# Conditional propagation of chaos for generalized Hawkes processes having alpha-stable jump heights

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# Outline

# Where are we ?

The Model

Strong existence of the limit

Strong convergence to the limit

This is a joint work with my colleague and friend Dasha Loukianova from Université d'Evry, France.

# Systems of interacting particles (neurons, agents, ...)

- $N$ interacting particles $X_t^{N,1}, \ldots, X_t^{N,N}$, taking values in $\mathbb{R}_+$

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \psi(X_{t-}^{N,i})dZ_t^{N,i} + \sum_{j \neq i} W_{j \to i}^N dZ_t^{N,j},$$

  $Z_t^{N,i}$ jump process, having rate $f(X_{t-}^{N,i})$ at time $t$ ($f$ bounded).
- Each particle has drift $b(x, \mu)$, where $x$ is its current position and $\mu$ the empirical measure of the total system (deterministic flow between jumps).
- Each particle jumps at rate $f(x)$ when it is in position $x$.
- When jumping, it goes from position $x$ to a new position $x + \psi(x)$.
- When it is the $i-$th particle that jumps, at the same time, all other particles $j \neq i$ receive a small kick $W_{i \to j}^N$, which is random (synaptic weight).

# Propagation of chaos/Mean field frame

- Mean field interactions : the kicks $W^N_{i \to j} = W^N$ do not depend on the pair of particles that is involved $\Rightarrow$ exchangeable systems of particles described by their empirical measures

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{N,i})_{t \geq 0}} :$$

random probability measures on càdlàg trajectories.

- For suitable scalings of $W^N$, $\mu^N \to \bar{\mu}$ : describes the typical behavior of a single particle within an infinite limit population.
- Propagation of chaos : Limit $\bar{\mu}$ is deterministic $\iff$ in the limit, particles are independent.
- Holds for our system if

$$W^N = W/N$$

($W$ may be random), see De Masi et al. (2015), Fournier and Lö. (2016), Cormier, Tanré, Veltz (2020) and many others.

# Conditional propagation of chaos

- Diffusive scaling of synaptic weights $W^N = W/\sqrt{N}$, with random $W \sim \nu$ centered, independent choices at each jump time : has been studied in a series of papers together with Xavier Erny and Dasha Loukianova.
- Gives rise to a limit Brownian motion representing a source of common noise in the limit system (due to the CLT).

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- Gives rise to a limit Brownian motion representing a source of common noise in the limit system (due to the CLT). In this case the limit empirical measure $\bar{\mu}$ will be random (see later).
- Today we ask the question: What happens if we are in the domain of attraction of a stable law instead of being attracted to a Gaussian law??? So when $W^N = U/N^\xi$, with random $U \geq 0$ (independent choices at each jump time) which is a (one sided) stable random variable of index $\alpha \in (0,1)$, that is

$$\mathbb{E}(e^{-sU}) = e^{-s^\alpha}, s \geq 0.$$

$N^\xi, \xi > 0$, is a suitable renormalization. (It was $\xi = 1$ in the frame of propagation of chaos and $\xi = \frac{1}{2}$ in the diffusive setting).

- The process is a PDMP (piecewise deterministic Markov process) with generator

$$A^N\varphi(x) = \sum_{i=1}^N \partial_{x^i}\varphi(x) b(x^i, \mu^{N,x})$$
$$+\sum_{i=1}^N f(x^i) \int_{\mathbb{R}_+} \nu(du) \left( \varphi(x + \psi(x^i)e_i + \sum_{j \neq i} \frac{u}{N^\xi} e_j) - \varphi(x) \right),$$

$x = (x^1, \ldots, x^N) \in \mathbb{R}_+^N$, $\mu^{N,x} = \frac{1}{N} \sum_{j=1}^N \delta_{x^j}$, $e_j$ denotes the $j-$th unit vector in $\mathbb{R}^N$.
- $\nu$ is the law of the strictly stable random variable $U \geq 0$.
- If $U_1, \ldots, U_n$ i.i.d., $\sim \nu$, then $n^{-1/\alpha}(U_1 + \ldots + U_n) \sim \nu$.

