

TORSION-FREE ABELIAN GROUPS ARE BOREL COMPLETE

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ABSTRACT. We prove that the Borel space of torsion-free Abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

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Post 1: Completeness of TFAB

- ▶ Pasolini + Sheloh, Torsion-free abelian groups are Borel complete. Archive.

Post 2: Short idea of the proof

Idea of the proof in 4 steps.

Part 3: G -topological groups

- ▶ Pasolini + Sheloh, On the existence of uncountable topological and G -topological AB -Anchors.
- ▶ Pasolini + Sheloh, G -topological groups are complete G -analytic. In preparation.

Part 1: Completeness of TFAB

Seminal paper on DST of Ab. structures



[3] H. Friedman and L. Stanley. *A Borel reducibility theory for classes of countable structures*. J. Symb. Log. **54** (1989), no. 03, 894-914.

~~possible, as an isomorphism relation.~~ The Borel completeness of countable abelian group theory is particularly interesting from the perspective of model theory, as this class is model theoretically “low”, i.e stable (in the terminology of [10]). In fact, as already observed in [3], Borel reducibility can be thought of as a weak version of $\mathfrak{L}_{\omega_1, \omega}$ -interpretability, and for other classes of countable structures such as groups or fields much stronger results than Borel completeness exist, as in such cases we can first-order interpret graph theory, but such classes are unstable, while abelian group theory is stable. ~~Reference [8] starts a systematic study of the relations between~~

Fact 1.1. *The set K_ω^L of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(K_\omega^L, \mathcal{B})$. Every Borel subset of this space $(K_\omega^L, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(K_\omega^L, \mathcal{B})$.*

For example, if take $L = \{e, \cdot, ()^{-1}\}$, and we let K' to be one of the following:

- (a) the set of elements of K_ω^L which are groups;
 - (b) the set of elements of K_ω^L which are abelian groups;
 - (c) the set of elements of K_ω^L which are torsion-free abelian groups;
 - (d) the set of elements of K_ω^L which are n -nilpotent groups, for some $n < \omega$;
- then we have that K' is a Borel subset of $(K_\omega^L, \mathcal{B})$, and so Fact 1.1 applies.

Notation 1.6. (1) We denote by Graph the class of graphs.

(2) We denote by Gp the class of groups.

(3) We denote by AB the class of abelian groups.

(4) We denote by TFAB the class of torsion-free abelian groups.

(5) Given a class K we denote by K_λ the set of structures in K with domain λ .

(λ a cardinal)

We now introduce two notions of resolutions:

- 
- ① Resolutions between ideals of Borel spaces;
 - ② Resolutions between equivalence relations def. on Borel spaces.

⑦ of previous slides
↓

Definition 1.2. Let X_1 and X_2 be two standard Borel spaces, and let also $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is Borel reducible to Y_2 , denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \rightarrow X_2$ such that for every $x \in X_1$ we have:

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Notice that Definition 1.2 covers in particular the case $X_1 = K' \times K'$ for K' as in Fact 1.1, and so for example Y_1 could be the isomorphism relation on K' . Also,

Definition 1.3. Let X_1 be a Borel space and $Y_1 \subseteq X_1$. We say that Y_1 is complete analytic (resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2 we have that $Y_2 \leqslant_R Y_1$.

② of previous slides
↓

Definition 1.4. Let X_1 and X_2 be two standard Borel spaces, and let also E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2 . We say that E_1 is Borel reducible to E_2 , denoted as $E_1 \leq_B E_2$, when there is a Borel map $\mathbf{B} : X_1 \rightarrow X_2$ such that for every $x, y \in X_1$ we have:

$$xE_1y \Leftrightarrow \mathbf{B}(x)E_2\mathbf{B}(y).$$

Remark 1.5. Notice that in the context of Definitions [1.2](#) and [1.4](#), $E_1 \leq_R E_2$ and $E_1 \leq_B E_2$ have two different meaning, as in the first case the witnessing Borel function has domain $X \times X$, while in the second case it has domain X . Furthermore, notice that $E_1 \leq_B E_2$ implies $E_1 \leq_R E_2$ (but the converse need not hold, see [1.7](#)).

↑ ↑ ↑ ↑ ↑
Important to avoid confusion !!!

