Propagation of chaos for stochastic particle systems in interaction of $L^q - L^p$ type

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A Random Walk in the Land of Stochastic Analysis and Numerical Probability
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Pour commencer...

... une photo de Denis:

Overview

Propagation of chaos

· Consider the following **particle system (PS)** in $\mathbb{R}^d$

$$\begin{cases} dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^N b(t, X_t^{i,N}, X_t^{j,N})\,dt + dW_t^i, \quad t>0, i\le N \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1\le i\le N), \end{cases} \tag{1}$$

where $b : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ a Borel measurable function.

· When b is "nice": $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_\cdot}$ converges, as $N\to\infty$, towards the law of the **non-linear stochastic process** given by

$$\begin{cases} dX_t = \int b(t, X_t, y)\rho_t(y)\,dy\,dt + dW_t, \quad t>0, \\ \rho_t(y)dy := \mathcal{L}(X_t), \quad X_0 \sim \rho_0(x)dx. \end{cases} \tag{2}$$

Of course, $(\mu_t^N)_{t\ge 0}$ converges to the corresponding Fokker Planck PDE.
Long history in the literature (from Kac, McKean to today...)

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Singular interactions: Physical examples

Probabilistic approach to **singular non-linear FP** equations such as:

- Boltzmann, Burgers, Navier-Stokes, Keller-Segel equations, ...

studied by many authors:

- Bossy, Calderoni, Fournier, Graham, Guérin, Hauray, Jabir, Jourdain, Méléard, Osada, Pulvirenti, Roelly, Sznitman, Talay, ...

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Our motivations: Krylov-Rockner condition

In [K-R, PTRF 05] the following linear SDEs are studied (among other)

$$X_t = x + \int_0^t b(r, x_r)dr + dW_t, \quad t \geq 0,$$

where $x \in \mathbb{R}^d$ and b satisfies for any $t > 0$

$$\int_0^t \|b(r, \cdot)\|^q_{L^p(\mathbb{R}^d)} \, dr < \infty \quad \text{with} \quad \frac{d}{p} + \frac{2}{q} < 1, p \geq 2, q > 2.$$

Strong well posedness is obtained. (General condition, not necessarily Lipschitz continuos coefficient, can be singular..)

Then, [Rockner-Zhang, Bernoulli 21] proved strong well posedness of the NLSDE

$$\begin{cases} dX_t = \int b(t, X_t, y)\rho_t(y)\ dy\ dt + dW_t, \quad t > 0, \\ \rho_t(y)dy := \mathcal{L}(X_t), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{3}$$

under the following assumption:

Assumption

For $x, y \in \mathbb{R}^d$ and $t > 0$, one has $|b(t, x, y)| \le h_t(x - y)$ for some $h \in L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d))$, where $p, q \in (2, \infty)$ satisfy $\frac{d}{p} + \frac{2}{q} < 1$.

and supposing $\int |x|^\beta \rho_0(dx) < \infty$ for some $\beta > 2$.
(can also be localised L^p in space: no need for integrability at infinity.)

Our goal: Prove **well-posedness** and **propagation of chaos** for the corresponding PS.

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Overview

Main result I

Define for $t > 0$

$$\mathcal{N}_b(t) = \Big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \lim_{(x',y')\to(x,y)} |b(t,x',y')| = \infty \text{ or } \lim_{(x',y')\to(x,y)} |b(t,x',y')| \text{ does not exist}\Big\}.$$

As $|b(t,x,y)| \leq h_t(x-y)$ and $h_t \in L^p(\mathbb{R}^d)$, the set $\mathcal{N}_b(t)$ is of Lebesgue's measure zero in $\mathbb{R}^d \times \mathbb{R}^d$.

PS now reads:

$$\begin{cases} dX_t^{i,N} = \frac{1}{N}\sum_{j=1, j\neq i}^N b(t, X_t^{i,N}, X_t^{j,N}) \mathbb{1}_{\{(X_t^{i,N}, X_t^{j,N}) \notin \mathcal{N}_b(t)\}} \, dt + \sqrt{2} dW_t^i, \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1 \leq i \leq N). \end{cases} \tag{4}$$

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Theorem ([T, ECP 23])

Let Assumption 1 hold. Given $0 < T < \infty$ *and* $N \in \mathbb{N}$*, there exists a weak solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t; 0 \le t \le T), \mathbb{Q}^N, W, X^N)$ *to the* N*-interacting particle system* (4) *that satisfies, for any* $1 \le i \le N$*,*

$$\mathbb{Q}^N \left(\int_0^T \left(\frac{1}{N} \sum_{j=1, j\neq i}^{N} b(t, X_t^{i,N}, X_t^{j,N}) \mathbb{1}_{\{(X_t^{i,N}, X_t^{j,N}) \notin \mathcal{N}_b(t)\}} \right)^2 dt < \infty \right) = 1.$$

Uniqueness in law holds in the class of solutions satisfying above equality.

Girsanov transform $\to$ Lebesgue measure of the set $i \neq j$, $\{t > 0, (X_t^{i,N}, X_t^{j,N}) \in \mathcal{N}_b(t)\}$ will thus be a.s. zero. Hence, the dynamics (1) and (4) are essentially the same.

