A lattice theoretical interpretation of generalized deep holes

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Based on a joint work with Masahiko Miyamoto (University of Tsukuba)

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Motivation

Main Aim

A **combinatorial approach** towards the classification of strongly regular holomorphic vertex operator algebras (VOAs) (of CFT type) of central charge $c = 24$.

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Holomorphic VOA of $c = 24$	$\longleftrightarrow$	Pairs (N, τ), N Niemeier lattice $\tau \in O(N)$ with +ve frame shape + some conditions

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Theorem

Each potential Lie algebra on Schellekens' list is realized by a strongly regular holomorphic VOA of central charge 24 *and this VOA is uniquely determined by its* V_1*-structure if* $V_1 \neq \{0\}$.

$X_{n,k}$ denotes a Lie algebra of type X_n and the level is k; $N = \dim V_1$.

N	algebra	rank	N	algebra	rank
0	$\emptyset$	0	24	$U(1)^{24}$	24
36	$C_{4,10}$	4	36	$A_{2,6}D_{4,12}$	6
36	${A_{1,4}}^{12}$	12	48	$A_{6,7}$	6
48	${A_{4,5}}^{2}$	8	48	${A_{2,3}}^{6}$	12
48	$A_{1,2}D_{5,8}$	6	48	$A_{1,2}A_{5,6}C_{2,3}$	8
48	$A_{1,2}{A_{3,4}}^{3}$	10	48	${A_{1,2}}^{16}$	16
60	${C_{2,2}}^{6}$	12	60	$A_{2,2}F_{4,6}$	6
60	${A_{2,2}}^{4}D_{4,4}$	12	72	$A_{1,1}C_{5,3}G_{2,2}$	8
72	${A_{1,1}}^{2}D_{6,5}$	8	72	${A_{1,1}}^{2}C_{3,2}D_{5,4}$	10
72	${A_{1,1}}^{3}A_{7,4}$	10	72	${A_{1,1}}^{3}A_{5,3}D_{4,3}$	12
72	${A_{1,1}}^{4}{A_{3,2}}^{4}$	16	72	${A_{1,1}}^{24}$	24
84	${B_{3,2}}^{4}$	12	84	${A_{4,2}}^{2}C_{4,2}$	12

N	algebra	rank	N	algebra	rank
96	$C_{2,1}{}^4 D_{4,2}{}^2$	16	96	$A_{2,1} C_{2,1} E_{6,4}$	10
96	$A_{2,1}{}^2 A_{8,3}$	12	96	$A_{2,1}{}^2 A_{5,2}{}^2 C_{2,1}$	16
96	$A_{2,1}{}^{12}$	24	108	$B_{4,2}{}^3$	12
120	$E_{6,3} G_{2,1}{}^3$	12	120	$A_{3,1} D_{7,3} G_{2,1}$	12
120	$A_{3,1} C_{7,2}$	10	120	$A_{3,1} A_{7,2} C_{3,1}{}^2$	16
120	$A_{3,1}{}^2 D_{5,2}{}^2$	16	120	$A_{3,1}{}^8$	24
132	$A_{8,2} F_{4,2}$	12	144	$C_{4,1}{}^4$	16
144	$B_{3,1}{}^2 C_{4,1} D_{6,2}$	16	144	$A_{4,1} A_{9,2} B_{3,1}$	16
144	$A_{4,1}{}^6$	24	156	$B_{6,2}{}^2$	12
168	$D_{4,1}{}^6$	24	168	$A_{5,1} E_{7,3}$	12
168	$A_{5,1} C_{5,1} E_{6,2}$	16	168	$A_{5,1}{}^4 D_{4,1}$	24
192	$B_{4,1} C_{6,1}{}^2$	16	192	$B_{4,1}{}^2 D_{8,2}$	16
192	$A_{6,1}{}^4$	24	216	$A_{7,1} D_{9,2}$	16
216	$A_{7,1}{}^2 D_{5,1}{}^2$	24	240	$C_{8,1} F_{4,1}{}^2$	16
240	$B_{5,1} E_{7,2} F_{4,1}$	16	240	$A_{8,1}{}^3$	24

