Optimal Revelated Utilities and Convex Pricing kernels:

A Forward Point of view of convexity propagation

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Regular utility function u and its conjugate

A regular utility function u is a

- (concave, increasing, positive) function on $[0,\infty)$, with $u(0)=0$.
- Inada condition on its derivative u_z: $u_z(+\infty)=0, u_z(0)=\infty$.
- power utility $u(z)=\frac{x^{1-\alpha}}{1-\alpha}, \alpha<1$, with conjugate $-\tilde{u}_y(y)= y^{-1/\alpha}$

The convex decreasing conjugate utility $\tilde{u}$ and Legendre inequalities

- $\tilde{u}(y)=\sup_{x>0}\big(u(x)-x\,y\big)$, with $u'_x(x^*)=y$,
- $\tilde{u}(y)=u(-\tilde{u}_y(y))+y\tilde{u}_y(y)$, $\quad u(x)=x\,u_x(x)-\tilde{u}(u_x(x))$
- Legendre inequality: $\tilde{u}(y)\,-u(x)-x\,y\;\geq 0\;\forall (x,y)>0$

A dynamic utility on $(\Omega,\mathbb{P},(\mathcal{F}_t))$

- is a family of optional random field $\mathrm{U}=\{U(t,z), z\in\mathbb{R}^+\}$
- such that $\forall t,(z\rightarrow U(t,z))$ is a standard utility function.
- Its conjugate is the field $\widetilde{\mathrm{U}}=\{\widetilde{U}(t,y), y\in\mathbb{R}^+\}$

Gap function

Definition Gap function(G.Charlier)

- Let $(u, \tilde{u})$ be a pair of conjugate utility functions $(\bar{R}^{+} \to \bar{R}^{+}$
- The bivariate Gap function, $G_u(x, y) = \tilde{u}(y) - u(x) + x\,y \geq 0$
- Fenchel: $G_u(x, u_x(x)) = 0, \forall x > 0$
- Convex in each argument (x, y) (separately), whith x-derivative $-u_x(x) + y$.

Integral Gap function and minimization

- Let μ be a probability measure and $Z \geq 0$ a r.v. with finite mean (or barycentre) $b_\mu = \mathbb{E}_\mu(Z) < +\infty$.
- $\mathbb{E}_\mu(G_u(Z,\, y)) = G_u(b_\mu, y)) + \mathbb{E}_\mu(u(Z)) - u(\mathbb{E}_\mu(Z))$
- The maximum in y is $\mathbb{E}_\mu(u(Z)) - u(\mathbb{E}_\mu(z))$, attained at $u_x(b_\mu))$.

Decision making in economics and finance

Decision making under uncertainty

- Most of decision making focuses on the selection of optimal sequence of actions given preferencies criterium
- In economy and finance the preferences are based on expected utility (concave criterium) of some terminal value, or its robust extension.
- Backward point of view

Decision making in e-commerce

- Learning based on observed data, over the time.
- Preferencies of the agent are "estimated" in view to propose an optimal offer
- Forward point of view

Samuelson (1930) C.P. Chambers

- Samuelson (1930) "Theory of revealed preference"
- Chambers : *"We can never see a utility function, but what we might be able to see are demand observations at a finite list of prices.*

Forward stochatic utilities(2002-2009) Musiela, Zariphopoulou

- Preferences are defined in isolation to the investment universe
- How use intertemporal diversification,(short, medium, long term strategies). With which utility function ?
- At the optimum the investor should become indifferent to the investment horizon.

Concave optimization with constraint

A backward toy model with conjugacy

- Data: utility concave function u, its conjugate $\tilde{u}$, a horizon T , $\mathscr{X}$ a convex family of r.v. X_T,
- $\{Y_t \geq 0\}$ a state price process with $\mathbb{E}(X_T Y_T) \leq X_0 Y_0, \forall X_T \in \mathscr{X}\}$.
- *Optimization problem* $\max\{\mathbb{E}(u(X_T))|X_T \in \mathscr{X}\}$, with budget constraint $\mathbb{E}(Y_T X_T) \leq x$.

Solution of the problem via Lagrange multiplier and conjugate

- Equivalent Pb: $\max\{\mathbb{E}(u(X_T) + y(x - Y_T X_T)|X_T \in \mathscr{X}\} = U_0(x, y)$
- If $-\tilde{u}_y(yY_T) \in \mathscr{X}$, then optimum is $X_T^* = -\tilde{u}_y(yY_T)$
- y is selected by $\mathbb{E}[-\tilde{u}_y(yY_T)Y_T] = x$
- $\tilde{U}_0(y) = \mathbb{E}[\tilde{u}(yY_T]$ decreasing convex conjugate utility in y.
- Its conjugate at 0, $U_0(x)$ is an increasing concave function.

