Osculating conics of a regular curve on $\mathbb{RP}^2$

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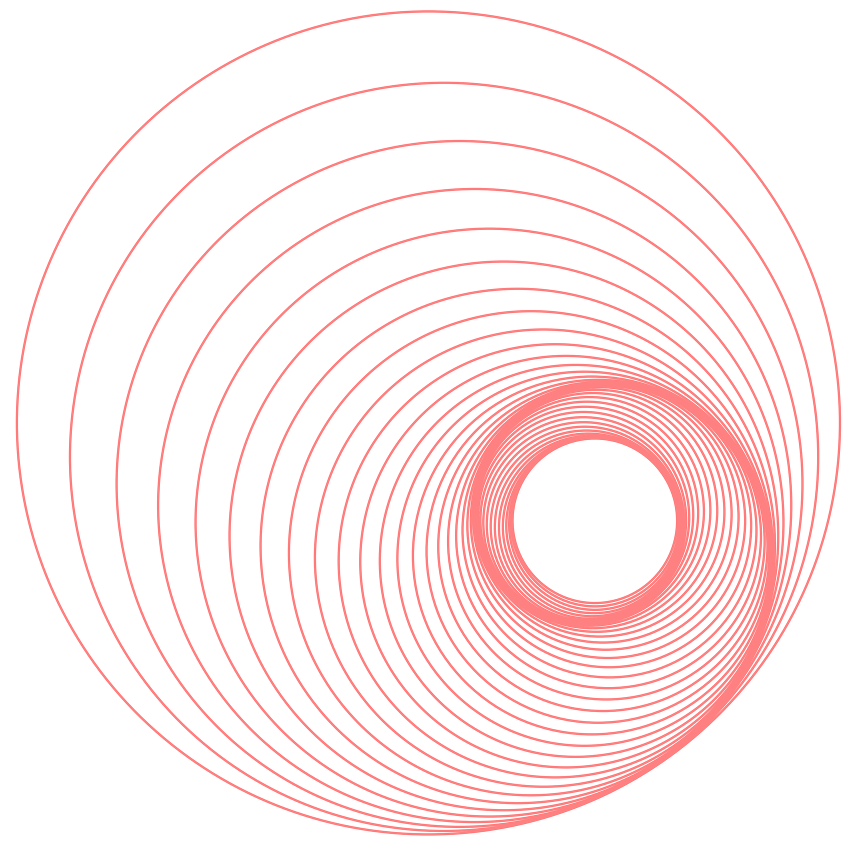
Introduction

The **Tait-Kneser theorem** is a well known result of differential geometry which states that the osculating circles of a plane curve with monotonic curvature and no inflection points are disjoint and nested. Therefore, an arc with no **vertex** gives rise to an interesting foliation of the region of the plane delimited by its largest and smallest osculating circles. In this work, we investigate the analogous result for **osculating conics**. It is already known that the osculating conics of a curve with no **sextactic** or **inflection** points are also disjoint and nested. However, we will show that the relative position of two such conics is actually more restricted than that, they are in some sense **"convexly nested"**.

Osculating Circles



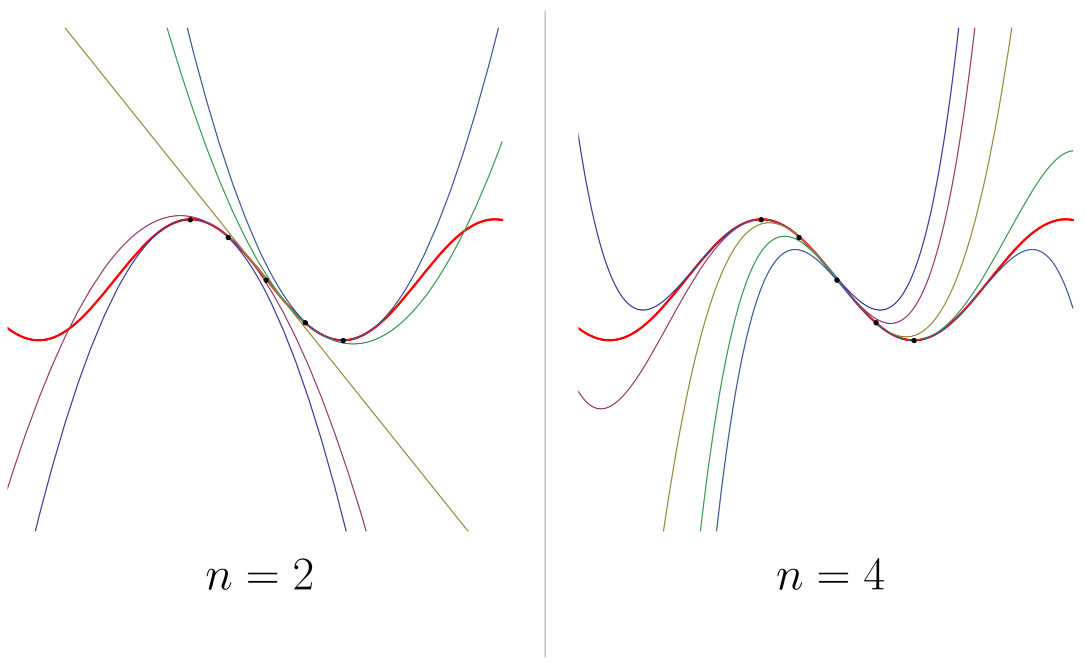
Taylor Polynomials

Let $f:\mathbb{R}\to\mathbb{R}$ be a real function and I an interval. Let $n=2r$ be an even integer. Suppose that f is $(n+1)$ times differentiable and that

$$\forall x\in I,\ f^{(n+1)}(x)\neq 0$$

Then, for every $a,b\in I$, the graphs of the Taylor polynomials of degree n on a and b, denoted T_a and T_b, are disjoint over the entire real line.

