WELCH GAMES TO LAVER IDEALS

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JOINT WORK WITH

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AND
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This is a nostalgic trip across the early days of Set Theory and Large cardinals

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Think of it as a remix of classical 60's and 70's rock, but using modern technology.

HERE ARE THE ELEMENTS

CARDINALS WITH COMPACTNESS PROPERTIES

- Techniques for proving determinacy: auxiliary games
- Games of transfinite length

STRONG IDEALS

- Precipitous ideals
- Saturated ideals
- *Laver Ideals:* Ideals on a cardinal such that the quotient has a dense closed set

FINE STRUCTURE

IN A MODERN GUISE

FORCING

- Creating non-reflecting stationary sets
- Killing them while preserving other stationary sets
- Laver “lottery style” forcing
- Coding

SO GET OUT YOUR DANCING SHOES AND GRAB A COPY OF KANAMORI-MAGIDOR

BACK TO THE 60'S: EVEN BEFORE THE BEATLES

Hanf, Kiesler and Tarski and others were interested in generalizing first order logic to stronger languages. In particular they were interested in compactness properties of L_κ.

LARGE CARDINALS

- ω
- weakly compact cardinals
- strongly compact cardinals
- supercompact cardinals

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The first three correspond to compactness properties of languages

WEAKLY COMPACT CARDINALS

Kiesler and Tarski proved the following characterization of weakly compact cardinals:

Theorem Let κ be inaccessible. The κ is weakly compact iff whenever

- $\mathcal{B} \subseteq P(\kappa)$ is a κ-complete Boolean subalgebra of size κ
- $U \subseteq \mathcal{B}$ is a κ complete filter

there is a κ-complete ultrafilter V on $\mathcal{B}$ extending U.

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I'm going to call the versions in this paper "Welch games"

WELCH GAMES

Fix a regular cardinal κ and an ordinal γ. Players I and II alternate moves:

I	$\mathcal{A}_0$	$\mathcal{A}_1$	$\dots$	$\mathcal{A}_\alpha$	$\mathcal{A}_{\alpha+1}$	$\dots$
II	U_0	U_1	$\dots$	U_α	$U_{\alpha+1}$	$\dots$

The game continues of some length $\ell \leq \gamma$.

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- $\mathcal{A}_i$ is a κ-complete subalgebra of $\mathcal{P}(\kappa)$ and $|\mathcal{A}_i| = \kappa$
- U_i is a κ-complete, non-principal ultrafilter on $\mathcal{A}_i$
- If $i < j$ then $\mathcal{A}_i \subseteq \mathcal{A}_j$ and $U_i \subseteq U_j$.

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The sequence $\langle \mathcal{A}_\delta : 0 \leq \delta < \ell \leq \gamma \rangle$ is an increasing sequence of κ-complete subalgebras of $\mathcal{P}(\kappa)$ of cardinality κ and $\langle U_\delta : 0 \leq \delta < \ell \rangle$ is sequence of uniform κ-complete filters on $\mathcal{A}_\alpha$ and $\alpha < \alpha'$ implies that $U_\alpha \subseteq U_{\alpha'}$.

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The winning condition. Player II wins if the game continues through all stages below γ.

WELCH GAMES

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γ

- If κ is weakly compact then Player II has a winning strategy in the game of length ω.
- If κ is measurable then Player II has a winning strategy in the game of length 2^κ.

What happens in between?

Nielsen and Welch showed that if Player II has a winning strategy in the game of length $\omega + 1$ then there is an inner model with a measurable cardinal.

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Welch's Question If Player II wins the game of length ω_1 is there a precipitous ideal?

NOTATION

Let $\mathcal{G}_\gamma^W$ denote the Welch game of length γ.

A PROOF THAT DOESN'T WORK

- Collapse 2^κ to have cardinality ω_1 with countable conditions.
- Use player II's strategy to build a countably complete ultrafilter U on $(2^\kappa)^V$
- Then V^κ/U is well-founded
- In V, let $\mathcal{I} = \{X : \|X \in U\| = 0\}$

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You forced an ultrafilter that gives a well-founded ultrapower without adding infinite sequences to V!

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Problem!

U is not generic for $\mathcal{P}(\kappa)/\mathcal{I}$ . . . !

In the early 1980's Laver gave a counterexample to exactly this type of construction.

OUR MAIN RESULT

Theorem Let κ be inaccessible and $2^\kappa = \kappa^+$. Then

1. If II wins the game of length $\omega + 1$ then there is a normal, κ-complete precipitous ideal on κ,

2. Assume that there are no κ^+-saturated ideals on κ, $\omega_1 \leq \gamma \leq \kappa$ and II wins the game of length γ. Then there is a normal, κ-complete ideal $\mathcal{I}$ such that

$$\mathcal{P}(\kappa)/\mathcal{I}$$

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"Laver ideals"

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Are there any models where II wins and kappa isn't measurable?

