Gambling for resurrection and the heat equation on a triangle

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Introduction

Controlling the diffusion coefficient of a process with negative drift and minimizing the probability to hit a lower barrier up to time T.

Minimizing the probability

$$J(t,x,\alpha) = P\left(\inf_{t\leqslant s\leqslant T} X_s^{t,x,\alpha} < 0\right).$$

where

$$X_s^{t,x,\alpha} = x - (s-t) + \int_t^s \alpha_r \, dW_r, \quad s \in [t,T],$$

and $\alpha \in \mathcal{A}$ the set of progressively measurable stochastic processes with values in $[0,1]$.

Aim of our work

Let the value function

$$V(t, x) = \inf \{ J(t, x, \alpha) : \alpha \in \mathcal{A} \},$$

for all $(t, x) \in [0, T] \times [0, \infty)$.

Outline

Motivation

Stylized model :

- First example: a manager who can control the volatility of a firm's value process, wants to minimize the bankruptcy probability up to time T.
- The optimal strategy is to choose 1 if the value process is less than the time to maturity and 0 otherwise.

An alternative representation of the value function

Remark : If $x \geq T - t$, $\alpha = 0$ is optimal and $V(t,x) = 0$
For all $(t,x) \in [0,T] \times [0,\infty)$ and $\alpha \in \mathcal{A}$ we define

$$\tau_l^{t,x,\alpha} = \inf\{s \geqslant t : X_s^{t,x,\alpha} \leqslant 0\},$$
$$\tau_u^{t,x,\alpha} = \inf\{s \geqslant t : X_s^{t,x,\alpha} \geqslant T - s\}.$$

Moreover, we set $\widehat{\alpha}_s^{t,x} = \alpha_s 1_{\{\tau_u^{t,x,\alpha} > s\}}$ for all $s \in [t,T]$. Notice that

$$J(t,x,\alpha) \geqslant J(t,x,\widehat{\alpha}^{t,x}) = P(\tau_l^{t,x,\alpha} < \tau_u^{t,x,\alpha}).$$

Therefore, the value function satisfies

$$V(t,x) = \inf\{P(\tau_l^{t,x,\alpha} < \tau_u^{t,x,\alpha}) : \alpha \in \mathcal{A}\}.$$

HJB equation

Value function V is a solution of the following HJB equation:

$$\partial_t V(t,x) - \partial_x V(t,x) + \inf_{a\in[0,1]} \frac{1}{2} a^2 \partial_{xx} V(t,x) = 0,$$

with the boundary conditions

$$V(t,0) = 1, \text{ for all } t \in [0, T),$$
$$V(t, T-t) = 0, \text{ for all } t \in [0, T].$$

The infimum in the above equation can only be attained for $a = 0$ or for $a = 1$. So the the candidate feedback control is

$$a^\star(t,x) = \begin{cases} 1, & \text{if } x < T - t, \\ 0, & \text{otherwise.} \end{cases}$$

and $Y_s^{t,x} = x - (s-t) + W_s^{t,x}$ be the state process controlled with feedback function $a^\star$.

The value function coincides with the probability for $Y^{t,x}$ to first exit form the cathetus

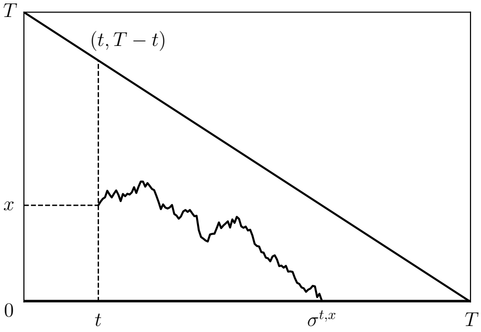

Figure: $Y^{t,x}$ exits the triangle from the cathetus.

It is equivalent to the probability of the BM $(W_s^{t,x})_{s\geqslant t}$ to exit the right from the hypothenuse.

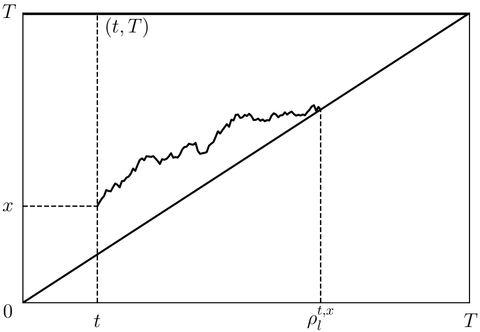

Figure: First exit time of $W^{t,x}$ from the hypothenuse.

First exit for a BM from the hypothenuse

For all $t \in [0, T)$ and $x \in (t, T)$

$$\rho_u^{t,x} = \inf\{s \geqslant t : \gamma W_s^{t,x} \geqslant T\},$$
$$\rho_l^{t,x} = \inf\{s \geqslant t : \gamma W_s^{t,x} \leqslant s\},$$

define the following quantity

$$H(t,x) := P(\rho_l^{t,x} < \rho_u^{t,x}),$$

which is the probability that $W^{t,x}$ exits from the hypothenuse .

Proposition

For all $(t,x) \in J = \{(t,x) \in [0,T) \times (0,T) : x \geqslant t\}$ we have

$$H(t,x) = 1 - \int_0^{T-t} f(T-x,v)g(T-x,v,(T-t-v),1)dv$$

where the functions f and g are defined as

$$f(z,v) = \frac{|z|}{\sqrt{2\pi}v^{3/2}} \exp(-\frac{z^2}{2v}), \quad (z,v) \in \mathbb{R} \times (0,\infty),$$

and

$$g(y,u,a,b) = \sum_{j=-\infty}^{\infty} (1 - 2j\frac{a+ub}{y})e^{-\frac{2a}{u}j^2(a+ub)}e^{\frac{2a}{u}jy}.$$

Moreover, $H \in \mathcal{C}(\bar{J} \setminus \{(T,T)\}) \cap \mathcal{C}^{1,2}(J)$. and satisfies the heat equation on J with the boundary conditions

$$H(t,t) = 1 \text{ and } H(t,T) = 0,$$

for all $t \in [0,T)$.

Ideas of the proof-I

- Function H verifies

$$1 - H(t,x) = \int_t^T P_x\left(W_s > s,\ s \in (0, v-t) | \rho_u = v\right) P(\rho_u^{t,x} \in dv),$$

$$1 - H(t,x) = \int_t^T P_{x,v-t,T}\left(W_s > s,\ s \in (0, v-t) | W_s < T,\ s \in (0, v-t)\right) f(T-x, v)dv,$$

where $P_{x,u,y}$ denotes the law of the Brownian bridge between x and y, with length $u > 0$.

Ideas of the proof-II

- The probability for a Brownian Bridge to exit from a wedge is given by Salminen and Yor [1]
- Tedious calculations+Itô calculus for the last point

The expression of V

Theorem

The value function satisfies, for all $(t,x) \in [0,T] \times (0,\infty)$ with $x < T-t$,

$$V(t,x) = 1 - \int_0^{T-t} f(T-t-x, v) g(T-t-x, v, T-t-v, 1) dv.$$

Proof

- A candidate for V, let $w(t,x) = H(t, x+T)$
- Verification arguments with classical arguments.

Thank you for your attention

P. Salminen and M. Yor.
On hitting times of affine boundaries by reflecting brownian motion and bessel processes.
Periodica Mathematica Hungarica, 62(1):75–101, Mar 2011.