LARGE RANDOM MATRICES AND PDE'S

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FOR DENIS

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I, II, III joint work with Ch. Bertucci, M. Debbah and J-M. Lasry

IV joint work with Ch. Bertucci and P.E. Souganidis

V joint work with Ch. Bertucci

I, II, III in last year's course at CdF (videos)

I. INTRODUCTION

- Classical topic going back to Wishart (1928) for correlation matrices and Wigner (1958) – Dyson (1962) for

$$D_N = \frac{1}{\sqrt{N}} \left(W_N + W_N^T \right)$$

(Wishart : $\frac{1}{N}\, W_N\, W_N^T$)

where $W_N = (G_{ij})$ G_{ij} i.i.d Gaussian R.V.

- Let $\lambda_1 \leqslant \ldots \leqslant \lambda_N$: $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda i} \to$ semi-circular distribution

(Wishart $1 < i \leqslant N, 1 \leqslant j \leqslant M, \frac{M}{N} \to c > 0$, limit is the Marcenko-Pastur distribution)

- Books by A. Guionnet for the classical theory (IMU 2022)

- Typical examples of situations arising in many contexts (free probability, statistics...)
- Main applications: Finance, Telecommunications (Mobile, Networks)
- Dyson: $A_N + \frac{1}{\sqrt{N}} \left(W_N(t) + W_N(t)^T \right)$

  where A_N symmetric, $\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_{i0}} \to m_0 \in P(\mathbb{R}), W_N = (W_{ij}(t))1 \leqslant i, j \leqslant N$ and W_{ij} ind$^{\mathrm{t}}$ Brownian motions.

$$d\lambda_i = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \, dt + \frac{\sqrt{2}}{\sqrt{N}} \, dB^i$$

  (M.F. Bru related equation for Wishart...)

- Formally $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda i} \to m \in P(\mathbb{R})$

$$(D) \qquad \frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(H(m)m\right) = 0 \quad t \geqslant 0, x \in \mathbb{R}$$

where $H(m) =$ "$\int \frac{1}{x-y} m(y)dy$" $= PV(\frac{1}{x}) * m$

- $m_0 = \delta_0$, $m = \frac{2}{\pi t}\sqrt{(t-x^2)}_+$
- Many proofs exist (explicit, moments, gradient flows...) but none carry over to general/nonlinear models such as

$$dX_N = \sigma(X_N)\, dD_N + dD_N\sigma(X_N) + b(X_N)dt$$

or

$$dX_N = \sigma(X_N)\, dD_N\sigma(X_N) + b(X_N)dt$$

- Uniqueness proofs for (D): Fourier, moments...!
- General approach possible!

II. SPECTRAL DOMINATION AND M.P.

- A, B symmetric $\qquad A$ is spectrally dominated by B if

$$\lambda_i(A) \leqslant \lambda_i(B) \;\; \forall i \quad (\lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_N)$$

equivalent to $m(A) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ is stochastically dominated

by $m(B)$ i.e. $F_A(x) = \int 1_{(-\infty,x]} \, dm_A \geqslant F_B(x) \; \forall x$

- If m solves (D), let $F = \int_{-\infty}^{x} dm$

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}(H\frac{\partial m}{\partial x}) = 0$$

and $H\frac{\partial m}{\partial x} = FP(\frac{1}{x^2}) * F = \int \frac{F(x)-F(y)}{(x-y)^2}\ dy = (-\frac{d^2}{dx^2})^{1/2}F$

or

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\ \ A_0 F = 0 \tag{1}$$

(with $\frac{\partial F}{\partial x} \geqslant 0$, or $\frac{\partial F}{\partial t} + (\frac{\partial F}{\partial x})_+\ \ A_0 F = 0$).

- Maximum Principle! Formally if $F_0^1 \leqslant F_0^2$ at $t = 0$ then $F^1 \leqslant F^2$ for all (x, t)!
- Thus, Viscosity Solutions...!

- General nonlinear models lead to

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \left(\int c(x,y) \, \frac{F(x)-F(y)}{(x-y)^2} dy \right) + b(x)\frac{\partial F}{\partial x} = 0 \quad (2)$$

with $c(x,x) > 0, c(x,y) = c(x,x) + 0(x-y)^2)$ i.e.

$$\frac{\partial F}{\partial t} + a(x)\frac{\partial F}{\partial x} \, A_0 F + \frac{\partial F}{\partial x} \, A_1 F + b(x) \, \frac{\partial F}{\partial x} = 0 \qquad (3)$$

$A_1 F = \int d(x,y)F(y)dy$ "d smooth, nice at ∞"

