Functional convex ordering for stochastic processes: a constructive (and simulable) approach

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Definitions

Definition (Convex order, peacock)

(*a*) Two $\mathbb{R}^d$-valued random vectors $U, V \in L^1(\mathbb{P})$ are ordered w.r.t. convex order, denoted

$$U \preceq_{cvx} V$$

if, for every $\varphi : \mathbb{R}^d \to \mathbb{R}$, convex, [$\varphi$ Lipschitz is enough (Jourdain-P. 2023) by an inf convolution argument],

$$\mathbb{E}\,\varphi(U) \leq \mathbb{E}\,\varphi(V) \in (-\infty, +\infty].$$

(*b*) A stochastic process $(X_u)_{u \geq 0}$ is a p.c.o.c. (for "processus croissant pour l'ordre convexe") if

$$u \longmapsto X_u \text{ is non-decreasing for the convex order.}$$

- Then $\mathbb{E}\,U = \mathbb{E}\,V$ [$\varphi(x) = \pm x$] and, if both lie in L^2 [$\varphi(x) = x^2$]

$$\mathrm{Var}(U) \leq \mathrm{Var}(V).$$

Examples and motivation

- If $(X_t)_{t\geq 0}$ is a martingale, then $(X_t)_{t\geq 0}$ is a p.c.o.c./peacock: let $0 \leq s \leq t$,
$$\mathbb{E}\,\varphi(X_s) = \mathbb{E}\left(\varphi(\mathbb{E}(X_t|X_s))\right) \underbrace{\leq}_{Jensen} \mathbb{E}\left(\mathbb{E}(\varphi(X_t)|X_s)\right) = \mathbb{E}\,\varphi(X_t).$$

- Example: Gaussian distributions (centered): Let $Z \sim \mathcal{N}(0, I_q)$ on $\mathbb{R}^q$ and let A, $B \in \mathbb{M}(d,q)$ be $d \times q$ matrices
$$\left(A \preceq B \;\; i.e. \;\; BB^* - AA^* \in \mathcal{S}^+(d)\right) \Longrightarrow AZ \preceq_{cvx} BZ$$
i.e. $\mathcal{N}(0, AA^*) \preceq_{cvx} \mathcal{N}(0, BB^*)$ [Still true if Z is radial: $Z \sim OZ$, $\forall\, O \in O(d)$, Jourdain-P. 2022].

- Proof: Let $Z_1, Z_2 \sim \mathcal{N}(0; I_q)$ be independent and set
$$X_1 = AZ_1, \quad X_2 = X_1 + (BB^* - AA^*)^{1/2} Z_2.$$
Then (X_1, X_2) is an $\mathbb{R}^d$-valued martingale and $X_2 \sim \mathcal{N}(0, BB^*)$.

- Scalar case $d = q = 1$: $|\sigma| \leq |\vartheta| \Longrightarrow \mathcal{N}(0, \sigma^2) \preceq_{cvx} \mathcal{N}(0, \vartheta^2)$.

- 1D-proof: $\varphi : \mathbb{R} \to \mathbb{R}$ convex and $Z \in L^1$, $Z \stackrel{d}{=} -Z$. Then, by Jensen's $\leq$,
$$u \mapsto \mathbb{E}\,\varphi(uZ) \text{ is even, convex and attains its minimum } \varphi(0) \text{ at } u = 0.$$
Hence $u \mapsto \mathbb{E}\,\varphi(uZ)$ is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$.

About the converse of "martingale $\Rightarrow$ p.c.o.c."

- Strassen's Theorem (1965): $\mu \preceq_{cvx} \nu \Longleftrightarrow \exists$ transition $P(x,dy)$ s.t.

$$\nu = \mu P \quad \text{and} \quad \forall\, x \in \mathbb{R}^d, \quad \int y\, P(x,dy) = x$$

- Kellerer's Theorem (1972): X is a p.c.o.c $\Longleftrightarrow$

$$\text{There exists a martingale } (M_t)_{t\ge 0} \text{ such that } X_t \stackrel{d}{=} M_t,\; t \ge 0,$$

  i.e. X is a "1-martingale".

- Both proofs are unfortunately non-constructive.
- In Hirsch, Roynette, Profeta & Yor's monography, many (many. . .) explicit "representations" of p.c.o.c. by true martingales.

A revival motivated by Finance...

