Differential Geometry, Billiards, and Geometric Optics
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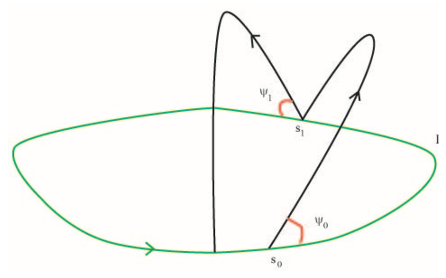
A non-integrable circular billiard

Cássio H.V. Morais   Sylvie Oliffson Kamphorst   Sônia Pinto de Carvalho
Departamento de Matemática, Universidade Federal de Minas Gerais, Brasil

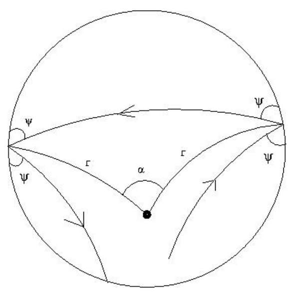
A Billiard can be defined on a closed region on a Riemannian surface as the free motion of a particle along a geodesic inside the region, making elastic collisions at the impacts with the boundary.



Surfaces with constant curvature

Let the three constant curvature surfaces be given by

$$\begin{array}{ll} \mathbb{E}^2: (\rho\cos\theta, \rho\sin\theta) & \rho > 0 \\ \mathbb{S}^2_+: (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho) & 0 < \rho < \frac{\pi}{2} \\ \mathbb{H}^2: (\sinh\rho\cos\theta, \sinh\rho\sin\theta, \cosh\rho) & \rho > 0 \end{array}$$



The billiard map in the geodesic circle with radius ρ_0 is is

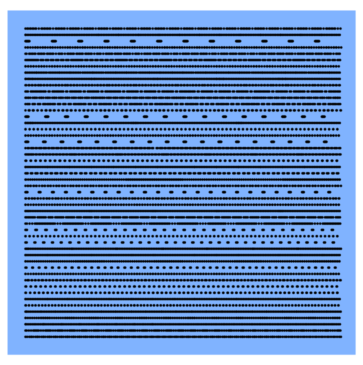
$$T(\theta_0, \varphi_0) = (\theta_0 + w(\varphi_0) \bmod 2\pi, \varphi_0)$$

where

$$w(\varphi_0) = \begin{cases} 2\varphi_0\rho_0 & \text{if } \mathbb{E}^2 \\ \arccos\left(\frac{\cos^2\rho_0 - \tan^2\varphi_0}{\sec^2\varphi_0 - \sin^2\rho_0}\right) & \text{if } \mathbb{S}^2_+ \\ \arccos\left(\frac{\cosh^2\rho_0 - \tan^2\varphi_0}{\sec^2\varphi_0 + \sinh^2\rho_0}\right) & \text{if } \mathbb{H}^2 \end{cases}$$

The curves $\varphi = \varphi_0$ are invariant under T.

The Geodesic Circular Billiard is totally integrable on surfaces with constant curvature. L.dos Santos, SPC, 2017



The circle is the only C^2, closed and strictly convex curve in $\mathbb{E}^2$, $\mathbb{S}^2_+$ or $\mathbb{H}^2$ such that its associated billiard map is totally integrable. Bialy, 1993, 2012

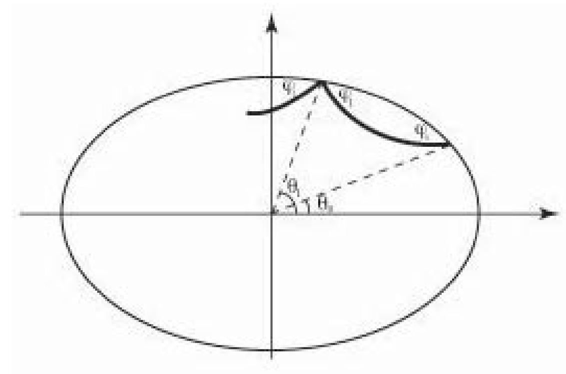
Surfaces with a symmetric metric

Let
S be a C^∞ Riemannian surface,
$V \subset S$ a totally normal neighbourhood,
$\gamma \subset V$ a geodesic circle,
(r, θ) geodesic polar coordinates centred at the center of γ.
In these polar coordinates the Riemannian metric is

$$g = dr^2 + G(r,\theta)^2 d\theta^2$$

Let $\alpha \subset V$ be a convex closed curve. The billiard on α is

$$\begin{array}{rcl} T: [0,2\pi) \times (0,\pi) & \longrightarrow & [0,2\pi) \times (0,\pi) \\ (\theta_0, \varphi_0) & \longmapsto & (\theta_1, \varphi_1) \end{array}$$