Definition 1.6. Let K_1 be a Borel class of structures with domain ω and let $\cong_1$ be the isomorphism relation on K_1 . We say that K_1 is **Borel complete** (or, in more modern terminology, $\cong_1$ is S_∞ -complete) if for every Borel class K_2 of structures with domain ω there is a Borel map $\mathbf{B} : K_2 \rightarrow K_1$ such that for every $A, B \in K_2$:

$$A \cong_2 B \Leftrightarrow \mathbf{B}(A) \cong_1 \mathbf{B}(B),$$

that is, the isomorphism relation on the space K_2 is Borel reducible (in the sense of Definition 1.4) to the isomorphism relation on the space K_1 .

Need: $\cong$ on K_1 is as simplified as possible as an isom. relation on th. struch

Fact 1.7 ([3]). Let K be a Borel class of structures with domain ω . If K is **Borel complete**, then its isomorphism relation is a **complete analytic subset** of $K \times K$, but the converse need not hold, as for example **abelian p -groups with domain ω have complete analytic isomorphism relation but they are not a Borel complete space.**

Examples

projective planes (look in)

- (i) countable graphs, linear orders and trees are Borel complete;
- (ii) torsion abelian groups have complete analytic $\cong$ but are *not* Borel complete;
- (iii) nilpotent groups of class 2 and exponent p (p a prime) are Borel complete¹;
- (iv) the isomorphism relation on finite rank torsion-free abelian groups is Borel.

↓ '89 paper

In [3] Friedman and Stanley state explicitly:

There is, alas, a missing piece to the puzzle, namely our **conjecture that torsion-free abelian groups are complete**. [...] We have not even been able to show that the isomorphism relation on torsion-free abelian groups is complete analytic, nor, in another direction, that the class of all abelian groups is Borel complete. We consider these problems to be among the most important in the subject.

This problem is the main objective of our work

A little bit of Lie group theory

Def. $\mathfrak{g} \in \mathfrak{AB}$ is torsion-free when $\forall \gamma \in \mathfrak{g} \setminus \{0\}$
and $0 < n < \infty$ we have that $n\gamma \neq 0$.

Def. $\mathfrak{g} \in \mathfrak{AB}$ is rigid when $\text{Aut}(\mathfrak{g})$
consists only of $\text{ad}_{\mathfrak{g}}$ and $-\text{ad}_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}$
 $\gamma \mapsto -\gamma$

Abelian no action as we work in $\mathfrak{AB}$!

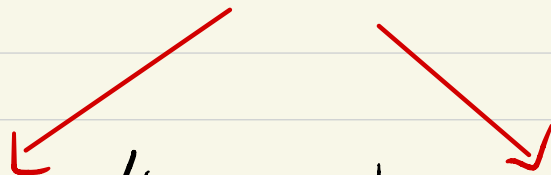
Examples (list) of TFA B:

(i) $\mathbb{Z}, \mathbb{Q}$;

(ii) $\forall \alpha \in \text{Ord}$, $\bigoplus_{b < \alpha} \mathbb{Z}$ and $\bigoplus_{b < \alpha} \mathbb{Q}$;

(iii) Subgroups of $\bigoplus_{b < \alpha} \mathbb{Q}$.

Direct sum



Def. $L \in AB$, $g \in L$, $0 < n < w$. We say that

g is n -divisible if $\exists h \in L$ s.t. $nh = g$.

L is divisible if $\forall g \in L$ and $0 < n < w$

g is n -divisible.

5. REMARKS. Every torsion free divisible group D of rank α is a direct sum of α copies of the additive group of rational numbers, and D contains an isomorphic copy of every torsion free Abelian group of


 Subgroups of $\bigoplus_{b < \alpha} \mathbb{Q}$.

$\wedge$
 rank $\leq \alpha$

is not an example but a description
of the torsion-free AB groups

is not divisible

is divisible

→ We say that $G \in AB$ is **TORSION**
when $\forall g \in G \exists 1 \leq n < \infty$ s.t. $n g = 0$.

→ Given $G \in AB$ and p a prime number:

$$\text{Tor}(G) = \{ g \in G \mid \exists 1 \leq n < \infty, n g = 0 \}$$

$$\text{Tor}_p(G) = \{ g \in G \mid \exists 1 \leq n < \infty, p^n g = 0 \}$$

→ $G \in AB$ is a p -group if $G = \text{Tor}_p(G)$.