N	algebra	rank	N	algebra	rank
264	${D_{6,1}}^4$	24	264	${A_{9,1}}^2 D_{6,1}$	24
288	$B_{6,1} C_{10,1}$	16	300	$B_{12,2}$	12
312	${E_{6,1}}^4$	24	312	$A_{11,1} D_{7,1} E_{6,1}$	24
336	${A_{12,1}}^2$	24	360	${D_{8,1}}^3$	24
384	$B_{8,1} E_{8,2}$	16	408	$A_{15,1} D_{9,1}$	24
456	$D_{10,1} {E_{7,1}}^2$	24	456	$A_{17,1} E_{7,1}$	24
552	${D_{12,1}}^2$	24	624	$A_{24,1}$	24
744	${E_{8,1}}^3$	24	744	$D_{16,1} E_{8,1}$	24
1128	$D_{24,1}$	24			

Main techniques:

Proposition (Dong and Mason (2004))

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(2) Suppose $V_1 \cong \oplus_{j=1}^{t} \mathcal{G}_{j,k_j}$ *is semisimple, where* $\mathcal{G}_{j,k_j}$ *are simple Lie algebras with the dual Coxeter number* $h_j^\vee$ *and level* k_j *for* $j = 1, \dots, t$*. Then*

$$\frac{h_j^\vee}{k_j} = \frac{\dim V_1 - 24}{24}. \tag{1}$$

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$\mathbb{Z}_n$-orbifold construction (cf. van Ekeren-Möller-Scheithauer)

- V: hol. VOA, $g \in \mathrm{Aut}\, V$ s.t. g has finite order.
- $V^g = \{v \in V \mid g(v) = v\}$: subVOA of V.
- $V[g^i]$: irreducible g^i-twisted V-module [Dong-Li-Mason '00].
- $\tilde{V} := V^g \oplus \bigoplus_{i=1}^{|g|-1} V[g^i](0)$: V^g-module.
- (Under some assumptions), $\tilde{V}$ is a holomorphic VOA.

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$$h = \begin{cases} 1 & \text{on } V^g \\ \exp(2k\pi\sqrt{-1}/r) & \text{on } V[g^k](0) \end{cases}$$

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We can apply the $\mathbb{Z}_r$-orbifold construction to $\tilde{V}_g$ and h, and the resulting VOA is V, i.e., $V = \widetilde{(\tilde{V}_g)}_h$.

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- Dimension formula (Montague, Möller- Scheithauer):

$$\dim(\widetilde{V}_g)_1 = 24 + \sum_{m|n} c_n(m) \dim(V_1^{g^m}) - R(g),$$

where the rest term $R(g)$ is non-negative.

Direct construction

Let V be a holomorphic VOA of central charge 24.
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Define an automorphism $\sigma = \exp(2\pi i u(0)) \in \mathrm{Aut}(V)$ and consider the VOA $\tilde{V}_\sigma$ obtained by orbifold construction associated with V and σ.

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There is also a more elementary proof by Chigira-L-Miyamoto, which uses the property of the Leech lattice.

Corollary

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$$V \cong (\widetilde{V_\Lambda})_g, \text{ where } g = \hat{\tau} \exp(2\pi i \beta(0)) \in \mathrm{Aut}(V_\Lambda), \tau \in O(\Lambda).$$

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The automorphism g is obtained by the reversed orbifold construction associated with the construction from
V and $\sigma = \exp(2\pi i u(0)) \in \mathrm{Aut}(V)$, where $u = \sum_{j=1}^{t} \frac{1}{h_j^\vee}\rho_j$.

Such a g is very special and is a generalized deep hole

Generalized deep hole (Möller- Scheithauer)

Dimension formula: $\dim(\widetilde{V}_g)_1 = 24 + \sum_{m|n} c_n(m) \dim(V_1^{g^m}) - R(g)$, where the rest term $R(g)$ is non-negative.