(2) nice perturbation (A_1) of MP equation

MP for (2) if $c(x,y) \geqslant 0$

- N (Dyson): $\lambda_i^0 \leqslant \mu_i^0 \Longrightarrow \lambda_i(t) \leqslant \mu_i(t)$ (classical)

III. VISCOSITY SOLUTIONS AND LIMIT THEOREMS

Extension of viscosity solutions theory allow

THEOREM 1 (D) : *i) Let* $m_0 \in P(\mathbb{R}), F_0 = \int 1_{(-\infty,x]}\, dm$ *then* $\exists!$ *viscosity solution of (1) (F usc,* $F_* = F(x_-)$*)*

ii) comparison principle

iii) $F \in C$ *if* $F_0 \in C$*, F Lip. if* F_0 *Lip.*

iv) F Lip. for $t > 0$ *(reg. effect!)*

v) $N \to \infty$: $m_N^0 \to m_0$ *(tightly) then* $m_N \to m = \frac{\partial F}{\partial x}$

Remarks : i) contraction for all Wasserstein distances ($\simeq$ Crandall-Tartar, $\nearrow$, inv. by translation, conservation of center of mass)

ii) simular for Wishart and for general models:

$$b(x) - b(y) \geqslant -C_0(x-y) \text{ if } x \geqslant y$$

c Lip., bded strictly positive

iii) $N \to \infty$ straightforward but with some technical difficulties due to the singularity of the interaction $(\frac{1}{x})$

iv) the general case is not covered by standard argument for viscosity solutions "à la Barles-Imbert", in fact new arguments which can be used to make a complete theory for jump (diffusion) process and viscosity solutions of integro-differential operators... (Ch. Bertucci-PL2 in preparation)

v) Conjecture : $F \in C^{1,1/2}$ for $t > 0$?

IV. LARGE DEVIATIONS AND HJB IN W

- previous $N \to \infty$ akin to the law of large numbers
- large deviations: partial results by A. Guionnet and O. Zeitouni, slightly extended by A. Guionnet with very delicate proofs. . .
- $N - SDE \to N - FP$: Log transform formally yields the following optimal control problem given $m_0, m_1 \in P_2(\mathbb{R})$

$$\mathrm{Inf} \left\{ \int_0^1 \int m\, \alpha^2 ds\, dx / \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \Big(m(\alpha + Hm) \Big) = 0, \right.$$
$$\left. m|_{t=0} = m_0, m|_{t=1} = m_1 \right\}$$

justified by A.G. if m_0, m_1 have five moments and finite entropy $E[m] = -\left(\int \mathrm{Log}\, |x - y| \;\; dm(x)\, dm(y) \right)$

- Dynamic programming approach allows to justify *LD* for any $m_0 \in P_2, m_1$, with finite entropy.

$$\text{(HJB)} \qquad \frac{\partial V}{\partial t} + \frac{1}{2}\left|\frac{\partial V}{\partial m}\right|^2 + \left\langle \frac{\partial V}{\partial m}, -\frac{\partial}{\partial x}\Big((Hm)m\Big)\right\rangle = 0$$

- Typical example of control problems for systems with large random matrices (dyn. optim. of mobile networks: 6G, nG...)

- $V|_{t=0} = V_0 \in C(P_2)$, or $= 1_{\{m_1\}}$ $(+\infty$ if $m \neq m_1, 0$ at $m_1)$

- Viscosity solutions approach combining i) the case of Crandall-PL2 perturbed test functions by singular functions $\pm\, \delta E(m)$ which allow to have max/min points in L^3, ii) Ch. Bertucci adaptation to P of the Hilbert formulation for non-singular HJB equation on P, and iii) Tataru's method to take advantage of the fact that $\frac{\partial}{\partial x}\,(m\;Hm)$ is a "monotone" operator in Wasserstein space...

- Existence/uniqueness/$N \to \infty$ theorem whose (strategy of) proof is transparent!

V. INTEGRO-DIFFERENTIAL OPERATORS AND JUMP (DIFFUSION) PROCESSES

- Markov generator:
  $Au = \int \{u(x) + \nabla u(x) \cdot z\, \mathcal{X}(z) - u(x+z)\}\, d\mu_x(z)$ $\mathcal{X} \sim 1_{|z|\leqslant 1}$, μ_x weekly cont. $\geqslant 0$ meas. on $\mathbb{R}^d - \{0\}$

$$\sup_x \int |z|^2 \wedge 1 \;\; d\mu_x(z) \quad (+\,\text{equiint.})$$

- Rks: $\mu_{x,t}$, $+$"elliptic op." $(-\frac{1}{2}\, Tr\sigma\sigma^T\; D^2u - bDu, \sigma, b$ Lip.) if non deg. C. Cancelier ("ADN")

