Optimal Revealed Utilities and Convex Pricing kernels:
A Forward Point of view of convexity propagation

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A regular utility function $u$ is a

- (concave, increasing, positive) function on $[0, \infty)$, with $u(0) = 0$.
- Inada condition on its derivative $u_z$: $u_z(+\infty) = 0$, $u_z(0) = \infty$.
- Power utility $u(z) = \frac{x^{1-\alpha}}{1-\alpha}$, $\alpha < 1$, with conjugate $-\tilde{u}_y(y) = y^{-1/\alpha}$.

The convex decreasing conjugate utility $\tilde{u}$ and Legendre inequalities

- $\tilde{u}(y) = \sup_{x>0} (u(x) - xy)$, with $u'(x^*) = y$,
- $\tilde{u}(y) = u(-\tilde{u}_y(y)) + y\tilde{u}_y(y)$, $u(x) = x u_x(x) - \tilde{u}(u_x(x))$,
- Legendre inequality: $\tilde{u}(y) - u(x) - xy \geq 0 \ \forall (x, y) > 0$.

A dynamic utility on $(\Omega, \mathbb{P}, (\mathcal{F}_t))$

- is a family of optional random field $U = \{U(t, z), z \in \mathbb{R}^+\}$
- such that $\forall t, (z \to U(t, z))$ is a standard utility function.
- Its conjugate is the field $\tilde{U} = \{\tilde{U}(t, y), y \in \mathbb{R}^+\}$.

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Gap function

Definition Gap function (G. Charlier)

- Let \((u, \tilde{u})\) be a pair of conjugate utility functions \((\tilde{R}^+ \rightarrow \tilde{R}^+)\)
- The bivariate Gap function, \(G_u(x, y) = \tilde{u}(y) - u(x) + xy \geq 0\)
- Fenchel: \(G_u(x, u_x(x)) = 0, \forall x > 0\)
- Convex in each argument \((x, y)\) (separately), with \(x\)-derivative \(-u_x(x) + y\).

Integral Gap function and minimization

- Let \(\mu\) be a probability measure and \(Z \geq 0\) a r.v. with finite mean (or barycentre) \(b_\mu = E_\mu(Z) < +\infty\).
- \(E_\mu(G_u(Z, y)) = G_u(b_\mu, y) + E_\mu(u(Z)) - u(E_\mu(Z))\)
- The maximum in \(y\) is \(E_\mu(u(Z)) - u(E_\mu(z))\), attained at \(u_x(b_\mu)\).
Decision making in economics and finance

Decision making under uncertainty

- Most of decision making focuses on the selection of optimal sequence of actions given preferences criterium
- In economy and finance the preferences are based on expected utility (concave criterium) of some terminal value, or its robust extension.

- Backward point of view

Decision making in e-commerce

- Learning based on observed data, over the time.
- Preferencies of the agent are ”estimated” in view to propose an optimal offer

- Forward point of view
Samuelson (1930) C.P. Chambers

- Samuelson (1930) "Theory of revealed preference"
- Chambers: "We can never see a utility function, but what we might be able to see are demand observations at a finite list of prices.


- Preferences are defined in isolation to the investment universe
- How use intertemporal diversification,(short, medium, long term strategies). With which utility function?
- At the optimum the investor should become indifferent to the investment horizon.
Concave optimization with constraint

A backward toy model with conjugacy

- **Data**: utility concave function $u$, its conjugate $\tilde{u}$, a horizon $T$, $\mathcal{X}$ a convex family of r.v. $X_T$,
- $\{Y_t \geq 0\}$ a state price process with $\mathbb{E}(X_T Y_T) \leq X_0 Y_0$, $\forall X_T \in \mathcal{X}$.
- **Optimization problem** $\max\{\mathbb{E}(u(X_T)) | X_T \in \mathcal{X}\}$, with budget constraint $\mathbb{E}(Y_T X_T) \leq x$.