Let $\mathcal{I}$ be a κ-complete ideal on $\mathcal{P}(\kappa)$ and $\gamma > \omega$ be a regular cardinal. Then $\mathcal{I}$ is γ-*densely treed* if there is a set $D \subseteq \mathcal{I}^+$ such that

1. $(D, \subseteq_{\mathcal{I}})$ is a downward growing tree,
2. D is closed under $\subseteq_{\mathcal{I}}$-decreasing sequences of length less than γ
3. D is dense in $\mathcal{P}(\kappa)/\mathcal{I}$.

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The first theorem says that
winning the game of length gamma gives
gamma-densely treed ideals

A CONVERSE

Theorem Let $\gamma \leq \kappa$ be uncountable regular cardinals and $\mathcal{J}$ be a uniform κ-complete ideal over κ such that $\mathcal{P}(\kappa)/\mathcal{J}$ is (κ^+, ∞)-distributive and $\mathcal{P}(\kappa)/\mathcal{J}$ has a dense γ-closed subset. Then Player II has a winning strategy $\mathcal{S}_\gamma$ in the game $\mathcal{G}^W_\gamma$.

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Proof Sketch Let $D \subseteq \mathcal{P}(\kappa)/\mathcal{I}$ be the dense closed set. Let $\dot{U}$ be a term for the generic ultrafilter on $\mathcal{P}(\kappa)/\mathcal{I}$. Player II builds a decreasing sequence of elements $d_\alpha \in D$. At stage α, player I presents II with $\mathcal{A}_\alpha \subseteq \mathcal{P}(\kappa)$. By distributivity there is a $d_{\alpha+1} \leq d_\alpha$ and a filter $U_\alpha \in V$ such that $d_{\alpha+1} \Vdash \dot{U} \cap \mathcal{A}_\alpha = \check{U}_\alpha$. Player II plays U_α. By the closure of D this strategy goes of length γ. $\dashv$

SO WE HAVE REDUCED THE PROBLEM TO BUILDING IDEALS WITH NICE PROPERTIES

But is it all or nothing?

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If you win the game for some uncountable
gamma do you win
the game of length kappa?

Theorem Assume κ is measurable and $V = L[E]$ is a fine structural extender model. Then there is a generic extension in which κ is inaccessible, carries no saturated ideals (in particular, κ is non-measurable) and for all regular γ with $\omega < \gamma \leq \kappa$ there is a uniform, normal γ-densely treed ideal $\mathcal{J}_\gamma$ on κ that is (κ^+, ∞)-distributive. The ideal $\mathcal{J}_\gamma$ is not γ^+-densely treed.

THE UPSHOT

- If you start in a fine structural $L[E]$ model, you can force to get a collection of ideals $\mathcal{J}_\gamma$ that construct winning strategies $\mathcal{S}_\gamma$ for player II in the games $\mathcal{G}_\gamma^W$.
- If $\gamma \neq \gamma'$ then $\mathcal{S}_\gamma$ is incompatible with $\mathcal{S}_{\gamma'}$.

MORE PRECISE INFORMATION

Theorem Assume κ is a measurable cardinal, $\gamma < \kappa$ is regular uncountable and $V = L[E]$ is a fine structural extender model. Then there is a generic extension in which κ is inaccessible, carries no saturated ideals (in particular, κ is non-measurable) and there is a uniform, normal γ-densely treed ideal $\mathcal{J}_\gamma$ on κ that is (κ^+, ∞)-distributive. The ideal $\mathcal{J}_\gamma$ is not γ^+-densely treed. Moreover, in the generic extension:

(a*) There does not exist any uniform ideal $\mathcal{J}'$ over κ such that $\mathcal{P}(\kappa)/\mathcal{J}'$ has a dense γ'-closed subset whenever $\gamma' > \gamma$.

(b*) Player II does not have any winning strategy in $\mathcal{G}^W_{\gamma'}$ where $\gamma' > \gamma$.

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(b*) Player II does not have any winning strategy in $\mathcal{G}^W_{\gamma'}$ where $\gamma' > \gamma$.

So you can create one strategy at a time of a given desired length.

WHY DOES FINE STRUCTURE CREEP IN?

In the simplest case of our construction we can work in $L[\mu]$. Then the core model has a canonical definable series of square sequences $\square_\infty = \langle \square_\alpha : \alpha \in OR \rangle$ such that if $j : L[\mu] \to M$ is the ultrapower by μ, then $\square_\kappa$ is the κ^{th} member of the j-image of the sequences of squares.

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This allows the definition of canonical sequences of non-reflecting stationary sets that can be used for coding purposes.

The theorems are proved by

- iterating up to κ with Easton support,
- using a "lottery" to choose γ_α's at inaccessibles $\alpha < \kappa$ along the way,
- adding generic non-reflecting stationary sets and killing other stationary sets of cofinality γ ordinals.
- these interact with the canonical non-reflecting stationary sets coding γ_α

HOW DO YOU GET THE RELEVANT IDEAL?