Closures of groups

Cmpl. of $\cong$

- ① Torision $G \in AB$
- ② $G \in TFA B$
- ③ Mixed $G \in AB$

NOT Borel
complete

?

?

.

[Notice that the torision subgroup $T_G(G)$ need
not be a direct summand of G , so
③ does not reduce to ① + ②.]

Problem Is TFAB_ω Borel complete?

History part Friedman-Stanley '89 paper

The challenge was taken by several mathematicians. The first to work on this problem was Hjorth, which in [6] proved that any Borel isomorphism relation is Borel reducible (*) to the isomorphism relation on countable torsion-free abelian groups, and that in particular the isomorphism relation on TFAB_ω is not Borel, leaving though open the question whether TFAB_ω is a Borel complete class, or even whether the isomorphism relation on TFAB_ω is complete analytic (**). (cf. Fact 1.5).

(*) in the sense of ② from previous slides
(**) in the sense of ① " " "

The problem resisted further attempts of the time and the interest moved to another very interesting problem on torsion-free abelian groups: for $1 \leq n < m < \omega$, is the isomorphism relation on torsion-free abelian groups of rank n strictly less complex than the isomorphism relation on torsion-free abelian groups of rank m ?

Very interesting problem with motivation

Whereas there are fairly large classes of torsion groups whose structures can be described in terms of satisfactory invariants, there are only a very few and rather restricted classes of torsion-free groups for which satisfactory structure theory is known. These include the torsion-free groups of rank 1 and their direct sums, but no other major classes; even for groups of finite rank no useful complete systems of invariants are known. Naturally, one can establish certain schemes for constructing torsion-free groups, which provide a certain amount of information about their structures, but the schemes so far known fail to give an acceptable solution to the basic problem of deciding when two groups given by different schemes are isomorphic.

- [6] Laszlo Fuchs. *Infinite abelian groups. Vol. II*. Pure and Applied Mathematics, Vol. 36-II Academic Press, New York-London 1973.

$\cong_n$ is iso relation on $G \in \text{TFAB}_w$ s.t. $\text{rk}(G) = n$.

$\cong_n^*$ is iso relation on $G \in \text{TFAB}_w$ s.t. $\begin{cases} \text{rk}(G) = n \\ G \text{ rigid} \end{cases}$

Theorem (Thomas) $\cong_n <_B \cong_{n+1}$

Theorem (Thomas) $\cong_{n+1}^* \not\leq_B \cong_n$

- [11] S. Thomas. *On the complexity of the classification problem for torsion-free abelian groups of rank two*. Acta Math. **189** (2002), no. 02, 287-305.
- [12] S. Thomas. *The classification problem for torsion-free abelian groups of finite rank*. J. Amer. Math. Soc. **16** (2003), no. 01, 233-258.

Ok, but what about $\cong$ on TFAB_ω ???
(= our major problem)

~~most important in the subject~~ in [3]. The problem remained dormant for various years (at the best of our knowledge), until Downey and Montalbán [2] made some important progress showing that the isomorphism relation on countable torsion-free abelian groups is complete analytic, a necessary but not sufficient condition for Borel completeness, as recalled in Fact 1.5. This was of course possible evidence that the isomorphism relation was indeed Borel complete, as conjectured in [3]. Despite this advancement, the problem of Borel completeness of countable torsion-free abelian groups resisted for other 12 years, until this day, when we prove:

Main Theorem. *The space TFAB_ω is Borel complete, in fact there exists a continuous map $\mathbf{B} : \text{Graph}_\omega \rightarrow \text{TFAB}_\omega$ such that for every $H_1, H_2 \in \text{Graph}_\omega$:*

$$H_1 \cong H_2 \text{ if and only if } \mathbf{B}(H_1) \cong \mathbf{B}(H_2).$$

Part 2 : Idea of the proof

Step 1

Let M be countable random graph

$$C_2 = \bigoplus_{v \in V} \{ \mathbb{Q} x \mid x \in X \}$$

Maximal
divisibility

"Minimal"

$\rightarrow$

$$C_1 = \bigoplus_{v \in V} \mathbb{Z} x$$

$$C_0 = \bigoplus_{v \in V} \{ \mathbb{Z} x \mid x \in X \}$$

Minimal
divisibility

X an infinite set "with structure sitting on it."