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- $\nu$ is the law of the strictly stable random variable $U \geq 0$.
- If $U_1, \ldots, U_n$ i.i.d., $\sim \nu$, then $n^{-1/\alpha}(U_1 + \ldots + U_n) \sim \nu$. And so the renormalization has to be $N^{-1/\alpha}$, that is, $\xi = 1/\alpha$.
- Notice : jumps are **simultaneous :** the jump caused by particle $i$ affects all other particles. So we have simultaneous small jumps.

# Heuristics : Limit process I

- Interactions felt by particle $i$ are given by

$$A_t^N = \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}_+} u\mathbf{1}_{\{z \le f(X_{t-}^{N,j})\}} \pi^j(dt, dz, du),$$

where the $\pi^i$ are i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ having intensity $dtdz\nu(du)$, $\nu = \mathcal{L}(U)$.

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- If $f \equiv \lambda$ is constant, then $A_t^N =$ renormalized compound Poisson process. The total jump rate is $N\lambda$,

$$A_t^N = \frac{1}{N^{1/\alpha}} Z_{P_t}, \; Z_n = U_1 + \ldots + U_n, \; (P_t)_t \; PP(N\lambda) :$$

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  **time changed random walk, having stable increments.**

# Time changed stable random walks

## Proposition

*Let $U_n$ i.i.d. $\sim \nu$, $Z_0 = 0$, $Z_n = U_1 + \ldots + U_n$, $n \geq 1$, the associated random walk. Let $P \in \mathbb{N}$ be a r.v., independent of $(U_n)_n$. Then the following almost sure equality holds.*

$$Z_P = \sum_{n=1}^{P} U_n = P^{1/\alpha} \tilde{U}_1, \; \tilde{U}_1 \sim \nu, \;\; P \perp\!\!\!\perp \tilde{U}_1!$$

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## Corollary

*If $f \equiv \lambda$ constant, $P_t$ $PP(N\lambda)$, then for our interaction term*

$$A_t^N = \frac{1}{N^{1/\alpha}} Z_{P_t} = \underbrace{\left(\frac{P_t}{Nt}\right)^{1/\alpha}}_{LLN \to \lambda^{1/\alpha}} \underbrace{t^{1/\alpha} \tilde{U}_1}_{\sim S_t^\alpha} \to \lambda^{1/\alpha} S_t^\alpha,$$

*$S_t^\alpha$ stable subordinator of index $\alpha$.*

- We will use this argument for $t = \delta$ small - some time discretization step that allows us to **freeze the jump rate over small time steps** to deal with the general case when $f$ is not constant.
- On each time interval $[k\delta, (k+1)\delta[$, $k \geq 0$, the typical contribution should be of type

  $(N^{-1} \times$ total jump intensity $)^{1/\alpha} \times$ increment stable subordinator

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  (on $[k\delta, (k+1)\delta[$).
- As $\delta \to 0$ (and $N \to \infty$) the sum of all such contributions should converge to

$$\int_0^t (\bar{\mu}_{s-}(f))^{1/\alpha} dS_s^\alpha.$$

# (Guess of the) Limit System

- Limit system should be an exchangeable system $(\bar{X}^1, \bar{X}^2, \ldots)$ s.t.

$$\bar{X}_t^i = \bar{X}_0^i + \int_0^t b(\bar{X}_s^i, \bar{\mu}_s) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}_{s-}^i) 1_{\{z \le f(\bar{X}_{s-}^i)\}} \bar{\pi}^i(ds, dz) + \int_{[0,t]} (\bar{\mu}_{s-}(f))^{1/\alpha} dS_s^\alpha.$$

- Presence of $S^\alpha$ = common source of noise for all particles $\Longrightarrow$ $\bar{X}^i, i \geq 1$, are NOT independent.

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- Limit system should be an exchangeable system $(\bar{X}^1, \bar{X}^2, \dots)$ s.t.

$$\bar{X}^i_t = \bar{X}^i_0 + \int_0^t b(\bar{X}^i_s, \bar{\mu}_s) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}^i_{s-}) 1_{\{z \le f(\bar{X}^i_{s-})\}} \bar{\pi}^i(ds, dz) + \int_{[0,t]} (\bar{\mu}_{s-}(f))^{1/\alpha} \, dS^\alpha_s.$$

- Presence of $S^\alpha$ = common source of noise for all particles $\Longrightarrow$ $\bar{X}^i, i \ge 1$, are NOT independent. But we will show that they are i.i.d. knowing $S^\alpha$.
- This is called conditional propagation of chaos.
- Limit measure $\lim_N \mu^N = \bar{\mu} = \lim_N \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}^i}$ is random expressing presence of common noise.