- A starter! t being fixed, $\sigma \mapsto e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a p.c.o.c. since

$$\forall\, \sigma > 0, \quad e^{\sigma W_t - \frac{\sigma^2 t}{2}} \stackrel{d}{=} e^{W_{\sigma^2 t} - \frac{\sigma^2 t}{2}} \;(\rightarrow \sigma\text{-martingale}).$$

- Application to Black-Scholes model $S_t^\sigma = s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}$. For every convex payoff function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

$$0 \le \sigma \le \sigma' \Longrightarrow \mathbb{E}\,\varphi(S_t^\sigma) \le \mathbb{E}\,\varphi(S_t^{\sigma'}).$$

- Vanilla options: *Call* and *Put* options: $\varphi(S_T) = (S_T - K)^+$, $\varphi(S_T) = (K - S_T)^+$, etc.
- Path-dependent options (Asian payoffs). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ convex

$$\sigma \longmapsto \text{Premium}(\sigma) = \mathbb{E}\Big[\varphi\Big(\frac{1}{T}\int_0^T \underbrace{s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}}_{=S_t^\sigma} dt\Big)\Big]\ ?$$

- P. Carr et al. (2008): Non-decreasing in σ when $\varphi(x) = (x-K)^+$ (Asian Call).
- M. Yor (2010): $\sigma \mapsto \dfrac{1}{T}\displaystyle\int_0^T s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}} dt$ is a p.c.o.c.

  (Hint: Representation using a a Brownian sheet).
- Yields bounds on the option prices of vanilla options.
- Extensions to American options (optimal stopping, P. 2016).

▷ This suggests many other (new or not so new) questions !

- Monotone (non-decreasing) convex order : $\exists$ drif b! [Hajek, 1985].
- Functional convex order I: switch from BS to local volatility models $i.e$ $\sigma = \sigma(x)$: $\sigma \mapsto \mathbb{E} f(X_T^{(\sigma)})$ [see e.g. El Karoui-Jeanblanc-Schreve, 1998].
- m-marginal path-dependent convex order: e.g. $\mathbb{E} f(X_{T_1}^{(\sigma)}, X_{T_2}^{(\sigma)})$ if $m = 2$. [see e.g.Brown, Rogers, Hobson 2001, Rüschendorf et al. 2008]
- "Functional" convex order II: from $\mathbb{E} f(X_T^{(\sigma)})$ to $\mathbb{E} F(X^{(\sigma)})$ path-dependent convex order [P.2016].
- Bermuda options [Pham 2005, Rüschendorf 2008], American options [P. 2016].
- Jump (risky asset) dynamics for $(X_t^{(\sigma)})$? [Rüschendorf-Bergenthum 2007, P. 2016]
- P.c.o.c. trough Martingale Optimal Transport. [Beïgelbock,Henry-Labordère et al, 2013, Tan, Touzi, Henry-Labordère 2015, Jourdain-P. 2022].

Aims and methods

1. Unify and generalize these results with of focus on functional aspects (path-dependent payoffs) (like Asian options) i.e. both functional convex order I and II.

2. Constraint: provide a constructive method of proof
   - based on time discretization of continuous time martingale dynamics (risky assets in Finance) .
   - using numerical schemes that preserve the functional convex order satisfied by the process under consideration...
   - e.g. to avoid "convexity arbitrages" in Finance.

3. Apply the paradigm to various frameworks:
   - American style options, jump diffusions, stochastic integrals,
   - McKean-Vlasov diffusions, MFG [Liu-P. 2022, SPA] and [Liu-P. 2023, AAP],
   - Volterra equations [Jourdain-P. 2022],
   - etc.

Martingale (and scaled) Brownian diffusions

- Pre-order $\preceq$ on $\mathcal{M}(d,q,\mathbb{R})$: let $A, B \in \mathbb{M}_{d,q}$.

$$A \preceq B \quad \text{if} \quad BB^* - AA^* \in \mathcal{S}^+(d,\mathbb{R}).$$

▷ If $d = q = 1$, $a \preceq b$ iff $a^2 \leq b^2$ iff $|a| \leq |b|$]

- $\preceq$ -Convexity: $\sigma : \mathbb{R}^d \to \mathbb{M}_{d,q}$ is $\preceq$-convex if

$\forall\, x, y \in \mathbb{R}^d$, $\lambda \in [0,1]$, there exists $O_{\lambda,x}$, $O_{\lambda,y} \in O(d)$ such that

$$\sigma\big(\lambda x + (1-\lambda)y\big) \preceq \lambda\sigma(x)O_{\lambda,x} + (1-\lambda)\sigma(y)O_{\lambda,y}$$

i.e.

$$\sigma\sigma^*\big(\lambda x+(1-\lambda)y\big) \leq \big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)\big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)^*$$

▷ $d = q = 1$ with $O_{\lambda,x} = \mathrm{sign}(\sigma(x))$ this simply reads

$$|\sigma| \text{ convex}.$$

Theorem (martingale case (weak), P. 2016, Fadili-P. 2017, Jourdain-P. 2022)

Let $\sigma, \theta \in \mathcal{C}_{lin_x}\big([0,T]\times\mathbb{R}, \mathbb{M}_{d,q}\big)$.

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^{1+\eta},\ \eta > 0$$

$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\ X_0^{(\theta)} \in L^{1+\eta}, \quad \text{both } (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ or & \\ (i)_\theta & \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ and & \\ (ii) & \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T] \end{cases}$$

then, for every functional $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$, *l.s.c. convex,*

(i) *The function* $x \mapsto \mathbb{E}\, F(X^{(\sigma),x})$ *is convex from* $\mathbb{R}^d$ *to* $(-\infty, +\infty]$,

(ii) *Convex ordering holds:* $\quad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}) \in (-\infty, +\infty]$.

