A De Giorgi Argument for L^∞ Solution to the Boltzmann Equation without Angular Cutoff

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Outline of the talk

(Boltzmann equation without angular cutoff)

- Setting and the problem
- Well-posedness with algebraic decay tail in L^2 framework
- Well-posedness with algebraic decay tail in L^∞ framework

(collaborators)

- Ricardo Alonso, Texas A&M University at Qatar
- Yoshinori Morimoto, Kyoto University
- Weiran Sun, Simon Fraser University

Boltzmann equation for non-equilibrium gas is about the time evolution of

$$F = F(t,x,v) \quad t \in \mathbb{R}^+, x \in \mathbb{T}^3, v \in \mathbb{R}^3,$$

which stands for the number density function of particles having position x and velocity v at time t:

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = Q(F,F)$$

Here Q, *the collision operator* describes the binary elastic collision of molecules.

$$Q(G,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{G(v_*')F(v') - G(v_*)F(v)\right\} d\sigma dv_*,$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ \ v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,\ \ \sigma\in\mathbb{S}^2,$$

satisfying the conservation of momentum and energy.

Angular cutoff and non-cutoff

- Cross-section in this talk:

$$B(v-v_*,\theta) \sim |v-v_*|^\gamma \theta^{-2-2s}, \quad 0 \le \theta \le \frac{\pi}{2},$$

  with $\gamma > 0$ and $0 < s < 1$. Motivated by inverse power law $U(r) \sim r^{-\rho}$ with $\gamma = 1 - \frac{4}{\rho}$ and $s = \frac{1}{\rho}$, $\rho > 1$.

- A lot of mathematical theories have been established under the Grad's angular cutoff assumption by removing the non-integrable angular singularity near grazing:

$$\theta \sim 0.$$

(Searching for function spaces with minimum spatial regularity)

- $L^2 - L^\infty$ approach for bounded domain for angular cutoff, Yan ('10)
- $H^s_x(L^2_v)$ with $s > 3/2$, local existence by Alexandre-Morimoto-Ukai-Xu-Y. ('13)
- Duan-Liu-Sakamoto-Strain ('19) consider the case without angular cutoff using the spatial Fourier-based norm (Wiener algebra in x)

$$\|f\| := \sum_k \sup_t \|\mathscr{F}_x\{f\}(t,k,\cdot)\|_{L^2_v}$$

(Algebraic structure)

For solutions in the perturbative framework, need estimate like

$$\|fg\|_X \lesssim \|f\|_X \|g\|_X,$$

for example, H^s, $s > 3/2$, the norm used in Duan-Liu-Sakamoto-Strain.

(L^∞ theory for cutoff, Ukai)

$$L = -\nu + K.$$

That is, the gain and loss parts in the collision kernel can be considered separately.

Key observation to fill in the gap of L^∞ theory without angular cutoff

(Alonso-Morimoto-Sun-Y., '20)

Weighted L^2 estimate on the level sets + a time localized strong averaging lemma + a De Giorgi argument

$$\Rightarrow \text{weighted } L^\infty \text{ estimate.}$$

Not to apply the 'algebraic structure' to close the bootstrap argument.

De Giorgi argument

(Related works)

- Diffusion equations: Caffarelli-Vasseur ('10), ...
- Landau equation and Fokker-Planck equation (hypoelliptic kinetic equations): Guerand('18), Golse-Imbert-Mouhot-Vasseur ('19), Kim-Guo-Huang ('20), Guerand-Mouhot ('21), ...[Harnack inequality, Hölder continuity]
- Spatially homogeneous Boltzmann equation: Alonso ('19), ...

Decay in the velocity variable

(Decay of tail for the perturbative solution)

- Most of the above results on the perturbative solutions are with Gaussian tail:

$$F = \mu + \sqrt{\mu} f;$$

- For algebraic decay

$$F = \mu + f$$

  with angular cutoff, Gualdani-Mischler-Mouhot ('18); without angular cutoff, Hérau,Tonon and Tristani (mild singularity, '20), and Alonso-Morimoto-Sun-Y. (strong singularity, '20).