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- Limit measure $\lim_N \mu^N = \bar{\mu} = \lim_N \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}^i}$ is random expressing presence of common noise. **Do we know more about $\bar{\mu}$ ?**

# More on $\bar{\mu}$

**Theorem of Hewitt-Savage:**
$\mathcal{L}\left((\bar{X}_t^i)_{i\geq 1}\right) =$ mixture of i.i.d.'s, directed by the law of $\bar{\mu}_t$.

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Conditional version of strong law of large numbers implies $\bar{\mu}_t(\cdot) = P(\bar{X}_t^i \in \cdot | S^\alpha)$ such that the limit equation for one typical particle is

$$d\bar{X}_t^i = b(\bar{X}_t^i, \bar{\mu}_t)dt + \psi(\bar{X}_{t-}^i)d\bar{Z}_t^i + \mathbb{E}(f(\bar{X}_t^i)|S^\alpha)^{1/\alpha} dS_t^\alpha, \qquad (1)$$

$\bar{Z}_t^i$ : jumps at rate $f(\bar{X}_{t-}^i)$. END HEURISTICS

# Plan of the talk

- Strong existence and uniqueness for the limit system
- Convergence of the finite system to the limit system by **coupling**.

# Where are we ?

The Model

Strong existence of the limit

Strong convergence to the limit

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- Small jumps ($\leq 1$) are integrable!
- So we need a distance which is a **mixture of $L^1$ for small jumps and $L^q$ for big jumps**.
- Similar problem has been tackled by Fournier IHP 2013 in the Markovian case, but his approach does not work here (among other things because of the dependence of the conditional law).
- So we do something else, still inspired by what I have learned from Nicolas : we introduce a space transform (bijection) $a \in C^2$, concave on $\mathbb{R}_+$, such that

  $a(x) \leq C(x \wedge x^q)$ if$x \geq 0$,

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  $$a(x) \le C(x \wedge x^q) \text{ if} x \ge 0, a(x) = x^q \text{ up to a constant, if } x \ge 1.$$

  (This function depends on $q$, but $q$ will be fixed in the sequel.)

## Assumptions

**Big jumps and jump rate:** *bounded and $a-$Lipschitz, $\psi$ is positive.*

**Drift:** *bounded and*

$$|b(x,\mu) - b(\tilde{x},\tilde{\mu})| \leq C(|a(x) - a(\tilde{x})| + \inf_{\pi \in \Pi(\mu,\tilde{\mu})} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \pi(dy,dy')|a(y) - a(y')|).$$

*Here, $\Pi(\mu,\tilde{\mu})$ is the set of all couplings of $\mu$ and $\tilde{\mu}$.*

***It suffices to suppose that $\mu \mapsto b(x,\mu)$ admits a functional derivative $\delta_\mu b(x,y,\mu)$ which is uniformly $a-$Lipschitz.***

**Minimal jump activity:** $f$ *lowerbounded.*

# A priori bounds on limit process

- Since $f, \psi, b$ bounded,

$$|\bar{X}_t| \leq |X_0| + \|b\|_\infty t + \|\psi\|_\infty N_t^{\|f\|_\infty} + \|f\|_\infty^{1/\alpha} S_t^\alpha,$$

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  where $N^{\|f\|_\infty}$ is a Poisson process of rate $\|f\|_\infty$.
- Since $\bar{\mu}_t = \mathcal{L}(\bar{X}_t | S^\alpha)$, this implies that $\bar{\mu}_t$ has as many moments as the initial condition (although this is not the case for $\mathbb{E}\mu_t$).

### Theorem

*Under our conditions (plus moment condition on initial condition) :*
*1. We have pathwise uniqueness for the limit equation.*
*2. There exists a unique strong solution* $(\bar{X}_t)_{t\geq 0}$ *of the limit equation, satisfying for every* $t > 0$,

$$\mathbb{E}(\sup_{s\leq t} |a(\bar{X}_s)|) < +\infty. \tag{2}$$

## Proof of the Uniqueness result.

– Let $\bar{X}$ and $\tilde{X}$ be two solutions, **driven by the same noise**, and starting from the same initial conditions.

– Consider $\Delta_t := |a(\bar{X}_t) - a(\tilde{X}_t)|$.