Step 2 For every $M \in \mathcal{M}$ infinite subgroup

we define subgroup $L(1, M) \leq L(1, M) = L_1$

Step 3 For $M, V \in \mathcal{M}$ as in Step 2:

$$M \cong V \iff L(1, M) \cong L(1, V)$$

(as graphs)

(as groups)

Step 4

$$\underline{B} : \text{Graph}_w \rightarrow \text{TFA } B_w$$

$$H \mapsto F[H] \subseteq M \mapsto G_{(1, F[H])} \subseteq G_1$$

$\uparrow$ continuous $\uparrow$ continuous

With some "Borel argument" we assume that on the right the domain of $G \in \text{TFA } B_w$ is an infinite subset of w instead of w .

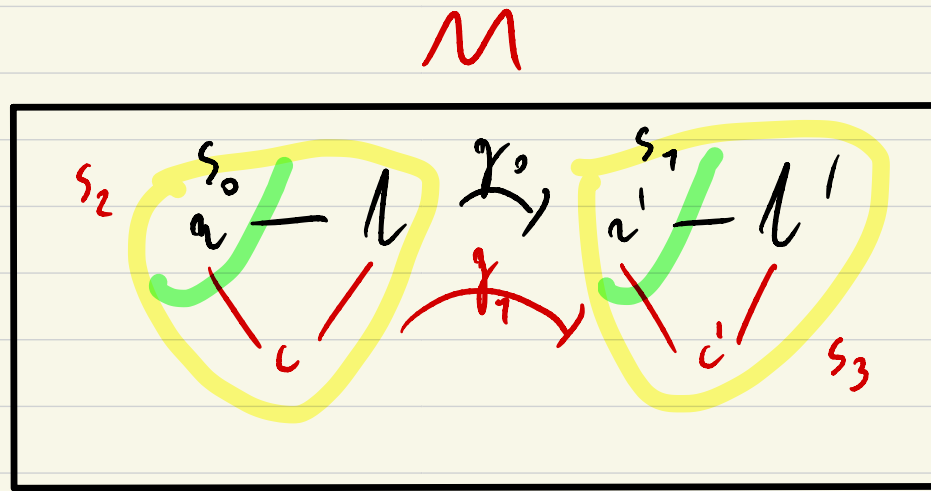
Finite subsets
of M

$$s_0 = \{a\}$$

$$s_1 = \{a'\}$$

$$s_2 = \{a, b, c\}$$

$$s_3 = \{a', b', c'\}$$

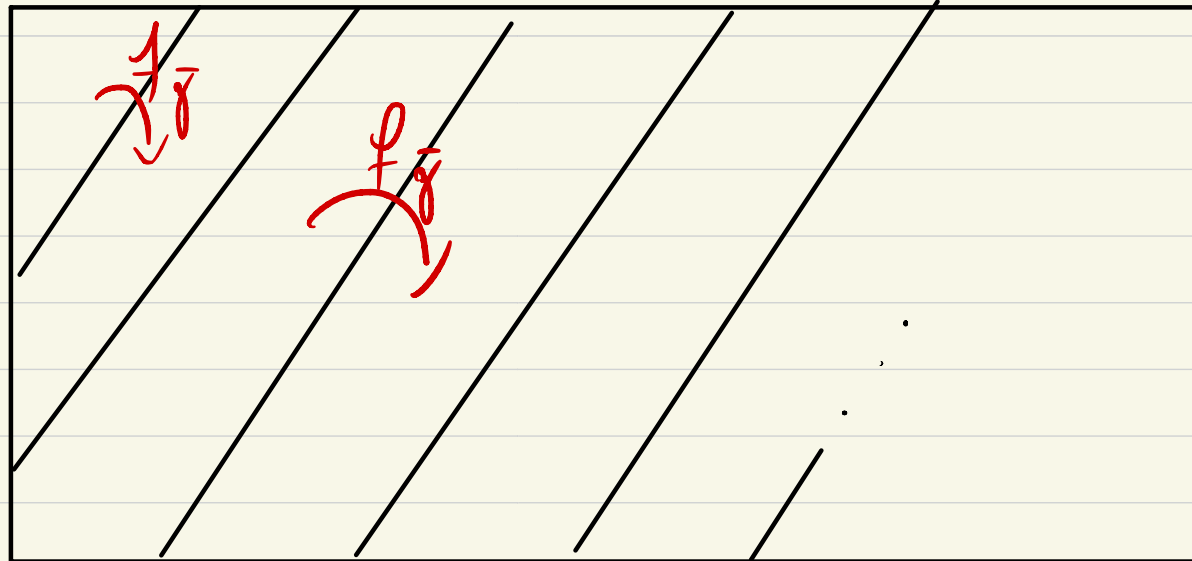


Partial automorphism
of M

$$g_0 \subseteq g_1$$

$$\bar{g} = (g_0, g_1)$$

X'_{s_0} X'_{s_1} X'_{s_2} X'_{s_3}



The line
of the $\rightarrow$
typ. G_e

X

$(X'_s : s \subseteq_\omega M)$ is a partition of X into infinite sets;

Part 3: Cr-hypofuran groups

Def. Let G be a group. We say that G is G -Hopfian (resp. Hopfian) when every $f \in \text{End}(G)$ which is injective (resp. surjective) is also surjective.
(resp. injective)

Example The group $\mathbb{Z}$ is Hbf-ism and not co-Hbf-ism. Why?

$$f: \mathbb{Z} \rightarrow 2\mathbb{Z} \quad (\text{non-co-Hbf-ism})$$
$$n \mapsto 2n$$

$$f: \mathbb{Z} \overset{\text{injection}}{\rightarrow} \mathbb{Z} \quad (\text{Hbf-ism})$$

Then $\ker(f) \leq \mathbb{Z} \approx \ker(f) = n\mathbb{Z}$ and as

$\mathbb{Z}/\ker(f) \cong \mathbb{Z}$ we must have that $n = 0$.

(*)

Example The group $\mathbb{Z}(p^\infty)$ is co-Hopfian and not Hopfian. Why? For not-Hopfian