$$f(x)=\cos(x)$$



$$T_t(x)=\sum_{i=0}^{n}\frac{f^{(i)}(t)}{i!}(x-t)^i$$

$$\frac{\partial T_t(x)}{\partial t}=\frac{f^{(n+1)}(t)}{n!}(x-t)^n\geq 0$$

Space of Forms and its Blenders

Motivating question: What functions can be such a difference $T_b(x)-T_a(x)$?

$$T_b(x)-T_a(x)=\int_a^b\frac{f^{(n+1)}(t)}{n!}(x-t)^n dt$$

This difference is given as a barycentre of function of type $(x-t)^n$ for an even n. Thus, it is positive, but also **convex** for $n\geq 2$.

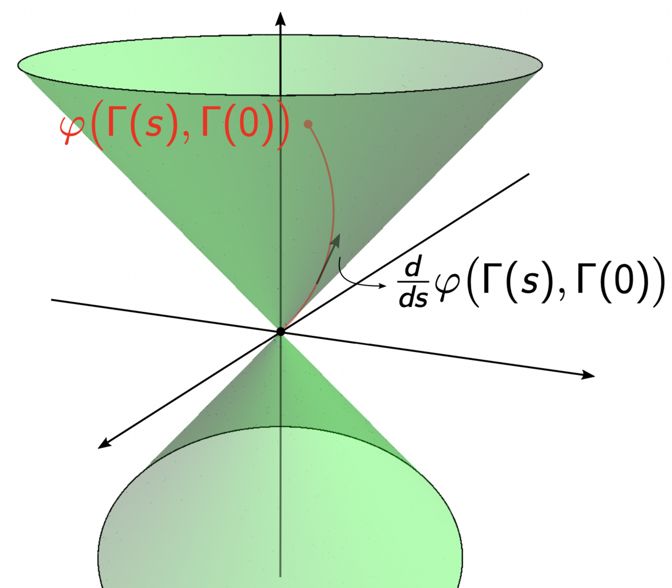
Definition
A subset B of $F_{n,d}$ is a **blender** if it is a closed convex cone invariant as a set under the $GL(n,\mathbb{R})$-action of linear change of coordinates.

Examples

$$P_{2,4}=\left\{p\in F_{2,4}:\forall(t,w)\in\mathbb{R}^2,\ p(t,w)\geq 0\right\}$$

$$K_{2,4}=\left\{p\in P_{2,4}:p\text{ is convex}\right\}$$

$$Q_{2,4}=\left\{p\in F_{2,4}:p=\sum_{k=1}^{m}(\alpha_k t+\beta_k w)^4,\ \alpha_k,\beta_k\in\mathbb{R}\right\}$$



Space of Conics of $\mathbb{RP}^2$

A conic is given implicitly by an algebraic equation of the following type:

$$ax^2+2hxy+by^2+2fxz+2gyz+cz^2=0$$

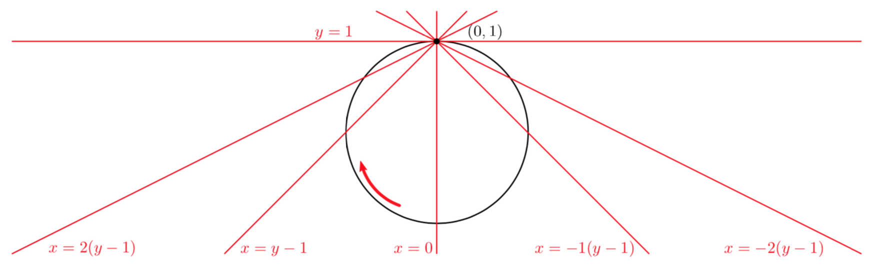
This can also be expressed in matrix form

$$\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}a & h & f\\ h & b & g\\ f & g & c\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0$$

Classification of real projective conics		
Signature	Normal form	Type
$(3,0)$ or $(0,3)$	$x^2+y^2+z^2=0$	Empty irreducible
$(2,1)$ or $(1,2)$	$x^2+y^2-z^2=0$	Non-empty irreducible
$(2,0)$ or $(0,2)$	$x^2+y^2=0$	Imaginary line-pair
$(1,1)$	$x^2-y^2=0$	Real line-pair
$(1,0)$ or $(0,1)$	$x^2=0$	Repeated line

We will construct a map $\varphi:\Omega\times\Omega\to F_{2,4}$ that compares the relative position of two conics by means of a binary quartic.