- By a functional inf-convolution argument, it suffices to consider $\|\cdot\|_{\sup}$-Lipschitz functionals.

Theorem (martingale case (strong), P. 2016, Fadili-P. 2017, Jourdain-P. 2022)

Let $\sigma, \theta \in \mathrm{Lip}_x\big([0,T]\times\mathbb{R}, \mathbb{M}_{d,q}\big)$, *W q-S.B.M.. Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the unique strong solutions to*

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t,\ X_0^{(\sigma)} \in L^1, (\textit{no more } \eta!)$$

$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t,\ X_0^{(\theta)} \in L^1, \quad (W_t)_{t\in[0,T]} \textit{ standard B.M.}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma \quad \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{or} \\ (i)_\theta \quad \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{and} \\ (ii) \quad \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T] \end{cases}$$

then, for every $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$, *l.s.c. convex,*

(i) *The function* $x \mapsto \mathbb{E}\, F(X^{(\sigma),x})$ *is convex from* $\mathbb{R}^d$ *to* $(-\infty, +\infty]$,

(ii) *Convex ordering holds:* $\quad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}) \in (-\infty, +\infty]$.

- By a functional inf-convolution argument, it suffices to consider $\|\cdot\|_{\sup}$-Lipschitz functionals.

Scaled/drifted martingale diffusions (extension to)

- The former theorems still hold true for

$$dX_t^{(\sigma)} = \alpha(t)\big(X_t^{(\sigma)} + \beta(t)\big)dt + \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},$$
$$dX_t^{(\theta)} = \alpha(t)\big(X_t^{(\theta)} + \beta(t)\big)dt + \theta(t, X_t^{(\theta)})dW_t^{(\theta)},$$

where $\alpha(t) \in \mathbb{M}_{d,d}(\mathbb{R})$ and $\beta(t) \in \mathbb{R}^d$ are continuous.

- Change of variable:

$$\widetilde{X}_t^{(\sigma)} = e^{-\int_0^t \alpha(s)ds}\big(X_t^{(\sigma)} + \beta(t)\big).$$

- Finance: spot interest rate $\alpha(t) = r(t)\mathbf{1}$ and $\beta(t) = 0$ since typical (risk-neutral) dynamics of traded assets read

$$dS_t = r(t)S_t dt + S_t\sigma(S_t, \omega)dW_t$$

- For more general drifts $b(t,x)$ when $d = q = 1$: functional version of Hajek's theorem: monotone functional convex order holds true if

$$\forall\, t \in [0, T], \quad b(t, .) \text{ is convex.}$$

Strategy (constructive)

- Time discretization (preferably) accessible to simulation: typically the Euler scheme.
- Propagate convexity (marginal or pathwise)
- Propagate comparison (marginal or pathwise)
- Transfer by functional limit theorems "à la Jacod-Shiryaev".

Step 1: discrete time ARCH models

- ARCH dynamics: Let $(Z_k)_{1\leq k\leq n}$ be a sequence of independent, symmetric r.v. on $(\Omega, \mathcal{A}, \mathbb{P})$. Two ARCH models: X_0, $Y_0 \in L^1(\mathbb{P})$,

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1}, \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad k = 0 : n-1, \end{aligned}$$

where σ_k, $\theta_k : \mathbb{R} \to \mathbb{R}$, $k = 0 : n-1$ have linear growth.

Proposition (Propagation result)

If σ_k, $k = 0 = n-1$ are $\preceq$-convex with linear growth,

$$X_0 = x \quad \textit{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then, for every convex funtion $F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ convex with linear growth

$$x \longmapsto \mathbb{E}\, F(x, X_1^x \ldots, X_n^x) \quad \textit{is convex}.$$

Partial proof (marginal) with Gaussian white noise

- $Z_k \sim \mathcal{N}(0, I_q)$, $1 \leq k \leq n$.
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Let

$$P_k^\sigma f(x) := \mathbb{E}f(x + \sigma_k(x)Z) = \left[\mathbb{E}\, f(x + AZ)\right]_{|A=\sigma_k(x)}.$$

- Set $A \in \mathbb{M}_{d,q} \mapsto Qf(A) := \mathbb{E}\, f(x + AZ)$ is right $O(d)$-invariant, convex and $\preceq$-non-decreasing by the starting example.
- Then $P_k^\sigma f$ is convex since $\forall\, x,\, y \in \mathbb{R}^d$ and $\forall\, \lambda \in [0,1]$

$$\begin{aligned} P_k^\sigma f\big(\lambda x + (1-\lambda)y\big) &= Qf\big(\sigma_k(\lambda x + (1-\lambda)y)\big) \\ &\leq Qf\big(\lambda\sigma_k(x) + (1-\lambda)\sigma_k(y)\big) \\ &\leq \lambda Qf\big(\sigma_k(x)\big) + (1-\lambda)Qf\big(\sigma_k(y)\big) \\ &= \lambda P_k^\sigma f(x) + (1-\lambda)P_k^\sigma f(y). \end{aligned}$$