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Möller and Scheithauer called an automorphism $g \in Aut(V_\Lambda)$ a generalized deep hole if

1. $\widetilde{(V_\Lambda)}_g$ is a VOA;
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Theorem (Möller and Scheithauer)

The cyclic orbifold construction $g \mapsto (\widetilde{V_\Lambda})_g$ defines a bijection between the algebraic conjugacy classes of generalized deep holes g in $\mathrm{Aut}(V_\Lambda)$ with $\mathrm{rank}(V_\Lambda^g)_1 > 0$ and the isomorphism classes of strongly regular holomorphic VOAs V of central charge 24 with $V_1 \neq \{0\}$.

Note $V > (V_\Lambda)^g > V_K \otimes V_{\Lambda_\tau}^\tau$, where $K = \{\alpha \in \Lambda^\tau \mid \langle \alpha, \beta \rangle \in \mathbb{Z}\}$

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Then τ belongs to one of the following conjugacy classes

$$\{1A, 2A, 2C, 3B, 4C, 5B, 6E, 6G, 7B, 8E, 10F\}$$

That τ belongs to one the 11 conjugacy classes was first observed by Höhn.

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- There are similar conditions for $\hat{\tau}^i$ - twisted modules for any i.

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For τ in these 10 classes, there is an isometry $\varphi_\tau : \sqrt{\ell}(\Lambda^\tau)^* \to \Lambda^\tau$, where ℓ is the level of Λ^τ, i.e, the smallest positive integer such that $\sqrt{\ell}(\Lambda^\tau)^*$ is an even lattice.

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Consider the Neighbor lattice

$$N = \Lambda^{[\tilde{\beta}]} = \mathrm{Span}\{\Lambda_{\tilde{\beta}}, \tilde{\beta}\},$$

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Theorem

Let $N = \Lambda^{[\tilde{\beta}]} = \Lambda_{\tilde{\beta}} + \mathbb{Z}\tilde{\beta}$. Then φ induces an isometry from $\sqrt{\ell}L^*$ to N^τ. In particular, we have $N^\tau \cong \sqrt{\ell}L^*$.

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$N = \Lambda^{[\varphi(\sqrt{\ell}\beta)]} = \mathrm{Span}\{\Lambda_{\varphi(\sqrt{\ell}\beta)}, \varphi(\sqrt{\ell}\beta)\}$ *is a Niemeier lattice and* $N \neq \Lambda$.

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Let $\tilde{\beta} = \sqrt{\ell}\varphi(\beta)$. *Then* $\tilde{\beta}$ *is a deep hole of the Leech lattice. We may also choose* β *such that* $\langle \tilde{\beta}, \tilde{\beta} \rangle = 2$.

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Let $\tilde{\beta} = \sqrt{\ell}\varphi(\beta)$. Then $\tilde{\beta}$ is a deep hole of the Leech lattice. We may also choose β such that $\langle \tilde{\beta}, \tilde{\beta} \rangle = 2$.

Consequence: The Coxeter number h of $N = \Lambda^{[\varphi(\sqrt{\ell}\beta)]}$ is $|g|$ and $|\tau|$ divides $|g| = h$.

Ideas about the proof

Since τ fixes $\tilde{\beta}$, we have

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$$R = \bigoplus_{m_i \| \tau|, m_i \neq 1} A_{m_i - 1}^{a_i}$$

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3. N_τ / R is cyclic and has order $|\tau|$.
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(C1) $\tau \in \mathcal{P}_0$ and $\tilde{\beta}$ is a τ-invariant deep hole of Leech lattice Λ with $\langle \tilde{\beta}, \tilde{\beta} \rangle = 2$;

(C2) the Coxeter number h of $N = \Lambda_{\tilde{\beta}} + \mathbb{Z}\tilde{\beta}$ is divisible by $|\tau|$;

(C3) N_τ contains a full rank sublattice $R = \bigoplus_{m_i||\tau|, m_i \neq 1} A_{m_i-1}^{a_i}$ if the frame shape of τ is $\prod m_i^{a_i}$ and N_τ/R is cyclic of order $|\tau|$.