Forward recovery bi-revealed utility problem

without optimization and control problem

Observed process and Choice of an adjoint process

To recover $\{U(t,z)\}$, the observable also must depend on z

The data

- An initial condition $U(0,z) = u(z)$ a given utility function
- An observed (data) positive adapted random field, $\mathrm{X} = \{X_t(x)\}$,
- The optimal adequation between $\{U(t,z)\}$ and $\{X_t(x)\}$ is the requirement that $\{U(t, X_t(x))\}$ is a martingale for any x.

To limit the size of family of dynamic utilities coherent with the data X, we add constraints on the conjugate utility $\{\tilde{u}(t,y)\}$ via an other family.

- $\mathrm{Y} = \{Y_t(y)\}$ of positive adjoint process,
- Orthogonal to $\{X_t(x)\}$ s.t $X_t(x)Y_t(y)$ is a supermartingale $\forall(x,y)$
- Coherent-optimal to $\tilde{u}(t,y)$, that is $\tilde{u}(t, Y_t(y))$ is a martingale.

Examples of bi-revealed problems

The constant case $X_t(x) = x$

- $\{U(t,x)\}$ is a revealed utility "if and only" if
- its marginal utility $\{U_z(t,x\}$ is a martingale.
- Then, U is bi-revealed, by two constant process

Linear characteristic process $X_t(x) := xX_t(1) = x\,X_t$

- Use X_t as numeraire and define $U^X(t,z) = U(t, zX_t)$, a martingale with characteristic process x
- Then " $U_z^X(t,z) = X_t\,U_z(t, zX_t) = X_t\,Y_t(u_z(z))$ is a martingale"
- U is a bi-revealed utility

Power utility: $u(z) = z^{1-\alpha}/(1-\alpha)$

- Then, $U_z'(t,z) = Y_t(u_z(z/X_t))$, and if u is a power utility, then U is a power utility, if and only if Y is linear

Pathwise first order condition

Bi-revealed utility problem

How to express on $(X_t(x), Y_t(y))$ and the previous (sur)-martingales conditions, that $\tilde{u}$ is the conjugate of U.

$$\tilde{u}(y) - u(x) + xy \geq 0, \text{ with equality for } y = u_x(x)$$

Pathwise First order condition

- A system is bi-revealed system, iff
  - A Pathwise first order condition holds $Y_t(u_z(x)) = U_z(t, X_t(x))$
  - $\{X_t(x)Y_t(u_x(x))\}$ is a martingale
- Sketch of the proof: Thanks to martingale-surmartingales conditions, and Legendre inequalitites
  - $0 \leq \mathbb{E}[\tilde{u}(\tau, Y_\tau(y)) - U(\tau, X_\tau(x) + Y_\tau(y)X_\tau(x)] \leq \tilde{u}(y) - u(x) + xy$
  - For $y = u_x(x)$, by Fenchel, the left and right sides are 0, as the non negative function on the expectation are 0
  - By strict monotony, first order condition: $Y_t(u_x(x)) = U_z(t, X_t(x))$

The utility U is bi-revealed by the triplet (u, X, Y) iff (approx)

- $\forall (x,y)$, $\{X_t(x)Y_t(y)\}$ is a supermartingale
- $\{X_t(x)Y_t(u_z(x))\}$ is a martingale
- First order $U_x(X_t(x)) = Y_t(u_x)$

Asymptotic behavior and intrinsic behavior

- Limit Conditions
  - $\lim_{x\to 0} \frac{X_t(x)}{x} = \Lambda_t^X > 0$ and $\lim_{y\to\infty} \frac{Y_t(y)}{y} = H_t^Y > 0$,
  - $L_t^{\text{int}} = \Lambda_t^X\, H_t^Y,\ L_0^{\text{int}} = 1$ is a $\mathbb{P}$-martingale.
- The intrinsic universe:
  - Change of proba : $d\mathbb{Q}^{\text{int}} = L_T^{\text{int}}.d\mathbb{P}$,
  - Change of numeraire $X_t^{\text{int}}(x) = X_t(x)/\Lambda_t^X,\ Y_t^{\text{int}}(y) = Y_t(y)/H_t^Y$
- $\mathbb{Q}^{\text{int}}$- Supermartingale properties
  - $\{X_t^{\text{int}}(x)\}$, $\{Y_t^{\text{int}}(y)\}$, $\{X_t^{\text{int}}(x)\, Y_t^{\text{int}}(y)\}$ are $\mathbb{Q}^{\text{int}}$-supermartingales.
  - Preference for the present with $\{U(t,z)\}$ $\mathbb{Q}^{\text{int}}$-supermartingale