Polynomial vs exponential

- Exponential: $F = \mu + \sqrt{\mu} f$
  - Linearized operator is self-adjoint:

$$L_\mu f = \frac{1}{\sqrt{\mu}} \left(Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu) \right)$$
$$= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) |v - v_*|^\gamma \left(\sqrt{\mu'_*} f' + \sqrt{\mu'} f'_* - \sqrt{\mu} f_* - \sqrt{\mu_*} f \right) \mathrm{d}\sigma \, \mathrm{d}v_*$$

  - Null space of L_μ:

$$\text{Null}(L_\mu) = \text{Span}\left\{ \sqrt{\mu}, \ \sqrt{\mu}\, v, \ \sqrt{\mu}\, |v|^2 \right\}.$$

  - Coercivity estimate [1]: $f \in (\text{Null}(L))^\perp$

$$\langle f, L_\mu f \rangle_{L^2_v} \le -c_0 \left(\|f\|^2_{H^s_{\gamma/2}} + \|f\|^2_{L^2_{s+\frac{\gamma}{2}}} \right).$$

[1] AMUXY, JFA, '12. See also Gressman-Strain, JAMS, '11.

Polynomial vs exponential

- Polynomial: $F = \mu + f$
  - Linearized operator is not self-adjoint:

$$\begin{aligned} Lf &= Q(\mu, f) + Q(f, \mu) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left(\mu'_* f' + \mu' f'_* - \mu f_* - \mu_* f\right) b(\cos\theta)|v - v_*|^\gamma \, \mathrm{d}\sigma \, \mathrm{d}v_* \, . \end{aligned}$$

  - Coercivity estimate [2]: Denote

$$J_1^\gamma(f) = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)|v - v_*|^\gamma \mu_* (f(v') - f(v))^2 d\sigma \, \mathrm{d}v_* \, \mathrm{d}v.$$

Then

$$\begin{aligned} \langle Q(\mu, f), f\rangle &= -c_1 J_1^\gamma(f) + \text{mod}\{\|f\|^2_{L^2_{\gamma/2}}\} \\ &\leq -c_0 \|f\|^2_{H^s_{\gamma/2}} + C\|f\|^2_{L^2_{\gamma/2}} \end{aligned}$$

[2] AMUXY, Kyoto J. Math. '12.

Function space for polynomial decay perturbation

- Weight function:

$$W(v) = \langle v \rangle^{m_0}, \quad m_0 > 4s.$$

- Define

$$Y_K = \left\{ h \in L^2_{x,v} \,\middle|\, W^{K-|\alpha|} \partial^\alpha f \in L^2_{x,v}, \quad |\alpha| \le 2 \right\},$$

and denote

$$\|f\|_{Y_K} = \sum_{|\alpha| \le 2} \left\| W^{K-|\alpha|} \partial_x^\alpha f \right\|_{L^2_{x,v}} .$$

Wellposedness theorem

Theorem (Alonso-Morimoto-Sun-Y, Revista Matematica Iberoamericana, '20)

Suppose $0 < s < 1$ *and* $0 < \gamma \leq 1$. *For some* K *being suitably large and* $\varepsilon_0 > 0$ *small enough, if* $F^{in} = \mu + f^{in} \geq 0$ *satisfies*

$$\left\|f^{in}\right\|_{Y_K} < \varepsilon_0 , \qquad \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f^{in}\phi(v)\,\mathrm{d}v\,\mathrm{d}x = 0$$

for any $\phi \in$ *Null*$\{L\}$, *then the non-cutoff Boltzmann equation has a unique non-negative solution* $F \in C([0,\infty), Y_K)$ *such that*

$$\|F - \mu\|_{Y_K} \lesssim e^{-\lambda t}\left\|f^{in}\right\|_{Y_K}, \quad \lambda > 0.$$

(Three key components in the proof)

- Propagation of moments:

$$\frac{d}{dt}\|f\|_{Y_K}^2 \le -(\cdots) + c\|f\|_{Y_K}^2;$$

- Regularization

$$\left\| S_L(t) h^{in} \right\|_{L^2(\langle v\rangle^k \mathrm{d}x\mathrm{d}v)} \lesssim \left(t^{-1/2}+1\right) \left\| \langle v\rangle^k h^{in} \right\|_{H_v^{-s}(\langle v\rangle^k \mathrm{d}x\mathrm{d}v)};$$

- Spectral gap (to show Gualdani-Mischler-Mouhot's result holds in the non-cutoff setting):

$$\left\| S_L(t) h^{in} \right\|_{L^2(\langle v\rangle^k \mathrm{d}x\mathrm{d}v)} \le e^{-\lambda t} \left\| h^{in} \right\|_{L^2(\langle v\rangle^k \mathrm{d}x\mathrm{d}v)}.$$

Can one relax the H_x^2 assumption?