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Set $\mathcal{T}$ be the set of pairs satisfying the conditions (C1) to (C3).
Define an equivalent relation $\sim$ on $\mathcal{T}$ as follows:
$(\tau, \tilde{\beta}) \sim (\tau', \tilde{\beta}')$ if and only if

1. $\tilde{\beta}$ and $\tilde{\beta}'$ are equivalent deep holes of the Leech lattice Λ, i.e., there are $\sigma \in O(\Lambda)$ and $\lambda \in \Lambda$ such that $\tilde{\beta}' = \sigma(\tilde{\beta} - \lambda)$;
2. τ is conjugate to $\sigma^{-1}\tau'\sigma$ in $O(N)$.

Note that τ and τ' are conjugate in $O(\Lambda)$ since they have the same frame shape by (2).

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Let $(\tau, \tilde{\beta}) \in \mathcal{T}$. Set $\beta = \frac{1}{\sqrt{\ell}}\varphi^{-1}(\tilde{\beta})$ and define

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Then one obtain a holomorphic VOA $V = V^{[\tilde{g}]}$ by an orbifold construction from V_Λ and $\tilde{g}$.

Need to show: If $(\tau, \tilde{\beta}) \sim (\tau', \tilde{\beta}')$, then they define isomorphic VOAs.

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Let h be the Coxeter number of $N = \Lambda_{\tilde{\beta}} + \mathbb{Z}\tilde{\beta}$. Then $|\tilde{g}| = h$.

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That means we can recover the pair $(\tau, \tilde{\beta})$ and a Niemeier lattice $N = \Lambda_{\tilde{\beta}} + \mathbb{Z}\tilde{\beta}$. $\tilde{\alpha}$ is obtained by modifying $\alpha = \sqrt{\ell}\varphi^{-1}(\pi(\rho)/h)$.

Orbit diagrams and Lie algebra structures of V_1

Recall: $\tilde{\beta} = \varphi(\sqrt{\ell}\beta)$ is a deep hole and $N = \cup_{i=0}^{h-1}(-k\tilde{\beta} + \Lambda_{\tilde{\beta}}) \not\cong \Lambda$. The Coxeter number $h = n = \mathrm{LCM}(r_i h_i^\vee)$ and $N^\tau \cong \sqrt{\ell}L^*$.

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$\tilde{\beta}$ is a deep hole $\implies$ R_1 is a disjoint union of the affine diagrams associated with N and τ acts on R_1.
By the choice, $\tau \in \mathrm{Weyl}(R)$ and preserves irreducible components of $R(N)$.

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We consider the quotient diagram as follows:

1. identify an orbit of nodes as one node and two nodes are connected if the nodes in the corresponding orbits are connected.
2. By removing the node associated with the extended node, one obtain a usual Dynkin diagram.

Diagram automorphisms of affine diagrams

Type	A_n	D_{2k}	D_{2k}	D_{2k+1}	D_{2k+1}	E_6	E_7
Root subsystem	$(A_{\frac{n+1}{k}-1})^k$	A_1^k	A_1^2	$A_3 A_1^{k-1}$	A_1^2	A_2^2	A_1^3
Frame Shape	$1^{-1}(\frac{n+1}{k})^k$	2^k	$1^{2k-4}2^2$	$1^{-1}2^{k-1}4$	$1^{2k-3}2^2$	3^2	$1^1 2^3$
Quotient diagram	A_{k-1}	B_k	C_{2k-2}	C_{k-1}	C_{2k-1}	G_2	F_4
Fixed sublattice	$\sqrt{\frac{n+1}{k}} A_{k-1}$	A_1^k	D_{2k-2}	A_1^{k-1}	D_{2k-1}	A_2	D_4
Fixed simple roots	$\emptyset$	A_1	A_{2k-3}	$\emptyset$	A_{2k-2}	A_1	A_2

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Therefore, the level k_j of a simple Lie subalgebra $\mathcal{G}_j$ is given by $k_j = \ell h_j^\vee / h$ for any j.
Note that the short roots of $\mathcal{G}_i$ correspond to an irreducible (connected) component S_i of N_2^τ.
Moreover, $S_i \cap R_1$ corresponds to the simple short roots of $\mathcal{G}_i$.
Therefore, S_i and $S_i \cap R_1$ determines the type of $\mathcal{G}_i$ uniquely.