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– Most difficult term comes from the stochastic integral

$$I_t = \int_{[0,t]\times\mathbb{R}_+} \left[ a\left(\bar{X}_{s-} + \bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\bar{X}_{s-}\right)\right] - \left[ a\left(\tilde{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{X}_{s-}\right)\right] M(ds, dx),$$

$M$ : jump measure of $S^\alpha$ : a PRM of intensity $dsm(dx)$,

$$m(dx) = C 1_{(0,+\infty)}(x) x^{-(\alpha+1)} dx.$$

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– Small jump $x \leq 1$ or big jumps $x > 1$ : $I_t =: I_t^1 + I_t^2$.

□

# Control of small jumps

## Proof.

- Since small jumps $x \leq 1$ are integrable, we can apply Taylor's formula and obtain

$$a(y + \bar{\mu}(f)^{1/\alpha} x) - a(y) \sim a'(\tilde{y}) \bar{\mu}(f)^{1/\alpha} x.$$

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$$|\bar{\mu}_{s-}(f)^{1/\alpha} - \tilde{\mu}_{s-}(f)^{1/\alpha}| \leq C|\bar{\mu}_{s-}(f) - \tilde{\mu}_{s-}(f)|,$$

since $z \mapsto z^{1/\alpha}$ is Lipschitz on $[0, \|f\|_\infty]$.

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- But $\bar{\mu}_{s-}(f) = \mathbb{E}[f(\bar{X}_{s-}) | S^\alpha]$, $\tilde{\mu}_{s-}(f) = \mathbb{E}[f(\tilde{X}_{s-}) | S^\alpha]$ (same driving noise) and $|f(x) - f(y)| \leq C |a(x) - a(y)|$...

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- Taking expectation, this gives

$$\mathbb{E} \sup_{t \leq T} |I_t^1| \leq C T \mathbb{E} \sup_{s \leq T} |a(\bar{X}_s) - a(\tilde{X}_s)|.$$

# Big jumps $x \geq 1$

- We use that $a$ is **concave** and compare after jump positions

$$
\begin{aligned}
&\left| a\left(\bar{X}_{s-} + \bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) \right| \\
&\quad \leq \left| a\left(\bar{X}_{s-} + \bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\bar{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) \right| \\
&\qquad + \left| a\left(\bar{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) \right| \\
&\leq \left| a\left(\bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{\mu}_{s-}(f)^{1/\alpha} x\right) \right| + \left| a\left(\bar{X}_{s-}\right) - a\left(\tilde{X}_{s-}\right) \right|.
\end{aligned}
$$

- Since $f$ is lower-bounded and $a(x) = x^q$ for $x > 1$, for large values of $x$, the dangerous red term equals

$$
x^q \left| \bar{\mu}_{s-}(f)^{q/\alpha} - \tilde{\mu}_{s-}(f)^{q/\alpha} \right|
$$

- and the mapping $y \mapsto y^{q/\alpha}$ is Lipschitz on $[\underline{f}, \infty[$, if $\underline{f}$ is the lowerbound on $f$.

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\begin{aligned}
&\left| a\left(\bar{X}_{s-} + \bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right)\right| \\
&\quad \leq \left| a\left(\bar{X}_{s-} + \bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\bar{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right)\right| \\
&\qquad + \left| a\left(\bar{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{X}_{s-} + \tilde{\mu}_{s-}(f)^{1/\alpha} x\right)\right| \\
&\leq \left| a\left(\bar{\mu}_{s-}(f)^{1/\alpha} x\right) - a\left(\tilde{\mu}_{s-}(f)^{1/\alpha} x\right)\right| + \left| a\left(\bar{X}_{s-}\right) - a\left(\tilde{X}_{s-}\right)\right|.
\end{aligned}
$$

- Since $f$ is lower-bounded and $a(x) = x^q$ for $x > 1$, for large values of $x$, the dangerous red term equals

$$
x^q \left| \bar{\mu}_{s-}(f)^{q/\alpha} - \tilde{\mu}_{s-}(f)^{q/\alpha} \right|
$$

- and the mapping $y \mapsto y^{q/\alpha}$ is Lipschitz on $[\underline{f}, \infty[$, if $\underline{f}$ is the lowerbound on $f$.