We need two ingredient, an **implicit equation** and a **polynomial parametrization** for each conic.



$$\rho(u)[t:w]=[-2tw:t^2-w^2:t^2+w^2]$$

Now we can compare two conics $u,v\in\Omega$ by taking

$$\varphi(v,u)=v\circ\rho(u)\in F_{2,4}$$

The Curve of Osculating Conics

Consider a smooth arc $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{RP}^2$ with no inflexion point. Its osculating conic at $\gamma(s_0)$ is the only one that has a contact of order greater than or equal to 5 at precisely this point.

$$\begin{cases} u\circ\gamma\,(s_0)=0\\ \frac{d}{ds}(u\circ\gamma)(s_0)=0\\ \frac{d^2}{ds^2}(u\circ\gamma)(s_0)=0\\ \frac{d^3}{ds^3}(u\circ\gamma)(s_0)=0\\ \frac{d^4}{ds^4}(u\circ\gamma)(s_0)=0\end{cases}$$

We have a path of osculating conics $\Gamma:(-\varepsilon,\varepsilon)\to\Omega$ such that its tangent vector $\frac{d}{ds}\lambda(s)\Gamma(s)$ correspond to the **double line** tangent to γ at $\gamma(s)$ due to the following lemma.

Lemma
If for every $s\in(-\varepsilon,\varepsilon)$ the conic given implicitly by $\Gamma(s)$ has a contact of order at least 5 with the curve $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{RP}^2$ at the point $\gamma(s)$, then the conic given by $\Gamma'(s)$ has a contact of order at least 4 with γ at the point $\gamma(s)$.

$$\frac{d}{ds}\lambda(s)\Gamma(s)=\lambda'(s)\Gamma(s)+\lambda(s)\Gamma'(s)$$

Also, the conics given by $\Gamma(s)$ and $\Gamma'(s)$ are the same if and only if $\gamma(s)$ is a **sextactic point**.

We may compare two osculating conics by analysing $\varphi\big(\Gamma(s_0),\Gamma(0)\big)$, but since we have the whole path, we can make s vary and study the curve $\varphi\big(\Gamma(s),\Gamma(0)\big)\subset F_{2,4}$.

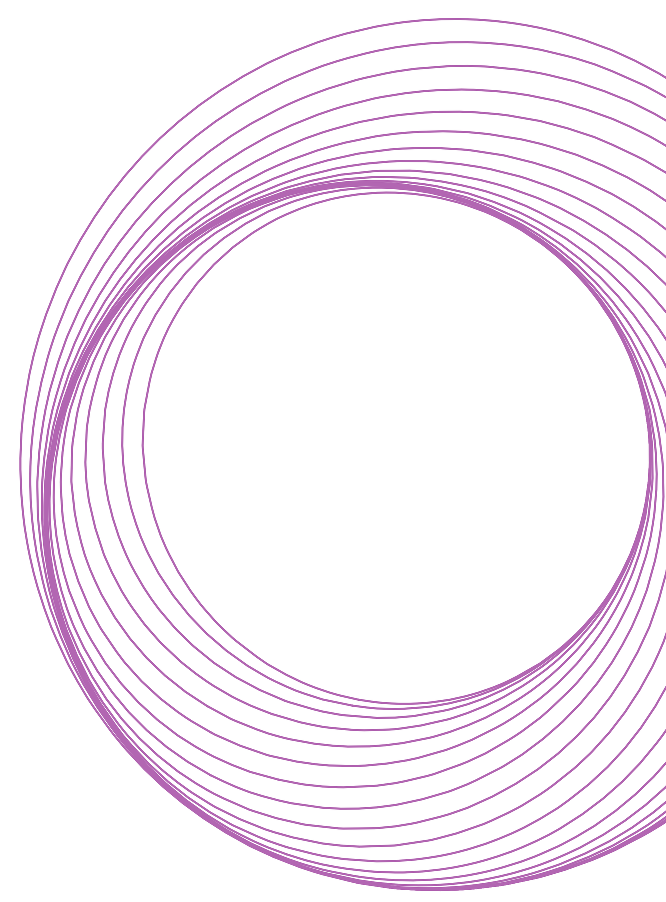
It starts at the origin and we wish to prove that its other extremity lies within $K_{2,4}$. It turns out to be simpler to study its tangent vector.

$$\frac{d}{ds}\varphi\big(\Gamma(s),\Gamma(0)\big)=\frac{d}{ds}\Gamma(s)\circ\rho\big(\Gamma(0)\big)$$

Using a geometric argument, we can show that $\frac{d}{ds}\varphi\big(\Gamma(s),\Gamma(0)\big)\in K_{2,4}$ and thus by integration we have that $\varphi\big(\Gamma(s_0),\Gamma(0)\big)\in K_{2,4}$.

Conclusion

Theorem
The osculating conics of a smooth arc $\gamma:(-\varepsilon,\varepsilon)\to\Omega\ \mathbb{RP}^2$ with no inflection or sextactic point are disjoint and **convexly nested**.



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