T is a Twist diffeomorphism with Lagrangian $H(\theta_0, \theta_1) = -d(\alpha(\theta_0), \alpha(\theta_1))$, the geodesic distance between two impacts.
If $G = G(r)$ then the Euler-Lagrange equations

$$\begin{cases} r'' - GG'\theta'^2 = 0 \\ \theta'' + 2\dfrac{G'}{G} r'\theta' = 0 \end{cases}$$

do not depend on θ. If $\eta(t) = (r(t), \theta(t))$ is a geodesic then $\eta_1(t) = (r(t), \theta(t) + \bar{\theta})$ and $\eta_2(t) = (r(t), -\theta(t))$ are also geodesics, which implies that
If the metric is invariant under rotations then the billiard on a geodesic circle centred at the origin is totally integrable.
Question: Bialy's result is still true in this case?

Breaking the symmetry: an example of a non-integrable circular billiard

There is a perturbation of the Euclidean metric such that the billiard on the geodesic circle is not integrable.

Sketch of the proof:
Let g_0 be the Euclidean metric in $\mathbb{E}^2$, T_0 the billiard map in the unity circle C.

Step1: preserving the unit circle:
In polar coordinates, let $g_\epsilon = dr^2 + (r^2 + \epsilon G_1(r,\theta))d\theta^2$. Taking $G_1(1,\theta) = 0$ the unity circle C for g_0 is still a unity circle for g_ϵ.

Step 2: breaking a (p,q)-resonant invariant curve:
For ϵ sufficiently small, let $g_\epsilon = g_0 + \epsilon g_1$ be the new metric such that in polar coordinates $G_1(1,\theta) = 0$, and T_ϵ the billiard map on C.
Let $T_\epsilon(s_1, \varphi_1) = (s_2, \varphi_2)$. The generating function of T_ϵ is minus the geodesic distance between two consecutive impacts $C(s_1)$ and $C(s_2)$. It can be written as minus

$$H_\epsilon(s_1, s_2) = cte + \epsilon H_1(s_1, s_2) + \mathcal{O}(\epsilon^2),$$

where, if γ_0 is the geodesic of the metric g_0 connecting $C(s_1)$ to $C(s_2)$,

$$H_1(s_1, s_2) = \frac{1}{2}\int_{\gamma_0} g_1(\gamma_0', \gamma_0')\, ds$$

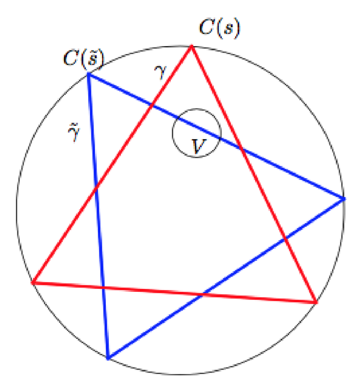
Let $\Upsilon_{(p,q)}$ be an invariant curve of T_0 with rotation number p/q and $(s, \varphi(s)) \in \Upsilon_{(p,q)}$. The action associated to this orbit is

$$L_\epsilon(s) = \sum_{k=1}^{q} H_\epsilon(\Pi_1(T_\epsilon^{k-1}(s,\varphi(s))), \Pi_1(T_\epsilon^{k}(s,\varphi(s)))) = cte + \epsilon L_1(s) + \mathcal{O}(\epsilon^2)$$

and the Melnikov Potential is

$$L_1(s) = \sum_{k=1}^{q} H_1(s, s + 2k\pi\frac{p}{q})$$

$L_1(s)$ is the sum of the integrals of g_1 along a polygonal trajectory τ of T_0, beginning at $C(s)$.
Let $\tilde{\tau}$ be another trajectory on $\Upsilon_{(p,q)}$, beginning at $C(\tilde{s})$ - which is just a rotation of τ - and let $L_1(\tilde{s})$ be its associated Melnikov Potential.
Let V be an open set inside the circle that intersects the trajectory $\tilde{\tau}$ but not τ. Choose $G_1 \equiv 0$ except on V. Then $g_1 \equiv g_0$ outside V but $g_1 \neq g_0$ inside V. Then $L_1(\tilde{s}) \neq L_1(s)$ and $\Upsilon_{(p,q)}$ do not persist under T_ϵ.



Remark: I.P.Toth, in a non-published text, gives a similar construction of the perturbed metric to build a non-integrable circular billiard.

Some References
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