Universal $\aleph_2$-Aronszajn trees

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⋇ The main question for this talk is the existence of a maximal (i.e. universal) element in these classes.

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⋇ Under *MA* $+ \neg$*CH*, $\mathcal{A}_{\aleph_1}$ is cofinal in $\mathcal{T}_{\aleph_1}$ (Dž.+ Shelah 2021).

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Question: What about universality ? For example, can we generalise the methods from $\aleph_1$?

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Consistent with any reasonable size of $2^{\aleph_1} > \aleph_2$.

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⋇ Unfortunately, as noticed by us and confirmed by Justin, the proof has a gap and moreover, the main Lemma (Lemma 18) is provably wrong modulo the previous known facts and arguments of the thesis.

⋇ So the question of a model in which CH holds and there are no universal $\aleph_2$-Aronszajn trees was left open.

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Theorem

For every tree $T \in \mathcal{A}_{\aleph_1}$, there is a ccc forcing which adds a tree T' in $\mathcal{A}_{\aleph_1}$ not weakly embeddable into T.

Definition

If $T \in \mathcal{A}_{\aleph_1}$, we define a forcing notion $\mathbb{Q} = \mathbb{Q}(T)$ to consist of all $p = (u^p, v^p, <_p, c^p)$ such that:

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2. if $\alpha \in v^p$ then there is $\beta \in u^p$ with $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$,
3. $<_p$ is a tree-like partial order on u^p such that $\alpha <_p \beta$ implies $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ and which fixes $\alpha \cap_{<_p} \beta \in u^p$ for every two different elements α, β of u^p and fixes the root $\langle\rangle$ of u^p,

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4. $c^p : \bigcup_{\delta \text{ limit } <\omega_1} \mathrm{lev}_\delta(\mathrm{u^p}) \times \mathrm{lev}_\delta(\mathrm{v^p}) \to \omega$ s.t.: if $c(x_1, y_1) = c(x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$, then $\alpha(x_1, y_1) \neq \alpha(x_2, y_2)$, $x_1 \perp_{u^p} x_2$, $y_1 \perp_{v^p} y_2$ and

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The order $p \leq q$ on $\mathbb{Q}$ is
$u^p \subseteq u^q, v^p \subseteq v^q, <_p \subseteq <_q, c^p \subseteq c^q$ and if $p \leq q$, then the intersection and the root given by $<_p$ are preserved in $<_q$.

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Theorem (BMR Lemma)

If T is tree of height and cardinality ω_1 with no uncountable branches and W is an uncountable collection of finite pairwise disjoint subsets of T, then there exist $s, s' \in W$ such that any $x \in s$ is incomparable with any $y \in s'$.

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Question: Can we generalise this theorem to $\aleph_2$?

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A *weak ascent path* in a tree T of height ω_2 is a a sequence $\langle \bar{x}^\alpha : \alpha < \omega_2 \rangle$ where:

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In fact, the same can be said about wide $\aleph_2$-Aronszajn trees.

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A still open Question: Is there a model with a universal Aronszajn tree ?

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Let $\leq^*$ stand for weak embeddings that preserve the splitting level and $\mathcal{T}_\lambda^\kappa$ for trees of size κ with no branches of length λ^+. Then the universality number of $(\mathcal{T}_\lambda^\kappa, \leq^*)$ is at least 2^λ.