$$\mathbb{Z}(p^\infty) \rightarrow \mathbb{Z}(p^\infty) \quad \left(\begin{array}{l} \text{surjective} \\ \text{not injective} \end{array} \right)$$

$$\gamma \longmapsto p\gamma$$

(*) Prüfer group: divisible p -group of rank 1

$$\mathbb{Z}(p^\infty) = \langle g_1, g_2, \dots \mid pg_1 = 0, pg_2 = g_1, \dots \rangle$$

Fact $G \in \text{TFAB}$ is co-hyp iff G is divisible of finite rank, i.e., $\exists m < \omega$

$$G \cong \bigoplus_{m < \omega} \mathbb{Q}.$$

Hence the co-hyp $G \in \text{TFAB}$ are a local subset!

Fact [2] Let $G \in \mathcal{AB}$ be co-hypfian. Then:
 $| \text{Tor}(G) | \leq 2^{\mathfrak{G}_0}$.

Fact [2] There are no co-hypfian p -groups
in $\mathcal{AB}$ of rank $\mathfrak{G}_0$.

[2] R. A. Beaumont and R. S. Pierce. *Partly transitive modules and modules with proper isomorphic submodules*. Trans. Amer. Math. Soc. **91** (1959), 209-219.

Question Are there co-Hopfian p -groups in AB
of size $\aleph_0 < \lambda \leq 2^{\aleph_0}$???

Fact [4] There are groups as $\uparrow$ for $\lambda = 2^{\aleph_0}$.

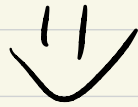
Question What about $\aleph_0 < \lambda < 2^{\aleph_0}$???

This is independent from ZFC [3] !!!

[3] G. Braun and L. Strüngmann. *The independence of the notions of Hopfian and co-Hopfian Abelian p -groups*. Proc. Amer. Math. Soc. 143 (2015), no. 8, 3331-3341.

[4] Peter Crawley *An infinite primary abelian group without proper isomorphic subgroups*. Bull. Amer. Math. Soc. **68** (1962), no. 5, 463-467.

Fact [24] $\forall \lambda \exists G \in \text{TFAB}$ which is endoregular, that
is $\forall f \in \text{End}(G) \exists m \in \mathbb{Z}$ s.t. $f(x) = mx$ and
 f is onto iff $m \in \{1, -1\}$.