Pathwise Utility construction

Hyp $x \to X_t(x)$ is strictly increasing in x with range $[0, \infty)$,

- $U_z(t,z) = Y_t(u_z(X_t^{-1}(z)))$, $\quad U(t,x) = \int_0^x Y_t(u_z(X_t^{-1}(z)))\,dz$
- Why $U(t, X_t(x)) = \int_0^x Y_t(u_x(z))\,d_z X_t(z)$ is a "martingale" ?
  - This last integral is a Stieljes integral, with explosion near to $z = 0$
  - One to one algebraic bijection between (u, X, U) and (u, X, Y)

Exemple of Differentiable characteristic process

- Assume $x \mapsto X_t(x)$ to be x-differentiable with differential $X_x(t,x)$.
- $U(t, X_t(x))$ is given by $U(t, X_t(x)) = \int_{(0,x]} Y_t(u_x(z))\,X_x(t,z)dz$
- Under some "regularity", the U-martingale condition is equivalent to $\{Y_t(u_x(x))X_x(t,x)\}$ is martingale
- In the bi-revealed case, $\{Y_t(u_x(x))X_x(t,x)\}$, $\{Y_t(u_x(x))X_t(x)\}$ are martingales

In the general case, similar argument from Darboux sums

Application to Itô's framework

Generalities on the Itô's framework

$(\Omega, (\mathcal{F}_t), \mathbb{P})$ is equipped with a d-dimensional Brownian motion (W_t)
Random field SDEs $(\mu_t(z), \sigma_t(z))$,

$$
\begin{aligned}
&\text{SDE form:} \quad dX_t(x) = \mu_t^X(X_t(x))dt + \sigma_t^X(X_t(x)).dW_t, \quad X_0 = x \\
&\text{Random Field} \quad dX_t(x) = \beta^X(t,x)dt + \gamma^X(t,x).dW_t, \quad X_0 = x \\
&\text{Parameter/Coeff} \quad \gamma^X(t,x) = \sigma_t(X_t(x)) \quad \beta^X(t,x) = \mu_t(X_t(x))
\end{aligned}
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Comments

- The SDE point of view is better suited for problem related to comparison
- RF Point of view is better suited to differentiability issues.
- All results are based on a strong differential regularity of coefficients or parameters, detailed in SIAM paper(2013)

SPDE's for compound SDE's

Itô Ventzel Formula for semimartingale Z

- A $\mathcal{K}^2_{loc}$-Itô semimartingale RF $F(t,z)$, (β^F, γ^F).
- $dF(t,Z_t) = \beta^F(t,Z_t)dt + \gamma^F(t,Z_t).dW_t$
- $+F_z(t,Z_t)dZ_t + \frac{1}{2}F_{zz}(t,Z_t)\langle dZ_t\rangle + \langle \gamma^F_z(t,Z_t).dW_t, dZ_t\rangle$

Second formulation

- $dF(t,z) = \beta^F(t,z)dt + \gamma^F(t,z).dW_t = d_tF(t,z)|_{z=Z_t}$
  $dF_z(t,z) = \beta^F_z(t,z)dt + \gamma^F_z(t,z).dW_t$.
- Then, $dF(t,Z_t) =$
  $d_tF(t,z)|_{z=Z_t} - \frac{1}{2}F_{zz}(t,Z_t)\langle dZ_t\rangle + d\big(F_z(t,Z_t)Z_t\big) - Z_t\, dF_z(t,Z_t)$

Bi-revealed utilities

- $dU(t,X_t(x)) = \gamma^U(t,X_t(x)).dW_t + d\big(U_z(t,X_t(x))X_t(x)\big)$
- $+\beta^U(t,X_t(x))dt - \frac{1}{2}U_{zz}(t,X_t(x)\langle dX_t(x)\rangle - X_t(x)dY_t(u_z(x))$