- Hence

$$x \longmapsto \mathbb{E}\, f(X_n^x) = P_{1:n}^\sigma f(x) := P_1^\sigma \circ \cdots \circ P_n^\sigma f(x) \quad \text{is convex}$$

Theorem (Discrete time comparison result)

If all σ_k, $k = 0 = n-1$ or all θ_k, $k = 0 : n-1$ are $\preceq$-convex with linear growth,

$$X_0 \preceq_{cvx} Y_0 \quad \textit{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then

$$(X_0, \ldots, X_n) \preceq_{cvx} (Y_0, \ldots, Y_n).$$

Partial proof (marginal) with Gaussian white noise

- Backward induction on k.
- For $k = n$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a (Lipschitz) convex function.

$$\sigma_n \preceq \theta_n \Longrightarrow P_n^\sigma f(x) = Qf\big(\sigma_n(x)\big) \leq Qf\big(\theta_n(x)\big) = P_n^\theta f(x)$$

by non-decreasing $\preceq$-monotony of Q.

- Assume $\underbrace{P_{k+1:n}^\sigma f}_{\text{convex}} \leq P_{k+1:n}^\theta f$.

$$A \in \mathbb{M}_{d,q} \longmapsto Q\big(P_{k+1:n}^\sigma f\big)(A) \quad \text{is } \preceq\text{-non-decreasing}$$

so that
$$\begin{aligned} P_{k:n}^\sigma f(x) = P_k^\sigma(P_{k+1:n}^\sigma)f(x) = Q\big(P_{k+1:n}^\sigma f\big)\big(\sigma_k(x)\big) &\overset{\downarrow}{\leq} Q\big(P_{k+1:n}^\sigma f\big)\big(\theta_k(x)\big) \\ &\leq Q\big(P_{k+1:n}^\theta f\big)\big(\theta_k(x)\big) \\ &= P_{k:n}^\theta f(x). \end{aligned}$$

- Hence

$$\mathbb{E}\, f(X_n^\sigma) = \mathbb{E}\, P_{1:n}^\sigma f(X_0) \leq \mathbb{E}\, P_{1:n}^\sigma f(Y_0) \leq \mathbb{E}\, P_{1:n}^\theta f(Y_0) = \mathbb{E}\, f(X_n^\theta).$$

Functional approach

- By “functional” we mean here : $F(X_0, \dots, X_n)$ with $F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ convex.
- Same strategy by induction
- But entirely backward.

Step 2 of the proof: Back to continuous time

▷ Euler scheme(s): Discrete time Euler scheme with step $\frac{T}{n}$, starting at x is an ARCH model. For $X^{(\sigma)}$: for $k = 0, \ldots, n-1$,

$$\bar{X}^{(\sigma),n}_{t^n_{k+1}} = \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})\big(W_{t^n_{k+1}} - W_{t^n_k}\big), \; \bar{X}^{(\sigma),n}_0 = x$$

Set

$$Z_k = W_{t^n_k} - W_{t^n_{k-1}}, \; k = 1, \ldots, n$$

$$\Downarrow$$

discrete time setting applies

Remark. Linear growth of σ and θ, implies

$$\forall\, p > 0, \qquad \sup_{n\geq 1} \Big\| \sup_{t\in[0,T]} |\bar{X}^{(\sigma),n}_t| \Big\|_p + \sup_{n\geq 1} \Big\| \sup_{t\in[0,T]} |\bar{X}^{(\theta),n}_t| \Big\|_p < +\infty.$$

From discrete to continuous time

▷ Interpolation ($n \geq 1$)

- *Piecewise affine interpolator* defined by

$$\forall\, x_{0:n} \in \mathbb{R}^{n+1},\ \forall\, k = 0, \ldots, n-1,\ \forall\, t \in [t_k^n, t_{k+1}^n], \quad .$$

$$i_n(x_{0:n})(t) = \frac{n}{T}\big((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}\big)$$

- $\widetilde{X}^{(\sigma),n} := i_n\Big((\bar{X}^{(\sigma),n}_{t_k^n})_{k=0:n}\Big)$ = piecewise affine Euler scheme.

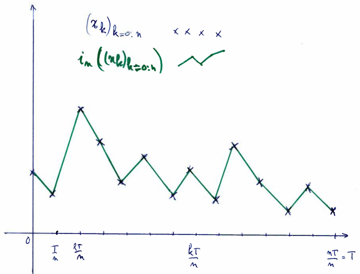


Figura: *Interpolator*

▷ Let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a convex functional (with r-poly. growth).

$$\forall\, n \geq 1, \qquad F_n : \mathbb{R}^{n+1} \ni x_{0:n} \longmapsto F_n(x_{0:n}) := F\big(i_n(x_{0:n})\big).$$

- Step 1 (Discrete time): $F\big(\widetilde{X}^{(\sigma),n}\big) = F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}$ and

$$F \text{ convex} \Longrightarrow F_n \text{ convex},\ n \geq 1.$$

Discrete time result implies since $\sigma(t^n_k,.) \leq \theta(t^n_k,.)$.