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Martingale problem

$\mathbb{Q} \in \mathcal{P}(C[0,T];\mathbb{R}^d)$ is a solution to (MP) if:

(i) $\mathbb{Q}_0 = \mu_0$;

(ii) For any $t \in (0,T]$ and any $r > 1$, the one dimensional time marginal $\mathbb{Q}_t$ of $\mathbb{Q}$ has a density ρ_t w.r.t. Lebesgue measure on $\mathbb{R}^d$ which belongs to $L^r(\mathbb{R}^d)$ and satisfies

$$\exists C_T, \quad \forall\, 0 < t \leq T, \quad \|\rho_t\|_{L^r(\mathbb{R}^d)} \leq \frac{C_T}{t^{\frac{d}{2}(1-\frac{1}{r})}};$$

(iii) Denoting by $(x(t);\, t \leq T)$ the canonical process of $C([0,T];\mathbb{R}^d)$, we have: For any $f \in C_b^2(\mathbb{R}^d)$, the process defined by

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \Big(\nabla f(x(s)) \cdot \Big(\int b(s,x(s),y)\rho_s(y)dy\Big) + \triangle f(x(s))\Big)\, ds$$

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Remark

1. *Under Assumption 1* $+ \int |x|^\beta \mu_0(dx) < \infty$ *for some* $\beta > 2$, *(MP) admits a unique solution according to Thm. 1.1 [Rockner-Zhang, 21].*
2. *Marginal densities satisfy some Gaussian estimates punctually. In our (MP),* L^r*-estimates* $+$ *Assumption 1* $\to$ *all the terms in* (M) *are well defined.*

$$\left| \int_0^t \nabla f(x(s)) \cdot \int b(s, x(s), y)\rho_s(y)dyds \right| \leq c_{f,t} \left(\int_0^t \int h_s^2(x(s) - y)\rho_s(y)dyds \right)^{1/2}$$

$$\leq c_{f,t} \|h\|_{L^q((0,t);L^p(\mathbb{R}^d))} \left(\int_0^t s^{-\frac{dq}{p(q-2)}} ds \right)^{1/2-1/q}$$

Finite if $d/p + 2/q < 1$.

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Theorem ([T, ECP 23])

In addition to Assumption 1, assume that for any $t > 0$, $b(t, \cdot, \cdot)$ ***is continuous outside of the set*** $\mathcal{N}_b(t)$*. Assume that the $X_0^{i,N}$'s are i.i.d. and that the initial distribution of $X_0^{1,N}$ is the measure μ_0 that for some $\beta > 2$ has finite β-order moment .*
Then, the ***empirical measure*** *of (4)* ***converges in the weak sense****, when $N \to \infty$, to the unique weak solution of (3).*

In practice, interaction kernels are **convolutions well defined and continuos almost everywhere** (like $\pm\frac{x}{|x|^r}$). Hence, it is not unreasonable to assume that $b(t, \cdot, \cdot)$ is continuous outside of $\mathcal{N}_b(t)$.

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Alternative hypothesis

Local integrability and boundedness at infinity

$$h \in L^q_{loc}(\mathbb{R}_+; L^p_{\boldsymbol{loc}}(\mathbb{R}^d)), \quad p, q \in (2, \infty) : \frac{d}{p} + \frac{2}{q} < 1$$

and the function $H(T) := \int_0^T \sup_{|x|>1} |h_t(x)|^2 dt$ is an increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Typical example for $d = 2$

$$b_t(x, y) = \frac{a_t(x, y)}{|x - y|^\alpha}, \quad |a_t(x, y)| \le \kappa |x - y|, \quad \alpha \in [1, 2), \kappa > 0.$$

Can't work for Keller-Segel or Navier-Stokes in $\mathbb{R}^2$: $\pm \frac{x}{|x|^2}$
(**Normal**: does not exploit sign for NS; particles collide more than BMs for KS.)

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- In [Jabir-Talay-T., ECP (2018)]: wellposedness and propagation of chaos for PS with both non-Markovian and singular interaction related to the parabolic-parabolic $1d$ Keller-Segel model.

$$\frac{1}{N}\sum_{j=1}^{N} b(t, X_t^{i,N}, X_t^{j,N}) \to \frac{1}{N}\sum_{j=1}^{N}\int_0^t K(t-s, X_t^{i,N} - X_s^{j,N})\, ds,$$

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Overview

Existence: Girsanov theorem

Start from
$$\bar{X}_t^{i,N} := X_0^{i,N} + W_t^i \ (t \leq T)$$
and $\bar{X} := (\bar{X}^{i,N}, 1 \leq i \leq N)$.
Denote the drift of X^i by $b_t^{i,N}(x)$, $x \in C([0,T];\mathbb{R}^d)^N$, and
$$B_t^N(x) = (b_t^{1,N}(x), \ldots, b_t^{N,N}(x)).$$
For a fixed $N \in \mathbb{N}$, consider
$$Z_T^N := \exp\left\{\int_0^T B_t^N(\bar{X}) \cdot dW_t - \frac{1}{2}\int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}.$$
Check the following Novikov condition: For any $T > 0$, $N \geq 1$, $\kappa > 0$, there exists $C(T,N,\kappa)$ such that
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Jensen's inequality leads to