We forced up to κ with $\mathbb{P}$. For the intended γ consider the condition in $j(\mathbb{P})$ that chooses γ at stage κ. Because everything is canonical, the forcing over M is the same as the forcing with the corresponding forcing in V. (The forcing is determined by the canonical square sequences.)

The ideal is the $\mathcal{I}_\gamma = \{X \subseteq \kappa : \|\kappa \in j(X)\| = 0\}$ in the forcing below the choice of γ.

By "duality" the quotient $P(\kappa)/\mathcal{I}_\gamma$ is isomorphic to the Boolean Algebra determined by the forcing at κ. Since the forcing at κ has the relevant properties you are done.

MAIN NEW TECHNICAL DIFFICULTIES

The main new technical difficulties are showing the closure and distributivity of the relevant forcing. Because of the lottery the usual distributivity arguments involve potential "ghost coordinates."

This is the main reason new fine structural techniques are involved. The second is that we are working over more complicated $L[E]$ models.

These results appear in Part 2 of the paper. The innovations are aimed at interations with Easton supports, but the seem to apply to a much broader collection of iterations.

LET'S GO BACK TO PLAYING GAMES

WHAT'S UP WITH THE INACCESSIBLES?

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They turn out to be necessary

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They turn out to be necessary

The next results are naive and easy.

Throwback math.

WHAT'S UP WITH THE INACCESSIBLES?

Proposition Suppose that κ is an infinite cardinal and either

- a singular strong limit cardinal or
- for some $\gamma < \kappa$, $2^\gamma > \kappa$ but for all $\gamma' < \gamma$, $2^{\gamma'} < \kappa$.

Then there is no Boolean subalgebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{A}| = \kappa$, $\mathcal{A}$ is κ-complete.

James Cummings was involved in discussions of part of this.

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Theorem Suppose that κ is an accessible infinite cardinal. Then either:

1. There is no κ-complete subalgebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = \kappa$ or
2. There is a κ-complete subalgebra $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = \kappa$ but every κ-complete ultrafilter U on $\mathcal{A}$ is principal

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Welch games can't be played
on cardinals that are not inaccessible

WHERE DO THE IDEALS COME FROM?

- They can't come directly from elementary embeddings.
- They have to come somehow from the filters being played.

HOPELESS IDEALS

Let $\mathcal{S}$ be a winning strategy for II in the Welch Game of length γ and $P = \langle (A_\alpha, U_\alpha) : \alpha < \beta \rangle$ be a partial play according to $\mathcal{S}$. The conditional *Hopeless Ideal* conditioned on P, $\mathcal{I}(P, S)$ is

$$\{X \subseteq \kappa : \text{for no extension of } P \text{ is } X \text{ in the filter played by II}\}.$$

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- A set is hopeless given history P if it never makes it into a filter in a play that extends P
- A set is *unconditionally hopeless* if it is in $\mathcal{I}(\emptyset, \mathcal{S})$.
- Hopeless ideals are κ-complete.

GOAL

Start with a strategy $\mathcal{S}$ in the Welch Game and build another strategy $\mathcal{S}^*$ in a different game such that

- $\mathcal{S}^*$ plays sets X_i diagonalizing normal ideals on $\mathcal{A}_i$,
- All of the conditional hopeless ideals for $\mathcal{S}^*$ are equal to the unconditionally hopeless ideals,
- The plays can be organized into a tree,

With these properties, the unconditionally hopeless ideal has a dense γ-closed tree.

THE INTERMEDIATE GAMES

We build a series of auxiliary games $\mathcal{G}_0, \mathcal{G}_1^-, \mathcal{G}_1, \mathcal{G}_2$ and a sequence of strategies starting with $\mathcal{S}_\gamma$ with later strategies simulating earlier games.

One particularly tricky transition is to require player II to play *normal* ideals. It looks simple: if you have a κ-complete ideal U you can take its normal derivative U^*. HOWEVER: it is **not** true that:

$$U \subseteq V \Longrightarrow U^* \subseteq V^*.$$

But the failures form a well-founded tree.

UPBEAT FINALE

Working on these Welch games was really *fun*. Lots of challenges came up, but the tools were well developed to solve them.

REMEMBER THE CONTEXT

- This is the modern remix of early 60's results of Kiesler and Tarski.
- What are the remaining open problems?
- Are they as tractable as the weakly compact vs. measurable questions?

OPEN PROBLEMS

Basically everything is open. This is new territory. A chance to be very creative.

- Are there analogous results for strongly compact cardinals?

  You can't normalize strongly compact ultrafilters so the proofs have to change.

- What is the analogue for successor cardinals?

- What about extenders?

- What happens in the games if you play ideals themselves?

- What's up with the saturated ideal hypothesis? $2^\kappa = \kappa^+$?

- Your question here ...

THANK YOU!