$$\mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) = \mathbb{E}\, F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big) \leq \mathbb{E}\, F_n\big((\bar{X}^{(\theta),n}_{t^n_k})_{k=0:n}\big) = \mathbb{E}\, F\big(\widetilde{X}^{(\theta),n}\big).$$

- Step 2 (Transfer): See e.g. [Jacod-Shiryaev's book, 2^{nd} edition, Theorem 3.39, p.551].

$$\widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\sigma)} \quad \text{as } n \to \infty.$$

$$\mathbb{E}\, F(X^{(\sigma)}) = \lim_n \mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) \qquad (\textit{idem} \ \text{ for } X^{(\theta)}).$$

The Euler scheme provides a simulable approximation

which preserves convex order.

Is convexity necessary ? $\sigma(t,x) = \sigma(x)$, $d = 1$

- Note that when $\vartheta = \sigma$, *a posteriori* $(ii) \Rightarrow (i)$ since
$$\delta_{\lambda x+(1-\lambda)y} \preceq_{cvx} \lambda\delta_x + (1-\lambda)\delta_y$$
so that, as $\sigma(\cdot) \leq \sigma(\cdot)$ (sic!),
$$\mathbb{E}\, F(X^{\lambda x+(1-\lambda)y}) \leq \lambda \mathbb{E}\, F(X^x) + (1-\lambda)\mathbb{E}\, F(X^y).$$
- One shows [Jourdain-P '23] that (when $d = 1$)
$$\sqrt{\frac{2}{\pi}}|\sigma(x)| = \lim_{t\to 0} \frac{1}{\sqrt{t}}\mathbb{E}|X^x_t - x| = \lim_{t\to 0} \frac{1}{\sqrt{t}}\mathbb{E}|X^x_0 - X^x_t| = \lim_{t\to 0} \frac{1}{\sqrt{t}}\mathbb{E}F(X^x)$$
with $F(\alpha) = |\alpha(t) - \alpha(0)|$ an (only) 2-marginal functional convex functional.
- If convexity propagation for 2-marginal functional holds true then $|\sigma|$ is convex !!
- The convexity assumption on σ or ϑ is mandatory ...except maybe for 1-marginal convex order when $d = q = 1$.

For the $1D$ diffusion (after [El Karoui et al.])

- Let $\varphi(x) = \mathbb{E}\, f(X_T^x)$ with $f : \mathbb{R} \to \mathbb{R}$ convex with right derivative f_r'.
- One has

$$\begin{aligned}\varphi'(x) &= \mathbb{E}\Big[f_r'(X_T^x)e^{\int_0^T \sigma'(X_s^x)dW_s - \frac{1}{2}\int_0^T (\sigma')^2(X_s^x)ds}\Big]\\ &= \dots\\ &= \mathbb{E}_{\mathbb{Q}} f_r'(Y_T^x) \quad \text{Girsanov !}\end{aligned}$$

- $x \mapsto Y_T^x$ is non-decreasing (cf. [Revuz-Yor])
- Finally

$$\varphi' : x \longmapsto \mathbb{E}_{\mathbb{Q}}\, f_r'(X_T^x) \text{ is non-decreasing}$$

so that (almost ...) whatever σ is

$$\varphi : x \longmapsto \mathbb{E}\, f(X_T^x) \text{ is convex.}$$

- So $1D$ setting for 1-marginal functionals is special !

Smooth σ & $d = q = 1$: get rid of convexity (with B. Jourdain '22)

- Assume $\sigma : \mathbb{R} \to \mathbb{R}_+$ C^2, Lipschitz ($\|\sigma'\|_\infty < +\infty$).
- True Euler operator, $Z \sim \mathcal{N}(0,1)$:

$$Pf(x) = \mathbb{E}\, f\big(x + \sqrt{h}\sigma(x)Z\big).$$

- Assume w.l.g. $f : \mathbb{R}^d \to \mathbb{R}$ C^2 and convex

$$\begin{aligned}(Pf)''(x) &= \mathbb{E}\big[f''(x+\sqrt{h}\sigma(x)Z\big)(1+\sqrt{h}\sigma'(x)Z)^2\big] \\ &\quad + \sqrt{h}\sigma'(x)\mathbb{E}\left[f'(x+\sqrt{h}\sigma(x)Z)Z\right] \\ &= \mathbb{E}\big[f''(x+\sqrt{h}\sigma(x)Z\big)(1+\sqrt{h}\sigma'(x)Z)^2\big] \\ &\quad + h\sigma\sigma''(x)\mathbb{E}\big[f''(x+\sqrt{h}\sigma(x)Z)\big] \quad \text{Stein I.P.} \\ &= \mathbb{E}\Big[f''(x+\sqrt{h}\sigma(x)Z)\underbrace{\big((1+\sqrt{h}\sigma'(x)Z)^2 + h\sigma\sigma''(x)\big)}_{\text{always } \geq 0\ \forall\, Z(\omega)??}\Big]\end{aligned}$$

