The Jackson analysis and the strongest hypotheses

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  - Some of these predictions can be verified.
  - One prediction is: $L(\mathbb{R})$ satisfies the Ultrapower Axiom.
    - Some consequences of UA can be shown to hold in $L(\mathbb{R})$.

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- A structure $\mathcal{N}$ is *interpretable* in a structure $\mathcal{M}$ if there is a surjection $f : \mathcal{M}^k \to \mathcal{N}$ such that the f-preimage of a definable subset of $\mathcal{N}$ is definable over $\mathcal{M}$.

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Therefore in actuality, $\delta^1_\omega = (\aleph_{\epsilon_0})^{L(\mathbb{R})}$.

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- (Jackson) $\aleph_{\omega\cdot 2+1}$ *is measurable, but* $\aleph_{\omega\cdot 3+1}$ *is singular.*

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Actually the first theorem can be proved using the second.

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If P and Q are transitive models of ZFC, $j : P \to Q$ is an *ultrapower embedding* if there is some $U \in P$ such that $Q = (M_U)^P$ and $j = (j_U)^P$.

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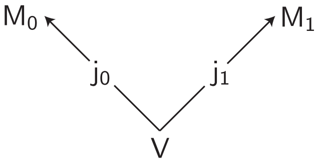

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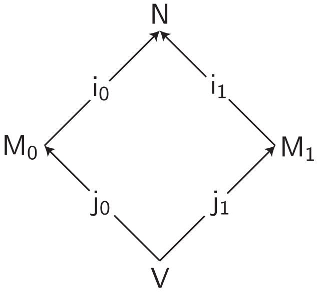

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- UA is equivalent to several natural combinatorial principles.
- Seems to yield an "optimal" theory of ω_1-complete ultrafilters (in the context of the Axiom of Choice).

The Rudin-Frolík order

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- *In fact, an ω_1-complete ultrafilter can have only finitely many Rudin-Frolík predecessors up to equivalence.*

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Proposition

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A uniform ultrafilter U on a cardinal κ is *Dodd sound* if the map $i : P(\kappa) \to M_U$ given by $i(A) = j_U(A) \cap [\mathrm{id}]_U$ belongs to M_U.

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An ultrafilter U on a family of nonempty sets $\mathcal{F}$ is *normal* if every choice function on $\mathcal{F}$ is constant on a set in U. If U is normal and $\lambda = \min_{A\in U} |A|$, then M_U is closed under λ-sequences.

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By a theorem of Menas, the least measurable limit of supercompact cardinals is strongly compact but not supercompact, so the corollary cannot be improved.

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- (Martin-Monk) $\leq_L$ *is wellfounded on subsets of $P(\omega)$.*

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Theorem (Kunen)

In $L(\mathbb{R})$, every ultrafilter on an ordinal is ordinal definable.

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Theorem (ZF + DC)

UA_1 *and* UA_2 *are equivalent.*

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Kunen's proof shows that for any ordinal α, there is no elementary $j : V_{\alpha+2} \to V_{\alpha+2}$. So if $j : V_\beta \to V_\beta$ is elementary with critical point κ, $\beta < \lambda + 2$ where

$$\lambda = \sup\{\kappa, j(\kappa), j^2(\kappa), j^3(\kappa), \dots\}$$

because $j(\lambda) = \sup\{j(\kappa), j^2(\kappa), j^3(\kappa), \dots\} = \lambda$.

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Going forward: λ denotes an I_0-cardinal, meaning there is an elementary $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\text{crit}(j) < \lambda$.

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- Every subset of λ^{+} is definable over $H(\lambda^{+})$ from parameters.

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The proof is by induction on λ^+-complete filters ordered by Ketonen reducibility.

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Conjecture

In $L(\mathbb{R})$, for all ordinals δ, every level of $(\beta_{\omega_1}(\delta), \leq_{\Bbbk})$ is finite.

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In $L(\mathbb{R})$, no ultrafilter on an ordinal has infinitely many Rudin-Keisler predecessors.

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Proofs use Steel's fine-structural analysis of $\text{HOD}^{L(\mathbb{R})}$ below $\Theta^{L(\mathbb{R})}$.

Products of ultrafilters

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- Usually, $U \times W$ is not an ultrafilter and $U \ltimes W \neq U \rtimes W$, so all three products are distinct.

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Question

Suppose $U \ltimes W = U \rtimes W$. Must $U \times W$ be an ultrafilter?

- $U \ltimes W = U \rtimes W$ iff the ultrafilter quantifiers commute:

$$\forall^U x\, \forall^W y\, R(x,y) \iff \forall^W y\, \forall^U x\, R(x,y)$$

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- Quantifiers commute iff the associated ultrapowers do:

$$\begin{aligned} U \ltimes W = U \rtimes W &\iff j_U(j_W) = j_W \restriction M_U \\ &\iff j_W(j_U) = j_U \restriction M_W \end{aligned}$$

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Theorem

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- In HOD_x:
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  - (Steel) GCH holds! UA holds!
  - By either of the previous theorems, $\bar{U} \times \bar{W}$ is an ultrafilter.

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  - For some $x \in \mathbb{R}$, $A \in \text{HOD}_x$.
  - $\bar{U} = U \cap \text{HOD}_x$ and $\bar{W} = W \cap \text{HOD}_x$ are in HOD_x.
- In HOD_x:
  - $\bar{U} \ltimes \bar{W} = \bar{U} \rtimes \bar{W}$.
  - (Steel) GCH holds! UA holds!
  - By either of the previous theorems, $\bar{U} \times \bar{W}$ is an ultrafilter.
  - So $\bar{U} \times \bar{W}$ measures A.

Products of ultrafilters, continued

Theorem

In $L(\mathbb{R})$, if U and W are ultrafilters on ordinals, $U \times W$ is an ultrafilter iff $U \ltimes W = U \rtimes W$.

Proof.

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  - Fix $A \subseteq \delta \times \delta$. Must show $U \times W$ measures A.
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  - So $\bar{U} \times \bar{W}$ measures A.
- This implies $U \times W$ measures A. □

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- *Every level of the Ketonen order is finite.*
- *No ultrafilter on an ordinal has infinitely many Rudin-Keisler predecessors.*

Thanks

Thanks!