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This notion of generalized hole diagrams essentially corresponds to the diagram associated with simple short roots of the full components (i.e., elements in $R_1 \cap N_2^{\tau}$).

Possible pairs for (N, τ)

Case: $\tau \in 2A \quad (1^8 2^8)$. $N_\tau = \mathrm{Span}_{\mathbb{Z}}\{A_1^8, \frac{1}{2}(\alpha_1 + \cdots + \alpha_8)\}$.
The vector $v = \frac{1}{2}(\alpha_1 + \cdots + \alpha_8)$ corresponds to a codeword $c \in N/R$, i.e, $v \in R^*$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_1^{24}	$(1^8, 0^8)$	$A_1^8 \hookrightarrow A_1^8$	A_1^{16}	$A_{1,2}^{16}$
A_3^8	(22022000)	$(A_1^2)^4 \hookrightarrow A_3^4$	$A_3^4(\sqrt{2}A_1)^4$	$A_{3,2}^4 A_{1,1}^4$
D_4^6	(233200)	$(A_1^2)^4 \hookrightarrow D_4^4$	$D_4^2 C_2^4$	$D_{4,2}^2 C_{2,1}^4$
$A_5^4 D_4$	(3300|1)	$(A_1^3)^2 + A_1^2 \hookrightarrow A_5^2 + D_4$	$A_5^2 C_2(\sqrt{2}A_2)^2$	$A_{5,2}^2 C_{2,1} A_{2,1}^2$
$A_7^2 D_5^2$	(44|00)	$(A_1^4)^2 \hookrightarrow A_7^2$	$D_5^2(\sqrt{2}A_3)^2$	$D_{5,2}^2 A_{3,1}^2$
$A_7^2 D_5^2$	(20|33)	$(A_1^4) + (A_1^2)^2 \hookrightarrow A_7 + D_5^2$	$A_7 C_3^2(\sqrt{2}A_3)$	$A_{7,2} C_{3,1}^2 A_{3,1}$
D_6^4	(2222)	$(A_1^2)^4 \hookrightarrow D_6^4$	C_4^4	$C_{4,1}^4$
D_6^4	(1230)	$(A_1^2) + (A_1^3)^2 \hookrightarrow D_6 + D_6^2$	$D_6 C_4 B_3^2$	$D_{6,2} C_{4,1} B_{3,1}^2$
$A_9^2 D_6$	(05|3)	$(A_1^5) + (A_1^3) \hookrightarrow A_9 + D_6$	$A_9(\sqrt{2}A_4)B_3$	$A_{9,2} A_{4,1} B_{3,1}$
$A_{11} D_7 E_6$	(620)	$A_1^6 + A_1^2 \hookrightarrow A_{11} + D_7$	$E_6 C_5(\sqrt{2}A_5)$	$E_{6,2} C_{5,1} A_{5,1}$
D_8^3	(033)	$(A_1^4)^2 \hookrightarrow D_8^2$	$D_8 B_4^2$	$D_{8,2} B_{4,1}^2$
D_8^3	(221)	$(A_1^2)^2 + A_1^4 \hookrightarrow D_8^2 + D_8$	$C_6^2 B_4$	$C_{6,1}^2 B_{4,1}$
$A_{15} D_9$	(80)	$A_1^8 \hookrightarrow A_{15}$	$D_9(\sqrt{2}A_7)$	$D_{9,2} A_{7,1}$
$E_7^2 D_{10}$	(11|2)	$(A_1^3)^2 + A_1^2 \hookrightarrow E_7^2 + D_{10}$	$C_8 F_4^2$	$C_{8,1} F_{4,1}^2$
$E_7^2 D_{10}$	(01|1)	$A_1^3 + A_1^5 \hookrightarrow E_7 + D_{10}$	$E_7 B_5 F_4$	$E_{7,2} B_{5,1} F_{4,1}$
D_{12}^2	(21)	$A_1^2 + A_1^6 \hookrightarrow D_{12} + D_{12}$	$C_{10} B_6$	$C_{10,1} B_{6,1}$
$E_8 D_{16}$	(01)	$A_1^8 \hookrightarrow D_{16}$	$B_8 E_8$	$B_{8,1} E_{8,2}$