# Control of the other terms

– The other terms are easier to be controlled....
– All in all we obtain

$$\mathbb{E} \sup_{s \leq T} |a(\bar{X}_s) - a(\tilde{X}_s)| \leq CT \mathbb{E} \sup_{s \leq T} |a(\bar{X}_s) - a(\tilde{X}_s)|.$$

Taking $T$ sufficiently small, we obtain uniqueness on $[0, T]$, and then we iterate this argument.
– Strong existence by Picard iteration, using the same distance function $a$.

# Where are we ?

### Theorem

*For suitable moment conditions on the initial condition, it is possible to construct the finite and the limit system on the same probability space such that for all $t \leq T$,*

$$\mathbb{E}(|a(X_t^{N,1}) - a(\bar{X}_t^1)|) \leq C_T N^{-\frac{\alpha}{2+\alpha}}.$$

**Attention** The above statement is not precise; actually we obtain the above rate only in the limit $q \uparrow \alpha$ (which is not a possible choice for $q$ since moments of order $\alpha$ of the stable random variables do not exist any more).

# Proof : Time discretization (Pseudo Euler)

- We freeze time during intervals of length $\delta = \delta(N) < 1$, $\delta \times N \to \infty$ as $N \to \infty$ (per time unit, the average number of jumps tends to infinity).
- Our approximation will be based on the observation that during each time interval of length $\delta$, the increment of the interaction term $A_t^N$ is approximately given by **a time changed random walk having stable increments** (increments: the random kicks).
- And the time change is the total number of jumps during this interval – which is (conditionally) Poisson distributed.

# More on discretising time

Freezing positions over time gives

- Approximation of the increment of the interaction term $A_t^N$ over $]k\delta, (k+1)\delta]$ :

$$A_\delta^k := \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{]k\delta,(k+1)\delta] \times \mathbb{R}_+ \times \mathbb{R}_+} u \mathbf{1}_{\{z \le f(X_{k\delta}^{N,j})\}} \pi^j(dt, dz, du).$$

- How many jumps ? Conditionally on $\mathcal{F}_{k\delta}$,

$$N_\delta^k = \sum_{j=1}^N \int_{]k\delta,(k+1)\delta] \times \mathbb{R}_+ \times \mathbb{R}_+} \mathbf{1}_{\{z \le f(X_{k\delta}^{N,j})\}} \pi^j(dt, dz, du),$$

$\sim Poiss(\bar{f}(X_{k\delta}^N)\delta), \; \bar{f}(x) = \sum_{i=1}^N f(x^i).$

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$\sim Poiss(\bar{f}(X_{k\delta}^N)\delta)$, $\bar{f}(x) = \sum_{i=1}^N f(x^i)$. Depends only on "$z$" : acceptance/rejection variable.

- Freezing time allows us to separate the randomness $\mathbf{z}$ of the acceptance/rejection and the one coming from the random height $\mathbf{u}$ of the jumps.
- For each time step $k$, we have

$$A_\delta^k = \frac{1}{N^{1/\alpha}} Z^k_{N_\delta^k},$$

  $Z^k$ the random walk built from the $\alpha-$stable increments, during the $k-$th interval.
  **Important :** $(Z_n^k)_n$ INDEPENDENT of $N_\delta^k$, and of $\bar{\pi}^j = \pi^j(\cdot, \cdot, \mathbb{R})$.
- We use the scaling argument of the beginning of this talk and take the time change "out as a factor".

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- We use the scaling argument of the beginning of this talk and take the time change "out as a factor". So

$$A_\delta^k = \left(\frac{N_\delta^k}{N\delta}\right)^{1/\alpha} S_\delta^{N,k,\alpha},$$

  where $S^{N,k,\alpha}$ is a stable subordinator, independent of $\mathcal{F}_{k\delta}$ and of $\bar{\pi}^j, j \geq 1$.