Solution of the problem via Lagrange multiplier and conjugate

- Equivalent Pb: $\max\{\mathbb{E}(u(X_T) + y(x - Y_T X_T)) | X_T \in \mathcal{X}\} = U_0(x, y)$
- If $-\tilde{u}_y(y Y_T) \in \mathcal{X}$, then optimum is $X_T^* = -\tilde{u}_y(y Y_T)$
- $y$ is selected by $\mathbb{E}[-\tilde{u}_y(y Y_T) Y_T] = x$
- $\tilde{U}_0(y) = \mathbb{E}[\tilde{u}(y Y_T)]$ decreasing convex conjugate utility in $y$.
- Its conjugate at 0, $U_0(x)$ is an increasing concave function.
Forward recovery bi-revealed utility problem

without optimization and control problem
Observed process and Choice of an adjoint process

To recover \( \{U(t, z)\} \), the observable also must depend on \( z \)

The data

- An initial condition \( U(0, z) = u(z) \) a given utility function
- An observed (data) positive adapted random field, \( X = \{X_t(x)\} \)
- The optimal adequation between \( \{U(t, z)\} \) and \( \{X_t(x)\} \) is the requirement that \( \{U(t, X_t(x))\} \) is a martingale for any \( x \).

To limit the size of family of dynamic utilities coherent with the data \( X \), we add constraints on the conjugate utility \( \{\tilde{u}(t, y)\} \) via an other family.

- \( Y = \{Y_t(y)\} \) of positive adjoint process,
- Orthogonal to \( \{X_t(x)\} \) s.t \( X_t(x)Y_t(y) \) is a supermartingale \( \forall(x, y) \)
- Coherent-optimal to \( \tilde{u}(t, y) \), that is \( \tilde{u}(t, Y_t(y)) \) is a martingale.
Examples of bi-revealed problems

The constant case $X_t(x) = x$

- $\{U(t, x)\}$ is a revealed utility "if and only" if
- its marginal utility $\{U_z(t, x)\}$ is a martingale.
- Then, $U$ is bi-revealed, by two constant process

Linear characteristic process $X_t(x) := xX_t(1) = x X_t$

- Use $X_t$ as numeraire and define $U^X(t, z) = U(t, zX_t)$, a martingale with characteristic process $x$
- Then " $U^X_z(t, z) = X_t U_z(t, zX_t) = X_t Y_t(u_z(z))$ is a martingale"
- $U$ is a bi-revealed utility

Power utility: $u(z) = z^{1-\alpha}/(1 - \alpha)$

- Then, $U'_z(t, z) = Y_t(u_z(z/X_t))$, and if $u$ is a power utility, then $U$ is a power utility, if and only if $Y$ is linear
Bi-revealed utility problem
How to express on \((X_t(x), Y_t(y))\) and the previous (sur)-martingales conditions, that \(\tilde{u}\) is the conjugate of \(U\).

\[
\tilde{u}(y) - u(x) + xy \geq 0, \text{ with equality for } y = u_x(x)
\]

Pathwise First order condition

- A system is bi-revealed system, iff
  - A Pathwise first order condition holds \(Y_t(u_z(x)) = U_z(t, X_t(x))\)
  - \(\{X_t(x)Y_t(u_x(x))\}\) is a martingale

- Sketch of the proof: Thanks to martingale-surmartingales conditions, and Legendre inequalities
  - \(0 \leq \mathbb{E}[\tilde{u}(\tau, Y_\tau(y)) - U(\tau, X_\tau(x) + Y_\tau(y)X_\tau(x)] \leq \tilde{u}(y) - u(x) + xy\)
  - For \(y = u_x(x)\), by Fenchel, the left and right sides are 0, as the non negative function on the expectation are 0
  - By strict monotony, first order condition: \(Y_t(u_x(x)) = U_z(t, X_t(x))\)
The utility $U$ is bi-revealed by the triplet $(u, X, Y)$ iff (approx)

- $\forall (x, y), \{X_t(x)Y_t(y)\}$ is a supermartingale
- $\{X_t(x)Y_t(u_x(x))\}$ is a martingale
- First order $U_x(X_t(x)) = Y_t(u_x)$

Asymptotic behavior and intrinsic behavior

- Limit Conditions
  - $\lim_{x \to 0} \frac{X_t(x)}{x} = \Lambda_t^X > 0$ and $\lim_{y \to \infty} \frac{Y_t(y)}{y} = H_t^Y > 0$,
  - $L_t^{\text{int}} = \Lambda_t^X H_t^Y$, $L_0^{\text{int}} = 1$ is a $\mathbb{P}$-martingale.