Case: $\tau \in 3B \quad (1^6 3^6)$. $R(N_\tau) \cong A_2^6$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_2^{12}	$(1^6 0^6)$	$A_2^6 \hookrightarrow A_2^6$	A_2^6	$A_{2,3}^6$
$A_5^4 D_4$	$(2220\|0)$	$(A_2^2)^3 \hookrightarrow A_5^3$	$A_5 D_4 (\sqrt{3}A_1)^3$	$A_{5,3} D_{4,3} A_{1,1}^3$
A_8^3	(630)	$(A_2^3)^2 \hookrightarrow A_8^2$	$A_8 (\sqrt{3}A_2)^2$	$A_{8,3} A_{2,1}^2$
E_6^4	(0111)	$(A_2^3)^3 \hookrightarrow E_6^3$	$E_6 G_2^3$	$E_{6,3} {G_{2,1}}^3$
$A_{11} D_7 E_6$	(401)	$A_2^4 + A_2^2 \hookrightarrow A_{11} E_6$	$D_7 (\sqrt{3}A_3) G_2$	$D_{7,3} A_{3,1} G_{2,1}$
$A_{17} E_7$	(60)	$A_2^6 \hookrightarrow A_{17}$	$E_7 (\sqrt{3}A_5)$	$E_{7,3} A_{5,1}$

Case: $\tau \in 5B \quad (1^8 4^4)$. $R(N_\tau) \cong A_4^4$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_4^6	(123400)	$A_4^4 \hookrightarrow A_4^4$	A_4^2	$A_{4,5}^2$
$A_9^2 D_6$	$(24\|0)$	$(A_4^2)^2 \hookrightarrow A_9^2$	$D_6 (\sqrt{5}A_1^2)$	$D_{6,5} A_{1,1}^2$

Case: $\tau \in 7B \quad (1^3 7^3)$. $R(N_\tau) \cong A_6^3$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_6^4	(0124)	$A_6^3 \hookrightarrow A_6^3$	A_6	$A_{6,7}$

Case: $\tau \in 2C \quad (2^{12})$. $R(N_\tau) \cong A_1^{12}$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_1^{24}	$(1^{12}0^{12})$	$A_1^{12} \hookrightarrow A_1^{12}$	A_1^{12}	$A_{1,4}^{12}$
D_4^6	(111111)	$(A_1^2)^6 \hookrightarrow D_4^6$	B_2^6	$B_{2,2}^6$
D_6^4	(2222)	$(A_1^3)^4 \hookrightarrow D_6^4$	B_3^4	$B_{3,2}^4$
D_8^3	(111)	$(A_1^4)^3 \hookrightarrow D_8^3$	B_4^3	$B_{4,2}^3$
D_{12}^2	(11)	$(A_1^6)^2 \hookrightarrow D_{12}^2$	B_2^6	$B_{6,2}^2$
D_{24}	(1)	$A_1^{12} \hookrightarrow D_{24}$	B_{12}	$B_{12,2}$
$A_5^4 D_4$	(3333|0)	$(A_1^3)^4 \hookrightarrow A_5^4$	$D_4\sqrt{2}A_2^4$	$D_{4,4}A_{2,2}^4$
$A_9^2 D_6$	(55|2)	$(A_1^5)^2 + A_1^2 \hookrightarrow A_9^2 + D_6$	$C_4\sqrt{2}A_4^2$	$C_{4,2}A_{4,2}^2$
$A_{17}E_7$	(9|1)	$A_1^9 + A_1^3 \hookrightarrow A_{17} + E_7$	$F_4\sqrt{2}A_8$	$A_{8,2}F_{4,2}$