Conditionally on $\mathcal{F}_{k\delta}$, $N_\delta^k \sim Poiss(\bar{f}(X_{k\delta}^N)\delta)$, $f$ bounded. So :

$$\frac{N_\delta^k}{N\delta} = \frac{\bar{f}(X_{k\delta}^N)}{N} + O(\frac{1}{\sqrt{N\delta}}) = \mu_{k\delta}^N(f) + O(\frac{1}{\sqrt{N\delta}}). \tag{3}$$

All in all :

## Proposition

*Representation of the increment of the interaction term*

$$A_\delta^k = \left(\frac{N_\delta^k}{N\delta}\right)^{1/\alpha} S_\delta^{N,k,\alpha} = \left(\frac{1}{N}\bar{f}(X_{k\delta}^N)\right)^{1/\alpha} S_\delta^{N,k,\alpha} + R_\delta^{N,k},$$

*where* $\mathbb{E}|R_\delta^{N,k}|^q \leq C(N\delta)^{-q/2}\delta^{q/\alpha}$ *(comes from the deviation bounds on the Poisson random variables plus* $q-$*th moment of the alpha-stable rv).*

# Concatenation

– We concatenate all these independent pieces of stable subordinators $S_\delta^{N,k,\alpha}$ and fill in subordinator bridges to obtain a global subordinator $S^{N,\alpha}$.
– By construction, $S^{N,\alpha}$ is independent of the projections of the PRM's $\bar{\pi}^i$ on the first two coordinates (time and acceptance/rejection variable).

## Theorem

*We obtain the representation of the interaction term*

$$A_t^N = \int_0^t \left( \frac{1}{N} \bar{f}(X_{s-}^N) \right)^{1/\alpha} dS_s^{N,\alpha} + R_t^N, \tag{4}$$

*where $R_t^N$ is an error term such that*

$$\mathbb{E}(|R_t^N|^q) \le C_T \left( \underbrace{N^{1-q/\alpha} \delta^{q/\alpha}}_{\textit{discretization}} + \underbrace{(N\delta)^{-q/2} \delta^{q/\alpha - 1}}_{\frac{T}{\delta} \textit{Poisson errors}} \right).$$

# Interpolating auxiliary system

– To prove the convergence to the limit system, we consider an auxiliary particle system which is a mean-field version of the limit system $(\bar{X}^{N,i})_{i\geq 1}$ :

$$d\tilde{X}_t^{N,i} = \ldots (\text{ jumps + drift terms}) + \left( \frac{1}{N} \sum_{j=1}^{N} f(\tilde{X}_{t-}^{N,j}) \right)^{1/\alpha} dS_t^{N,\alpha},$$

driven by same stable subordinator and same PRM's, starting from the same initial values.

– To control the distance of $\tilde{X}^N$ to the limit system, **we need to control the distance between $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_t^{N,i}}$, the empirical measure of the limit system, and $\bar{\mu}_t = \mathcal{L}(\bar{X}_t^{N,i} \,|S^{N,\alpha})$.**

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– Since $|b(x,\mu) - b(x,\tilde{\mu})| \ \leq CW_q(\mu,\tilde{\mu})$, it suffices to control the Wasserstein-$q-$distance $(q < 1)$. This is done by using Fournier-Guillin (2015), conditionally on $S^{\alpha}$.

# Open questions

- Jump heights not strictly stable, but only in the domain of attraction of $S^\alpha$ ??? Weak approach via martingale problem ?
- Is it possible to obtain weak rates of convergence ?
- Typical behavior of the limit system ?
- Notice that when there are no big jumps (and when the initial values are deterministic), then $\bar{X}_t^i$ is $\sigma\{S_s^\alpha, s \le t\}-$measurable.

# Open questions

- Jump heights not strictly stable, but only in the domain of attraction of $S^\alpha$ ??? Weak approach via martingale problem ?
- Is it possible to obtain weak rates of convergence ?
- Typical behavior of the limit system ?
- Notice that when there are no big jumps (and when the initial values are deterministic), then $\bar{X}_t^i$ is $\sigma\{S_s^\alpha, s \le t\}-$measurable. So $\bar{\mu}_t = \delta_{\bar{X}_t^i}$, such that the limit system is

  $$d\bar{X}_t^i = b(\bar{X}_t^i, \bar{\mu}_t)dt + (f(\bar{X}_{t-}))^{1/\alpha} dS_t^\alpha,$$

  which is Markovian.
- $\alpha \ge 1$?

# Some literature

– Andreis, L., Dai Pra, P., Fischer, M. *McKeanVlasov limit for interacting systems with simultaneous jumps. SPA* 2018.
– Erny, X. Löcherbach, E. Loukianova, D. *Strong error bounds for the convergence to its mean field limit for systems of interacting neurons in a diffusive scaling.To appear in AAP*, 2023.
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**Thank you for your attention !**