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2\,dt\right\}\right] \leq$$
$$\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1,j\neq i}^N \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\boldsymbol{\kappa N}\int_0^T |b(t,\bar{X_t}^i,\bar{X_t}^j)|^2\,dt\right\}\right].$$

For $i,j\leq N$ such that $j\neq i$ we can get

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\boldsymbol{\kappa N}\int_0^T |b(t,\bar{X_t}^i,\bar{X_t}^j)|^2\,dt\right\}\right] \leq C(T,\boldsymbol{N})$$

developing the exponential and controlling for any $k\geq 1$

$$\frac{(\kappa N)^k}{k!}\mathbb{E}_{\mathbb{W}}\left(\int_0^T |b(t,\bar{X_t}^i,\bar{X_t}^j)|^2\,dt\right)^k.$$

Iterate the integral and use the BMs and their independence.

For example, $k=1$:

$$\mathbb{E}_{\mathbb{W}}\left(\int_0^T |b(t,\bar{X_t}^i,\bar{X_t}^j)|^2\,dt\right) \leq \mathbb{E}_{\bar{X}^j}\int_0^T\int h_t^2(x-\bar{X_t}^j)g_t(x)dxdt$$
$$\leq C_T\|h\|^2_{L^q((0,t);L^p(\mathbb{R}^d))}.$$

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For $i,j\leq N$ such that $j\neq i$ we can get

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developing the exponential and controlling for any $k\geq 1$

$$\frac{(\kappa N)^k}{k!}\mathbb{E}_{\mathbb{W}}\left(\int_0^T|b(t,\bar{X}_t{}^i,\bar{X}_t{}^j)|^2\,dt\right)^k.$$

Iterate the integral and use the BMs and their independence.

For example, $k=1$:

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Partial transforms

Above transforms are **not useful** for proving tightness of the empirical measure as for any $\alpha \in \mathbb{R}$

$$\mathbb{E}(Z_T^N)^\alpha \leq C(T, N, \alpha) \text{ and } C(T, N, \alpha) \to \infty, N \to \infty.$$

For example, for $m \geq 1$

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A way out: **Partial transforms**! Fix $1 \leq r_0 < N$ and control the exponential martingale between (PS) and

$$\begin{cases} d\widehat{X}_t^{l,N} = dW_t^l, \quad 1 \leq l \leq r_0, \\ d\widehat{X}_t^{i,N} = \left\{ \frac{1}{N} \sum_{j=r_0+1}^N b(t, \widehat{X}_t^{i,N}, \widehat{X}_t^{j,N}) \right\} dt + dW_t^i, \quad r_0 + 1 \leq i \leq N, \\ \widehat{X}_0^{i,N} \text{ i.i.d. and independent of } (W) := (W^i, 1 \leq i \leq N). \end{cases}$$

$(\widehat{X}^{l,N}, 1 \leq l \leq r_0)$ are **BM independent of** $(\widehat{X}^{i,N}, r_0 + 1 \leq i \leq N)$

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For $x \in C([0,T];\mathbb{R}^d)^N$ the **change of drift** is given by

$$\beta_t^{(r_0)}(x) := \Big(b_t^{1,N}(x), \ldots, b_t^{r_0,N}(x), \frac{1}{N}\sum_{i=1}^{r_0} b(t, x_t^{r_0+1}, x_t^i), \ldots, \frac{1}{N}\sum_{i=1}^{r_0} b(t, x_t^N, x_t^i)\Big).$$

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Proposition

For any $\gamma > 0$ and $1 \le r_0 < N$ there exists $N_0 \ge r_0$ and $C(T,\gamma,r_0)$ s.t.

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Cauchy Schwarz, multiple Holder and $\frac{N-r_0}{N} < 1$, lead to

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&\mathbb{E}_{\mathbb{Q}^{r_0,N}} \exp\Big\{\gamma \int_0^T |\beta_t^{(r_0)}(\widehat{X})|^2 dt\Big\} \\
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As above you have b evaluated in a **BM and an independent process**, so you control the expectation. Advantage: no N in the exponential.

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Some concluding remarks

- For tightness $r_0 = 1$, for passage to the limit $r_0 = 1, 2, 3, 4$.
- To get (MP)-(ii) we also use the partial transforms.
  Let $\varphi \in C_c(\mathbb{R}^d)$ and fix $r > 1$. Let $\alpha \in (1, r')$ where r' is the conjugate of r.

$$< \nu_t^1, \varphi >= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N} < \mu_t^N, \varphi >= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N}(\varphi(X_t^{1,N}))$$

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All in all

- $L^p - L^q$ is a general formulation which is a limit for Girsanov to work.
- It works for singular convolution kernels of order $\frac{1}{|x|^\beta}, \beta < 1$.
- Question: Is time integration beneficial $\int_0^t K(X_t^1 - X_s^2)ds$?

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