- Proba.: existence/uniq. law/path

  PDE : $\frac{\partial u}{\partial t} + Au = 0 \quad x \in \mathbb{R}^d, t > 0; u|_{t=0} = u_0 \in$ BUC (Lip ...) existence/uniq $\iff$ existence/uniq. in law, pathwise $\iff$ "doubled equation"

- "classical" (and easy):

  $sup_x \int d\mu_x(z) < \infty, W_1(\mu_x, \mu_y) \leqslant C|x-y|$

  (and relatively easy):

  $sup_x \int |x| \wedge 1) d\mu_x(x) < \infty, \exists\ \delta > 0\ W_1(\mu_x 1_{|z| \geqslant \delta}, \mu_y 1_{|z| \geqslant \delta}) \leqslant C(|x-y|, \|\mu_x |z| 1_{|z| \leqslant \delta} - \mu_y |z| 1_{|z| \leqslant \delta}\| \leqslant C|x-y|)$

- interesting case:

  $\int |z| 1_{|z| \leqslant 1} d\mu = +\infty$, ex. $\mu = \frac{1}{|z| d+\alpha}, 1 \leqslant \alpha < 2$

  Rk: μ indt of x is easy...

- Image measures (classical proba, Arisawa-Barles-Imbert)

  $Au = \int \{u(x) + \nabla u(x) \cdot T(x,z)\, \mathcal{X} - u(x + T(x,z)) d\mu(z)$
  with $|T(x,z) - T(y,z)| \leqslant C|x-y||z| \dots$

  $\approx$ Ito's proof, visc. sol. doubling var. is clear

- but $\mu_x = c(x,z)\mu$ with strong singularities was open (except for a remark by Bass non-degenerate fractional Laplacian)

- why? singularity and how to double variables (coupling)

$$w(x,y)(= u(x) - v(y), E[|X_t^x - X_t^y|^2] \dots)$$

  image measure clear

$$\int \Big\{ w(x,y) + \nabla_x w T(x,z)\mathcal{X} + \nabla_y w T(y,z)\mathcal{X} - w(x + T(x,z), y + T(y,z)) \Big\} d\mu$$

- answer (thanks S.) "maximal coupling"

$$\int \left\{ w+(\nabla_x w+\nabla_y w)+z\mathcal{X}-w(x+z,y+z) \right\} c(x,z)\Lambda c(y,z) d\mu$$

$$+\int +\left\{ w+\nabla_x w\cdot\mathcal{X}-w(x+z,y) \right\}(c(x,z)-c(y,z))_+ d\mu$$

$$+\int +\left\{ w+\nabla_y w\cdot\mathcal{X}-w(x,y+z) \right\}(c(y,z)-c(x,z))_+ d\mu$$

- strategy: i) Lip estimate + adaptation of Bernstein's method,

$$\text{ii)} \int \left\{ u(x)+\nabla u(x)\cdot z-u(x+z) \right\} c(x) \frac{dz}{|z|^{d+\alpha}}$$

$$= \int u(x)+\nabla u(x)\cdot b(x)\zeta-u(x+b(x)\zeta\} \frac{d\zeta}{|\zeta|^{d+\alpha}}$$

with $b(x)=c^{1/\alpha}$,

iii) integration by parts: $\frac{1}{|z|^{d+\alpha}} = -\frac{1}{\alpha}\ \mathrm{div}\ \left(\frac{z}{|z|^{d+\alpha}}\right)$

- leads to a collection of results (regularity of $c(x,z)$, cancellation of $\int z \cdot d\mu$)
- a few samples (OK with diffusion, $\mu_{x,t}$, more general μ than $\frac{dz}{|z|^{d+\alpha}}, Aij(x)\frac{z_i z_j}{|z|^{d+2+\alpha}}\ dz, \alpha(x)\dots$)

  existence/uniqueness of viscosity solutions in BUC (doubled equation OK $\Rightarrow$ law and pathwise)
- some can be translated in proba. but all?

Sample 1: $c(x,z) = c(x)d(x,z) + b(x,z),\ \mu = \frac{1}{|z|^{d+\alpha}}, c^{1/\alpha}$ Lip.($\alpha \to 2, c^{1/2}$ Lip.!), $|b(x,z) \leqslant C|z|^2 \ldots$
$d(x,z)$ "*smooth*", $d(x,0) \equiv 1\ \ (c(x) = c(x,0))$

Sample 2: $c^{1/2}$ Lip. in $x, \partial^2_{x,z} c$ bded, $\mu = \frac{1}{|z|^{d+\alpha}}$

In all cases, one needs to know (for each x) the singularity at 0 of μ_x!