Case: $\tau \in 4C \quad (1^4 2^2 4^4)$. $R(N_\tau) \cong A_1^2 A_3^4$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
A_3^8	(32001011)	$A_3^4 + A_1^2 \hookrightarrow A_3^4 + A_3$	$A_3^3\sqrt{2}A_1$	$A_{3,4}^3 A_{1,2}$
$A_7^2 D_5^2$	(02|13)	$A_3^2 + (A_3A_1)^2 \hookrightarrow A_7 + D_5^2$	$A_7 2A_1 A_1^2$	$A_{7,4}A_{1,1}^3$
$A_7^2 D_5^2$	(22|20)	$A_3^2 + A_3^2 + A_1^2 \hookrightarrow A_7 + A_7 + D_5$	$D_5 C_3 2A_1^2$	$D_{5,4}C_{3,2}A_{1,1}^2$
$A_{11}D_7E_6$	(310)	$A_3^3 + A_3A_1^2 \hookrightarrow A_{11} + D_7$	$E_6 B_2 2A_2$	$E_{6,4}B_{2,1}A_{2,1}$
$A_{15}D_9$	(4|2)	$A_3^4 + A_1^2 \hookrightarrow A_{15} + D_9$	$C_7 2A_3$	$C_{7,2}A_{3,1}$

Case: $\tau \in 6E \quad (1^2 2^2 3^2 6^2)$. $R(N_\tau) \cong A_1^2 A_2^2 A_5^2$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_5^4 D_4$	(0255|1)	$A_5^2 + A_2^2 + A_1^2 \hookrightarrow A_5^2 + A_5 + D_4$	$A_5\sqrt{3}A_1 B_2$	$A_{5,6}B_{2,3}A_{1,1}$
$A_{11}D_7E_6$	(222)	$A_5^2 + A_1^2 + A_2^2 \hookrightarrow A_{11} + D_7 + E_6$	$\sqrt{6}A_1 C_5 G_2$	$C_{5,3}G_{2,2}A_{1,1}$

Case: $\tau \in 8E \quad (1^2 2^1 4^1 8^2)$. $R(N_\tau) \cong A_1 A_3 A_7^2$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_7^2 D_5^2$	$(37\|10)$	$A_7^2 + A_3 A_1 \hookrightarrow A_7^2 + D_5$	$D_5 A_1$	$D_{5,8} A_{1,2}$

Case: $\tau \in 6G$

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_5^4 D_4$	(31110)	$A_5^3 + A_1^3 \hookrightarrow A_5^3 + A_5$	$D_4 \sqrt{2} A_2$	$D_{4,12} A_{2,6}$
$A_{17} E_7$	$(3\|1)$	$A_5^3 + A_1^3 \hookrightarrow A_{17} + E_7$	$F_4 \sqrt{6} A_2$	$F_{4,6} A_{2,2}$

Case: $\tau \in 10F$

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_9^2 D_6$	$(79\|2)$	$A_9^2 + A_1^2 \hookrightarrow A_9^2 + D_6$	C_4	$C_{4,10}$

Remark

Since τ corresponds to an isometry associated with a codeword c of the glue code N/R,

Case: $\tau \in 8E \quad (1^2 2^1 4^1 8^2)$. $R(N_\tau) \cong A_1 A_3 A_7^2$.

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_7^2 D_5^2$	$(37\|10)$	$A_7^2 + A_3 A_1 \hookrightarrow A_7^2 + D_5$	$D_5 A_1$	$D_{5,8} A_{1,2}$

Case: $\tau \in 6G$

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_5^4 D_4$	(31110)	$A_5^3 + A_1^3 \hookrightarrow A_5^3 + A_5$	$D_4 \sqrt{2} A_2$	$D_{4,12} A_{2,6}$
$A_{17} E_7$	$(3\|1)$	$A_5^3 + A_1^3 \hookrightarrow A_{17} + E_7$	$F_4 \sqrt{6} A_2$	$F_{4,6} A_{2,2}$

Case: $\tau \in 10F$

Type	Codeword c	Embedding	$R(N^\tau)$	V_1
$A_9^2 D_6$	$(79\|2)$	$A_9^2 + A_1^2 \hookrightarrow A_9^2 + D_6$	C_4	$C_{4,10}$

Remark

Since τ corresponds to an isometry associated with a codeword c of the glue code N/R, we can recover the same information as in [Höhn, Table 3]. In particular, there are exactly 46 possible Lie algebra structures for V_1 if $0 < \mathrm{rank}(V_1) < 24$. This gives an alternative proof for the Schellekens list.

Thank you.