Utility issued from SDEs

- X regular solution of SDE (μ^X, σ^X) with inverse flow $\{\xi(t,z)\}$
- Y regular solution of SDE (μ^Y, σ^Y) decoupled of X
- suborthogonality : $z\mu_t^Y(y) + y\,\mu_t^X(x) + \langle \sigma_t^X(x), \sigma_t^Y(y)\rangle \leq 0,$

U differential coefficients

$$\gamma_z^U(t,z) = \sigma_t^Y(U_z(t,z)) - U_{zz}(t,z)\sigma_t^X(z).$$
$$\beta^U(t,z) = \frac{1}{2}U_{zz}(t,z)||\sigma_t^X(z)||^2 + z\,\mu_t^Y(U_z(t,z)) \quad \text{or}$$
$$\beta^U(t,z) + \langle \gamma_z^U(t,z), \sigma_t^X(z)\rangle = -[\frac{1}{2}U_{zz}(t,z)||\sigma_t^X(z)||^2 + U_z(t,z)\mu_t^X(z)].$$

In the intrinsic case, $\mu_t^Y(y) \leq 0$, $\beta^U(t,z) \leq 0$ and $U(t,z)$ is a supermartingale

Revealed utility and optimization

Generally utility criteria are associated with optimization problem;
General utility of test process: $dZ_t = \phi^Z_t dt + \psi^Z_t . dW_t$

- $dU(t, Z_t) =$
  $\frac{1}{2} U_{zz}(t, Z_t) ||\sigma^X_t(Z_t) - \psi^Z_t||^2 dt + \gamma^U(t, Z_t) + U_z(t, Z_t)\psi^Z_t . dW_t$
  $+ \big[Z_t\, \mu^Y_t(U_z(t, Z_t)) + U_z(t, Z_t)\phi^Z_t + \langle \sigma^Y_t(U_z(t, Z_t)), \psi^Z_t \rangle \big] dt$
- The last term is the drift of the product $\{Z_t\, Y_t(y)\}$ taken at $y = U_z(t, Z_t)$, negatif for $\{Z_t\, Y_t(y)\}$ supermartingale

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Controlled processes

- A controlled process Z is suborthogonal to $\{Y_t(y), \forall y\}$, that is $Z_t Y_t(y)$ is supermartingale (ex $X_t(x)$)
- A controlled process Z is suboptimal
  - $U(t, Z_t)$ is a supermartingale.
  - $U(t, Z_t)$ is a martingale iff Z is a solution of the SDE(μ, σ).

Applications to convex pricing forward dynamics

Work in progress

Merci Stephane.....

Stephane Comment

- OK, it is interesting, but utility framework is not so useful in pratice, in mathematical finance
- Pricing problems are more challenging, even the more classical with convex pay-off h.
- In particular, given a price derivative today how simulate coherent price in the future

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- Very similar to utility problem where concave is replaced by convex
- But, only if the price today of convex derivative is a convex function.

A old question studied with S.Shreve and M.Jeanblanc in 1995

Classical BS Formula

- $X_t(x)$=Geometric Brownian motion=$xX_t(1)$
- Option pay-off= Target $=h(T, X_T(x))$
- h convex option, positive, (nondecreasing)
- Intrinsic universe:$(\Omega, (\mathcal{F}_t), \mathbb{Q})$

BS Formula

- By change of numeraire, and probability measure, $X_t(x)$ can be assumed to be $\mathbb{Q}$ martingale
- $X_t = x \exp(\sigma W_t^Q - \frac{1}{2}\sigma^2 t) = xL_T$
- The pricing rule is risk-neutral, that is

$$BS(h)(x, T) = \mathbb{E}_Q(h(xL_T)) = \Phi_h(0, x) = \Phi_h(x)$$

- $BS(h)(x)$ is a convex function $\Phi_h(x)$ with $\Phi_h(T, x) = h(x)$

Robustness: Main theorem EJS

Model with local volatility in risk neutral universe

- $dX_t = X_t(\sigma(t, X_t)dW_t^Q), \quad X_0 = x$
- $\gamma(t, x) = x\sigma(t, x)$
- $\sigma(t, x)$ continuous and bounded above
- $\partial_x\gamma(t, x)$ is continuous in (t, x) and Lipschitz continuous and bounded in x uniformly in t

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Theorem The Euopean price $\Phi_h(0, x) = \mathbb{E}_Q(h(X_T(x))$ is a convex function, with bounded derivatives

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Then, it is possible to adapt the "machinery" of concave utility.

Conclusion

Que Sau, God of health and longevity

Te garde sous son regard bienveillant

Merci pour tout, Denis...