- No ! But... If we truncate : $Z \rightsquigarrow Z^h = Z\mathbf{1}_{\{|Z|\leq A_h\}}\ldots$

- ... Then, the same Stein-I.P. transform yields

$$(P^h f)''(x) = \mathbb{E}\Big[f''(x+\sqrt{h}\sigma(x)Z^h)\underbrace{\Big(\big(1+\sqrt{h}\sigma'(x)Z^h\big)^2 + h\big(1-e^{-(A_h^2-(Z^h)^2)}\big)\mathbf{1}_{\{Z^h\neq 0\}}\sigma\sigma''(x)\Big)}_{\text{always }\geq 0\ \forall\, Z^h(\omega)??}\Big]$$

- YES !! If $A_h = A/\sqrt{h}$ with $A < \frac{1}{\|\sigma'\|_\infty}$ for h small enough, provided

$$(*)\qquad \sup_{x\in\mathbb{R}} \frac{\sigma(\sigma'')^-}{|\sigma'|} < +\infty \qquad (\Longrightarrow \text{Ok if } \sigma \text{ convex since } = 0!!)$$

- Hence truncated Euler scheme propagates convexity, $\rightarrow$ comparison, etc !
- Truncated Euler scheme with time step $h = T/n$ does converge (almost) "as usual" toward the diffusion as $n \to \infty$.
- Smoothness of σ and $(*)$ can be relaxed into $\sigma^2(x) + ax^2$ convex for some $a > 0$ (semi-convexity).

Theorem (Jourdain-P. 2023)

Under this semi-convexity assumption on σ^2 both propagation & comparison theorems hold for 1-marginal convex ordering.

- Similar results for monotone convex ordering for diffusions sharing the same convex drift.
- Applications to local volatility models (like CEV) extending results by El Karoui-Jeanblanc-Shreve to continuous time path-dependent options.
- Extension to m-marginal directionally convex functionals F (see also Rüshendorf & Bergenthum but ... with restrictions).

Directionally convex functionals

- A function $f : \mathbb{R}^m \to \mathbb{R}$ is *directionally convex* if
  - $\forall\, i,\ x_i \mapsto f(x_1, \dots, x_i, \dots x_m)$ is convex
  - $\forall\, j,\ x_j \mapsto \partial_{x_i} f(x_1, \dots, x_i, \dots x_m)$ is non-decreasing.
- Functional version (smooth directionally convex functionals): $f : C([0,T],\mathbb{R}) \to \mathbb{R}$

$$\forall x, u,\ v \in C([0,T],\mathbb{R}), \quad u,\ v \geq 0 \Longrightarrow DF(x).(u,v) \geq 0$$

Theorem

The 1D version of both functional comparison-propagation theorems remains true under the assumption that σ^2 (or ϑ^2) is semi-convex, for the class of continuous directionally convex functionals on $C([0,T],\mathbb{R})$ with r-polynomial growth if $X_0^{(\sigma)}$ and $X_0^{(\vartheta)} \in L^r(\mathbb{P})$.

Examples

Convexité, convexité marginale et convexité directionnelle

On considère la fonction

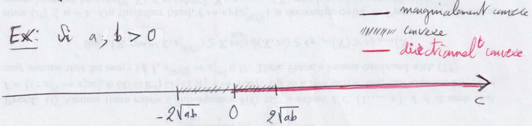
$$f(x,y) = \frac{1}{2}\left(a x^2 + b y^2 + c x y\right)$$

- f marginalement convexe ssi $a, b \geq 0$
- f convexe ssi $D^2 f(x,y) \in \mathcal{S}^+(d,\mathbb{R}) \iff \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \in \mathcal{S}^+(d,\mathbb{R})$

$$\iff a, b \geq 0 \text{ et } c^2 \leq 4ab$$

- f directionnellement convexe ssi $a, b, c \geq 0$

Ex: Si $a, b > 0$



Examples

- Let
$$F(x) = \Phi\left(\int_0^T \varphi(x(s))ds\right).$$
- F is convex iff φ is convex and Φ is non-decreasing convex.
- F is directionally convex iff both φ and Φ are non-decreasing convex.

Extensions

This provides as systematic approach which successfully works with

- Jump diffusion models,
- Path-dependent American style options,
- BSDE (without "Z" in the driver),
- ...