- The intrinsic universe:
  - Change of proba : $dQ^{\text{int}} = L_T^{\text{int}}.d\mathbb{P}$,
  - Change of numeraire $X_t^{\text{int}}(x) = X_t(x)/\Lambda_t^X$, $Y_t^{\text{int}}(y) = Y_t(y)/H_t^Y$

- $Q^{\text{int}}$- Supermartingale properties
  - $\{X_t^{\text{int}}(x)\}$, $\{Y_t^{\text{int}}(y)\}$, $\{X_t^{\text{int}}(x)Y_t^{\text{int}}(y)\}$ are $Q^{\text{int}}$-supermartingales.
  - Preference for the present with $\{U(t, z)\}$ $Q^{\text{int}}$-supermartingale
Algebraic Construction of bi-revealed utilities

Pathwise Utility construction

Hyp $x \rightarrow X_t(x)$ is strictly increasing in $x$ with range $[0, \infty)$,

\[ U_z(t, z) = Y_t(u_z(X_{t}^{-1}(z))), \quad U(t, x) = \int_{0}^{x} Y_t(u_z(X_{t}^{-1}(z))) \, dz \]

Why $U(t, X_t(x)) = \int_{0}^{x} Y_t(u_x(z)) \, d_z X_t(z)$ is a "martingale"?

- This last integral is a Stieljes integral, with explosion near to $z = 0$
- One to one algebraic bijection between $(u, X, U)$ and $(u, X, Y)$

Exemple of Differentiable characteristic process

- Assume $x \mapsto X_t(x)$ to be $x$-differentiable with differential $X_x(t, x)$.
- $U(t, X_t(x))$ is given by $U(t, X_t(x)) = \int_{(0, x]} Y_t(u_x(z)) \, X_x(t, z) \, dz$
- Under some "regularity", the $U$-martingale condition is equivalent to
  \[ \{ Y_t(u_x(x))X_x(t, x) \} \text{ is martingale} \]
- In the bi-revealed case, \{ $Y_t(u_x(x))X_x(t, x)$ \}, \{ $Y_t(u_x(x))X_t(x)$ \} are martingales

In the general case, similar argument from Darboux sums
Application to Itô’s framework
Generalities on the Itô’s framework

$(\Omega, (\mathcal{F}_t), \mathbb{P})$ is equipped with a $d$-dimensional Brownian motion $(W_t)$

Random field SDEs $(\mu_t(z), \sigma_t(z))$,

SDE form: $dX_t(x) = \mu_t^X(X_t(x))dt + \sigma_t^X(X_t(x)).dW_t$, $X_0 = x$

Random Field $dX_t(x) = \beta^X(t, x)dt + \gamma^X(t, x).dW_t$, $X_0 = x$

Parameter/Coeff $\gamma^X(t, x) = \sigma_t(X_t(x))$ $\beta^X(t, x) = \mu_t(X_t(x))$

Comments

- The SDE point of view is better suited for problem related to comparison
- RF Point of view is better suited to differentiability issues.
- All results are based on a strong differential regularity of coefficients or parameters, detailed in SIAM paper(2013)
SPDE’s for compound SDE’s

Itô Ventzel Formula for semimartingale $Z$

$\triangleright$ A $\mathcal{K}^2_{loc}$-Itô semimartingale RF $F(t, z)$, $(\beta^F, \gamma^F)$.

$\triangleright$ $dF(t, Z_t) = \beta^F(t, Z_t)dt + \gamma^F(t, Z_t).dW_t$

$\triangleright$ $+ F_z(t, Z_t)dZ_t + \frac{1}{2}F_{zz}(t, Z_t)\langle dZ_t \rangle + \langle \gamma^F_z(t, Z_t).dW_t, dZ_t \rangle$

Second formulation

$\triangleright$ $dF(t, z) = \beta^F(t, z)dt + \gamma^F(t, z).dW_t = d_t F(t, z)|_{z=Z_t}$

$dF_z(t, z) = \beta^F_z(t, z)dt + \gamma^F_z(t, z).dW_t$.

$\triangleright$ Then, $dF(t, Z_t) = d_t F(t, z)|_{z=Z_t} - \frac{1}{2}F_{zz}(t, Z_t)\langle dZ_t \rangle + d(F_z(t, Z_t)Z_t) - Z_t dF_z(t, Z_t)$

Bi-revealed utilities

$\triangleright$ $dU(t, X_t(x)) = \gamma^U(t, X_t(x)).dW_t + d(U_z(t, X_t(x))X_t(x))$

$\triangleright$ $+ \beta^U(t, X_t(x))dt - \frac{1}{2}U_{zz}(t, X_t(x)\langle dX_t(x) \rangle - X_t(x)dY_t(u_z(x))$
Utility issued from SDEs

- $X$ regular solution of SDE $(\mu^X, \sigma^X)$ with inverse flow $\{\xi(t, z)\}$

- $Y$ regular solution of SDE $(\mu^Y, \sigma^Y)$ decoupled of $X$

- suborthogonality: $z \mu^Y_t(y) + y \mu^X_t(x) + \langle \sigma^X_t(x), \sigma^Y_t(y) \rangle \leq 0$,