L^∞ solution to Boltzmann equation without cutoff

(Theorem, Alonso-Morimoto-Sun-Y., 2020)

Suppose $\gamma \in (0,1]$ and $s \in (0,1)$ and the initial data F_0 conserves the mass, momentum and energy of the equilibrium. Then for k_0, k large enough with $k > k_0$, there exists $\delta_0 > 0$ such that if

$$\left\|\langle v\rangle^{k_0}(F_0(x,v)-\mu)\right\|_{L^2_{x,v}\cap L^\infty_{x,v}} \le \delta_0, \; \left\|\langle v\rangle^{k}(F_0(x,v)-\mu)\right\|_{L^2_{x,v}} < \infty.$$

Then there exists a unique solution $F \in L^\infty(0,\infty; L^2_x L^2_k(\mathbb{T}^3\times\mathbb{R}^3))$. Moreover, $\exists\, \delta,\ \lambda > 0$, the solution F satisfies

$$\|\langle v\rangle^{k_0}(F(t,x,v)-\mu)\|_{L^\infty_{x,v}} < \min\{\delta,\, Ce^{-\tilde{\lambda}t}\}$$

and

$$\left\|\langle v\rangle^{k}(F(t,x,v)-\mu)\right\|_{L^2_{x,v}} < Ce^{-\lambda t}.$$

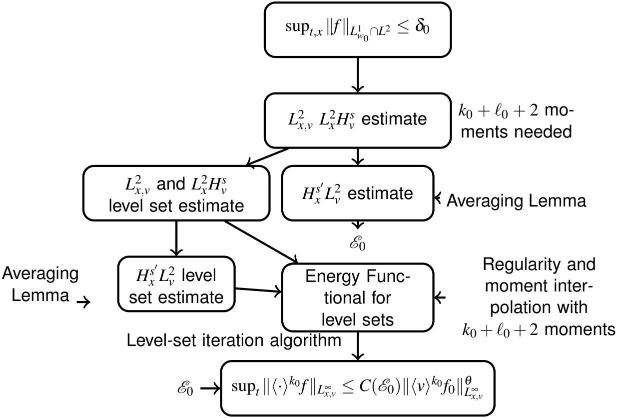

Figure: Flow chart of the strategy. Moments are related as $k_0 > w_0 > 0$ and so does regularity as $s > s' > 0$. The constant $C(\mathcal{E}_0)$ is independent of the smallness parameter δ_0.

L^2 estimates on level sets

Suppose $G = \mu + g \geq 0$, $F = \mu + f$ and $s \in (0,1)$. Suppose in addition G satisfies that

$$\inf_{t,x} \|G\|_{L^1_v} \geq D_0 > 0, \qquad \sup_{t,x} \left(\|G\|_{L^1_2} + \|G\|_{L \log L} \right) < E_0 < \infty.$$

Then for any $\ell > 8 + \gamma$, the (bilinear) collision term satisfies,

$$\begin{aligned}
&\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} Q(G,F) f^{(\ell)}_{K,+} \langle v \rangle^{\ell} \, \mathrm{d}v \, \mathrm{d}x \\
&\leq -\gamma_0 \left(1 - C \sup_x \|g\|_{L^1_\gamma}\right) \left\| f^{(\ell)}_{K,+} \right\|^2_{L^2_x L^2_{\gamma/2}} - \delta \left\| f^{(\ell)}_{K,+} \right\|^2_{L^2_x H^s_{\gamma/2}} \\
&\quad + C_\ell \left(1 + \sup_x \|g\|_{L^1_{\ell+\gamma}} + \sup_x \|g\|^{b_0}_{L^1_{3+\gamma+2s} \cap L^2}\right) \left\| f^{(\ell)}_{K,+} \right\|^2_{L^2_{x,v}} \\
&\quad + C_\ell (1+K) \left(1 + \sup_x \|g\|_{L^1_{\ell+\gamma}}\right) \left\| f^{(\ell)}_{K,+} \right\|_{L^1_x L^1_\gamma}.
\end{aligned}$$

Note: The weight of g can be improved if $g = f$.

L^1 estimate for the collision operator for applying the averaging lemma

Let $G = \mu + g \geq 0$ and $F = \mu + f$ then, for any

$$[T_1, T_2] \subseteq [0,T),\ s \in (0,1),\ \varepsilon \in [0,1],\ j \geq 0,\ \ell > 8+\gamma,\ \kappa > 2,\ K > 0,$$