McKean-Vlasov diffusions:

- The *MKV* dynamics

$$(E) \equiv \quad dX_t = b(t, X_t, \mu_t)dW_t + \sigma(t, X_t, \mu_t)dW_t, \quad t \in [0, T]$$

with $\mu_t = \mathcal{L}(X_t)$, $W = (W_t)_{t\in[0,T]}$ a standard B.M. and

$$b, \sigma : [0, T] \times \mathbb{R} \times \mathcal{P}_p(\mathbb{R}) \to \mathbb{R} \text{ are continuous satisfying}$$

$(Lip) \equiv$ *$b(t, \cdot, \cdot)$, $\sigma(t, \cdot, \cdot)$ is $\big(|\cdot|, \mathcal{W}_p\big)$-Lipschitz, uniformly in $t \in [0, T]$.*

Wasserstein distance: $\mathcal{W}_p^p(\mu, \nu) = \inf\Big\{\int |x-y|^p m(dx, dy),\ m(dx, \mathbb{R}^d) = \mu,\ m(\mathbb{R}^d, dy) = \nu\Big\}.$

$$\Big(= \sup\Big\{\int f d\mu - \int f d\nu, [f]_{\rm Lip} \le 1\Big\} \text{ when } p = 1\Big).$$

- Under this assumption a strong solution exists for this equation.
- "Scaled" Martingality "requires" a drift term

$$b(t, X_t, \mu_t) = \alpha(t)(X_t + \beta(t, \mathbb{E}\, X_t))$$

$\alpha(t), \beta(t, \xi)$ Hölder continuous in t, β Lipschitz in ξ, uniformly in t. (From now on all zero for convenience. . .)

Understanding *MKV*

- Vlasov framework ($p = 1$). If σ has the following linear representation in μ

$$\sigma(x, \mu) = \int_{\mathbb{R}} \varpi(x, \xi)\mu(d\xi).$$

- Non linear framework. E.g.

$$\sigma(x, \mu) = \varphi_0\left(\int_{\mathbb{R}} \varpi(x, \xi)\mu(d\xi)\right).$$

MKV propagates convex order

Theorem (Liu-P., *AAP* 2023)

Let $\sigma, \theta \in Lip\big([0,T]\times\mathbb{R}\times\mathcal{P}_p(\mathbb{R}), \mathbb{R}^d\big)$, $p \geq 2$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *solutions to*

$$dX_t = \sigma(t, X_t, \mu_t)dW_t,\ \ X_0 \in L^p$$

$$dY_t = \theta(t, Y_t, \nu_t)dW_t,\ \ Y_0 \in L^p \quad \text{with } (W^{(\cdot)}_t)_{t\in[0,T]} \text{ standard B.M.}$$

If
- $(i)_\sigma$ $\sigma(t,x,\mu)$ *is* x-$\preceq$-*convex and* μ-$\uparrow_{cv}$ *for every* $t \in [0,T]$,
- *or*
- $(i)_\theta$ $\theta(t,x,\mu)$ *is* x-$\preceq$-*convex and* μ-$\uparrow_{cv}$ *for every* $t \in [0,T]$,
- *and*
- (ii) $\sigma(t,x,\mu) \preceq \theta(t,x,\mu)$ $\quad [|\sigma(t,x,\mu)| \leq |\theta(t,x,\mu)|$ *if* $d=1]$

and $X_0 \leq_{cv} Y_0$, *then, for every* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$, *convex with* $\|\,.\,\|_{\sup}$*-polynomial growth,*

$$x \mapsto \mathbb{E}\,F(X^x) \text{ is convex (if } X_0 = x \text{ and } (i)_\sigma \text{ holds) and } \mathbb{E}\,F(X) \leq \mathbb{E}\,F(Y).$$

Specificty of the proof

- The "regular" Euler scheme is again the main tool ... although not simulatable.
- Specificity for convexity propagation: two steps
  - Forward "marginal " approach necessary
    prior to
  - a backward "functional" approach.

Stochastic Volterra equation (for $X_0 \in L^1(\mathbb{P})$)

- Let $(X_t)_{t\in[0,T]}$ be a solution to the scaled stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t,s)\big(\alpha(s)+\beta(s)X_s\big)ds + \int_0^t K(t,s)\sigma(s,X_s)dW_s,\ \ t\in[0,T]$$

where the non-negative kernel $\big(K(t,s)\big)_{0\le s\le t\le T}$ is measurable and integrable, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{M}_{d,q}$ and $(W_t)_{t\in[0,T]}$ is a standard q-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$.

Theorem (Strong solution, Zhang (2005), Joudain-P'22)

If If $\sup_{t\in[0,T]} \int_0^t K^{2\rho}(t,s)ds < +\infty$ *for some* $\rho > 1$,

$$(\mathcal{K}^{cont}_\theta)\ \ \exists\,\kappa < +\infty,\ \forall\,\delta \in (0,T),\ \sup_{t\in[0,T]} \Big[\int_0^t |K((t+\delta)\wedge T, s) - K_i(t,s)|^i ds\Big]^{\frac{1}{i}} \le \kappa\,\delta^\theta$$

and $b(t,.)$ *and* $\sigma(t,.)$ *are Lipschitz uniformly in* $t\in[0,T]$ *then, for any* $X_0 \in L^1(\mathbb{P})$, $X_0 \perp\!\!\!\perp W$, *the equation has a unique* $\mathcal{F}^{X_0,W}$*-adapted pathwise continuous strong solution.*

Non-Markovian dynamics

- Main features:
  - Such a process is centered, $(\mathcal{F}_t^W)$-adapted but, in general,
  - it is not a martingale (not even a semi-martingale),
  - nor a Markov process.
  - Used to mimick Fractional Brownian motion driven SDEs when $K(t,s) = (t-s)^{H-\frac{1}{2}}$ (Rough stochastic volatility models à la Gatheral-Rosenbaum).