$U$ differential coefficients

$$
\gamma^U_z(t, z) = \sigma^Y_t(U_z(t, z)) - U_{zz}(t, z)\sigma^X_t(z).
$$

$$
\beta^U(t, z) = \frac{1}{2} U_{zz}(t, z)\|\sigma^X_t(z)\|^2 + z \mu^Y_t(U_z(t, z))
$$

or

$$
\beta^U(t, z) + \langle \gamma^U_z(t, z), \sigma^X_t(z) \rangle = -\left[\frac{1}{2} U_{zz}(t, z)\|\sigma^X_t(z)\|^2 + U_z(t, z)\mu^X_t(z)\right].
$$

In the intrinsic case, $\mu^Y_t(y) \leq 0$, $\beta^U(t, z) \leq 0$ and $U(t, z)$ is a supermartingale.
Generally utility criteria are associated with optimization problem;

General utility of test process: \( dZ_t = \phi^Z_t dt + \psi^Z_t . dW_t \)

\[
\begin{align*}
\Delta dU(t, Z_t) &= \\
&= \frac{1}{2} U_{zz}(t, Z_t) \| \sigma_t^X(Z_t) - \psi_t^Z \|^2 dt + \gamma^U(t, Z_t) + U_z(t, Z_t) \psi_t^Z . dW_t \\
&\quad + \left[ Z_t \mu_t^Y(U_z(t, Z_t)) + U_z(t, Z_t) \phi_t^Z + \langle \sigma_t^Y(U_z(t, Z_t)), \psi_t^Z \rangle \right] dt
\end{align*}
\]

The last term is the drift of the product \( \{ Z_t Y_t(y) \} \) taken at \( y = U_z(t, Z_t) \), negative for \( \{ Z_t Y_t(y) \} \) supermartingale.

### Controlled processes

- A controlled process \( Z \) is suborthogonal to \( \{ Y_t(y), \forall y \} \), that is \( Z_t Y_t(y) \) is supermartingale (ex \( X_t(x) \)).

- A controlled process \( Z \) is suboptimal:
  - \( U(t, Z_t) \) is a supermartingale.
  - \( U(t, Z_t) \) is a martingale iff \( Z \) is a solution of the SDE(\( \mu, \sigma \)).
Applications to convex pricing forward dynamics

Work in progress
Stephane Comment

- OK, it is interesting, but utility framework is not so useful in practice, in mathematical finance
- Pricing problems are more challenging, even the more classical with convex pay-off $h$.
- In particular, given a price derivative today how simulate coherent price in the future

- Very similar to utility problem where concave is replaced by convex
- But, only if the price today of convex derivative is a convex function.

A old question studied with S.Shreve and M.Jeanblanc in 1995
Classical BS Formula

- $X_t(x) =$ Geometric Brownian motion $= x X_t(1)$
- Option pay-off $= \text{Target} = h(T, X_T(x))$
- $h$ convex option, positive, (nondecreasing)
- Intrinsic universe: $(\Omega, (\mathcal{F}_t), \mathbb{Q})$

BS Formula

- By change of numeraire, and probability measure, $X_t(x)$ can be assumed to be $\mathbb{Q}$ martingale
- $X_t = x \exp(\sigma W^Q_t - \frac{1}{2} \sigma^2 t) = x L_T$
- The pricing rule is risk-neutral, that is
  \[ BS(h)(x, T) = \mathbb{E}_Q(h(x L_T)) = \Phi_h(0, x) = \Phi_h(x) \]
- $BS(h)(x)$ is a convex function $\Phi_h(x)$ with $\Phi_h(T, x) = h(x)$
Model with local volatility in risk neutral universe

- \( dX_t = X_t(\sigma(t, X_t)dW_t^Q), \quad X_0 = x \)
- \( \gamma(t, x) = x\sigma(t, x) \)
- \( \sigma(t, x) \) continuous and bounded above
- \( \partial_x \gamma(t, x) \) is continuous in \((t, x)\) and Lipschitz continuous and bounded in \(x\) uniformly in \(t\)

**Theorem** The European price \( \Phi_h(0, x) = \mathbb{E}_Q(h(X_T(x))) \) is a convex function, with bounded derivatives

Then, it is possible to adapt the “machinery” of concave utility.
Conclusion

**Que Sau, God of health and longevity**

Te garde sous son regard bienveillant

**Merci pour tout, Denis...**