$\forall \lambda \exists G \in \text{TFAB}_\lambda$ which is Hopfian



But this uses STATIONARY SETS, so

The construction is non-effective.

What about an EFFECTIVE version ???

↓
on a specific notion of “effectiveness” which was suggested for abelian groups by Nadel in [16], i.e., the preservation under any forcing extension of the universe V . We refer to this as the problem of absolute existence (of a group satisfying a certain property). These kind of problems were considered by Fuchs, Göbel, Shelah and others (see e.g. [7, 10, 11]), probably the most important problem in this area is the problem of existence of absolutely indecomposable groups in every cardinality which remains open to this day (despite several partial answers are known).

Problem. (1) *Despite the known necessary restrictions, can we improve (in ZFC!) the result from [2] that there are no co-Hopfian p -groups of size $\aleph_0$ or $> 2^{\aleph_0}$?*
 (2) *Are there co-Hopfian groups in every (resp. arbitrarily large) cardinality?*
 (3) *Are there absolutely Hopfian groups in every cardinality?*

Theorem 1.1. *Suppose that $G \in \text{AB}$ is reduced and $\aleph_0 \leq |G| < 2^{\aleph_0}$. If $\mathfrak{p} > |G|$ and there is a prime p such that $\text{Tor}_p(G)$ is infinite, then G is not co-Hopfian. In particular there are no infinite reduced p -groups G of size $\aleph_0 \leq |G| < \mathfrak{p}$.*

Theorem 1.2. *If $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$, and $G \in \text{AB}_\lambda$, then G is not co-Hopfian.*

Theorem 1.3. *For all $\lambda \in \text{Card}$ there is $G \in \text{TFAB}_\lambda$ which is absolutely Hopfian.*

Our Theorems (Podelski + Shelah)

What about countable ω -hyperfinite groups?

Question (Thomas) Are the ω -hyperfinite groups complete ω -analytic in G_w ?
(The base space of groups with dom w)

Def. We say that G is nilpotent of class 2 if $\forall g, h, k \in G$ we have

$$[[g, h], k] = e,$$

where $[x, y] = x^{-1}y^{-1}xy$ (commutator).



Theorem 1.4. In $\text{NipGp}(2)_\omega$ the set of co-Hopfian groups is complete co-analytic.

Question 1.6. *Are the co-Hopfian groups complete co-analytic in AB_ω ?*

We leave this open but we prove:

Theorem 1.5. *If $G \in \text{AB}_\omega$ is co-Hopfian and reduced, then for every prime p we have that $\text{Tor}_p(G)$ is finite and G embeds in the profinite group $\prod_{p \in \mathbb{P}} \text{Tor}_p(G)$.*

Resolution to a concrete problem!

Result: $G \in \mathbf{A} \mathbf{B}$ is rigid when

$$\text{Aut}(G) = \{ \text{id}_G, -\text{id}_G \}.$$

Theorem 1.3. *The rigid $G \in \mathbf{TFAB}_\omega$ are complete co-analytic in $\mathbf{TFAB}_\omega$. In fact, there exists a Borel map $\mathbf{B}$ from Tr_ω to $\mathbf{TFAB}_\omega$ s.t. for $T \in \text{Tr}_\omega$ we have:*

- (1) $\mathbf{B}(T)$ is Hopfian;
- (2) if T is well-founded, then $\mathbf{B}(T)$ has only trivial endomorphisms;
- (3) if T is not well-founded, then $\mathbf{B}(T)$ has a non-trivial (free) automorphism.

$G \in AB$ is boundedly enlarged when
 $\text{End}(G) / \text{BEnd}(G) \cong \mathbb{Z}$, where $\text{BEnd}(G)$
 are those $f \in \text{End}(G)$ such that
 $\exists n < \omega$ such that $\text{ran}(f^n) = \{0\}$.

$\exists$ of $G \in AB$ as $\uparrow$ above with
 $\text{Ton}(G) = \{ \mathbb{Z}_{p^n} \mid p \text{ prime}, n \geq 1 \}$.


Corollary 1.8. If $2^{\aleph_0} < \lambda$, then there is a co-Hopfian $G \in AB$ iff $\lambda = \lambda^{\aleph_0}$.

THE END

THANK YOU