Theorem (convex propagation, (Jourdain-P. '22))

Assume $X_0 \in L^p(\mathbb{P})$, $p \in 51, +\infty)$ and

$$\forall\, t \in [0,T], \quad x \mapsto \sigma(t,x) \text{ is } \preceq\text{-convex}$$

then, for every convex functional $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$ with $\|\,.\,\|_{\sup}$-p-pol.growth

$$x \longmapsto \mathbb{E}\, F(X^x) \quad \text{is convex.}$$

Functional convex ordering

- Let

$$Y_t = Y_0 + \int_0^t K(t,s)\big(\alpha(s) + \beta(s)Y_s\big)ds + \int_0^t K(t,s)\theta(s, Y_s)dW_s, \; t \in [0,T]$$

Theorem (convex ordering (Jourdain-P. '22))

If

$$\begin{cases} (i)_\sigma & \sigma(t,x) \text{ is } x\text{-}\preceq\text{-convex for every } t \in [0,T], \\ or & \\ (i)_\theta & \theta(t,x) \text{ is } x\text{-}\preceq\text{-convex for every } t \in [0,T], \\ and & \\ (ii) & \sigma(t,x) \preceq \theta(t,x) \quad [|\sigma(t,x)| \le |\theta(t,x)| \;\; if \;\; d=1] \end{cases}$$

and $X_0 \le_{cv} Y_0$, *then, for every* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$, *convex (with* $\|\,.\,\|_{\sup}$*-polynomial growth),*

$$\mathbb{E}\,F(X) \le \mathbb{E}\,F(Y).$$

Methods of proof

- ($\alpha = \beta = 0$ for simplicity).
- We consider its Euler scheme with time step $\frac{T}{n}$ ($t_k = \frac{kT}{n}$):

$$\bar{X}_{t_k} = X_0 + \sum_{\ell=0}^{k-1} \sigma(t_\ell, \bar{X}_{t_\ell}) \int_{t_\ell}^{t_{\ell+1}} K(t_k, s)dW_s, \quad \bar{X}_0 = X_0.$$

- Not enough due to lack of Markovianity since $\bar{X}_{t_k}$ is not (in general) a function of $(\bar{X}_{t_{k-1}}, (W_s - W_{t_{k-1}})_{s\in[t_{k-1},t_k]})$.
- Markovianization: introduce for $k \in \{1, \cdots, n\}$, $(X^k_{t_\ell})_{0\le\ell\le k}$ starting from $X^k_0 = X_0$ and evolving inductively according to

$$X^k_{t_{\ell+1}} = X^k_{t_\ell} + \sigma(t_\ell, \bar{X}_{t_\ell}) \int_{t_\ell}^{t_{\ell+1}} K(t_k, s)dW_s, \quad 0 \le \ell \le k-1,$$

so that $\bar{X}_{t_k} = X^k_{t_k}$ for $k \in \{1, \cdots, n\}$ and $X^n = \bar{X}$.

- "Extend" the backward propagation proof to functionals

$$F\big((X^n_{t_\ell})_{\ell=0:n}, \dots, (X^k_{t_\ell})_{\ell=0:k}, \dots, (X^1_{t_\ell})_{\ell=0:1}\big).$$

- Transfer to continuous time by letting $n \to \infty$ (using e.g. Richard et al. '20). $\square$

- Extension to (one-dimensional) non-decreasing convex ordering when the drift b is $\preceq$-convex.

Applications to Vix options in rough Heston model

- Let us consider the auxiliary variance process in the quadratic rough Heston model (see Gatheral-Jusselin-Rosenbaum '20):

$$V_t = a(Z_t - b)^2 + c \quad \text{with} \quad a,\ b,\ c \geq 0$$

and, for $H \in (0, 1/2)$,

$$Z_t = Z_0 + \int_0^t (t-s)^{H-\frac{1}{2}} \lambda(f(s) - Z_s)ds + \sigma \int_0^t (t-s)^{H-\frac{1}{2}} \sqrt{a(Z_s - b)^2 + c}\, dW_s.$$

- $z \mapsto \sqrt{a(z-b)^2 + c}$ is convex and Lipschitz.
- Let $(Z_t^\sigma)_{t\geq 0}$ be its unique strong solution and V^σ the resulting squared volatility.
- For $\sigma \in (0, \tilde{\sigma}]$, one has $(Z_t^\sigma)_{t\in[0,T]} \preceq_{cvx} (Z_t^{\tilde{\sigma}})_{t\in[0,T]}$.
- Convexity of $L^2(dt)$ norm and (again) of $z \mapsto \sqrt{a(z-b)^2 + c}$ imply that

$$\mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T V_t^\sigma dt}\right) \leq \mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T V_t^{\tilde{\sigma}} dt}\right).$$