and 2011

- **Thank you for your attention**