it holds that

$$\int_{T_1}^{T_2} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left| \langle v \rangle^j (1-\Delta_v)^{-\kappa/2} \left(Q(G,F) \langle v \rangle^\ell f_{K,+}^{(\ell)} \right) \right| \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t$$
$$\lesssim \| \langle v \rangle^{j/2} f_{K,+}^{(\ell)}(T_1,\cdot,\cdot) \|_{L^2_{x,v}}^2 + C_\ell \left(1 + \sup_{t,x} \|g\|_{L^1_{\ell+\gamma}} \right) \left\| f_{K,+}^{(\ell)} \right\|_{L^2_{t,x} L^2_j}^2$$
$$+ \left(1 + \sup_{t,x} \|g\|_{L^1_{3+\gamma+2s} \cap L^2} \right) \left\| f_{K,+}^{(\ell)} \right\|_{L^2_{t,x} H^s_{\gamma/2}}^2 + \left(1 + \sup_{t,x} \|g\|_{L^1_{j+2+\gamma}} \right) \left\| f_{K,+}^{(\ell)} \right\|_{L^2_{t,x} L^2_{j+\gamma/2+1}}^2$$
$$+ (1+K) \left(1 + \sup_{t,x} \|g\|_{L^1_{\ell+\gamma}} \right) \left\| f_{K,+}^{(\ell)} \right\|_{L^1_{t,x} L^1_{j+\gamma}},$$

where C_ℓ are independent of ε and T_1, T_2.

Localized strong averaging lemma

(velocity averaging lemma)

- Golse-Perthame-Sentis('85), Golse-Lions-Perthame-Sentis('88), DiPerna-P. L. Lions-Meyer('91); Bézard ('94), P.L. Lions ('95), Perthame-Souganidis ('98);
- Bouchut-Desvillettes, ('99); Bouchut-Golse-Pulvirenti('00), Bournaveas-Perthame ('01), Bouchut, ('02);
- ...

(Localized version of strong velocity averaging lemma by Bouchut)

Fix $0 \le T_1 < T_2$, $p \in (1,\infty)$, $\beta \ge 0$, assume $f \in C\big([T_1,T_2];L^p_{x,v}\big) \cap L^p_{t,x,v}$ with $(-\Delta_v)^{\beta/2} f \in L^p_{t,x,v}$ satisfies

$$\partial_t f + v \cdot \nabla_x f = \mathscr{F}, \qquad t \in (0,\infty).$$

Then, for any $r \in [0,\frac{1}{p}]$, $m \in \mathbb{N}$, $\beta_- \in [0,\beta)$, define

$$s^\flat = \frac{(1-rp)\beta_-}{p(1+m+\beta)},$$

and $\tilde{f} = f\, 1_{(T_1,T_2)}(t)$, $\tilde{\mathscr{F}} = \mathscr{F}\, 1_{(T_1,T_2)}(t)$,

(Localized strong velocity averaging lemma, continued)

then

$$
\begin{aligned}
&\left\|(-\Delta_x)^{\frac{s^\flat}{2}}\tilde{f}\right\|_{L^p_{t,x,v}}+\left\|(-\partial_t^2)^{\frac{s^\flat}{2}}\tilde{f}\right\|_{L^p_{t,x,v}}\\
&\leq C\Big(\left\|\langle v\rangle^{1+m}(1-\Delta_x)^{-\frac{r}{2}}(1-\Delta_v)^{-\frac{m}{2}}f(T_1)\right\|_{L^p_{x,v}}\\
&+\left\|\langle v\rangle^{1+m}(1-\Delta_x)^{-\frac{r}{2}}(1-\Delta_v)^{-\frac{m}{2}}f(T_2)\right\|_{L^p_{x,v}}\\
&+\left\|\langle v\rangle^{1+m}(1-\Delta_x-\partial_t^2)^{-\frac{r}{2}}(1-\Delta_v)^{-\frac{m}{2}}\tilde{\mathscr{F}}\right\|_{L^p_{t,x,v}}\\
&\qquad+\left\|(-\Delta_v)^{\beta/2}\tilde{f}\right\|_{L^p_{t,x,v}}+\|\tilde{f}\|_{L^p_{t,x,v}}\Big),
\end{aligned}
$$

where the constant C only depends on d,β,r,m and p.

A De Giorgi argument

(Energy functional)

For $s'' \in (0,s) \subseteq (0,1)$, $\ell \geq 0$, $p > 1$,

$$\mathscr{E}_p(K,T_1,T_2) := \sup_{t\in[T_1,T_2]} \left\| f_{K,+}^{(\ell)} \right\|_{L^2_{x,v}}^2 + c_0 \int_{T_1}^{T_2} \int_{\mathbb{T}^3} \left\| \langle \cdot \rangle^{\gamma/2} f_{K,+}^{(\ell)} \right\|_{H^s_v}^2 \mathrm{d}x\,\mathrm{d}\tau$$
$$+ \frac{1}{C_0} \left(\int_{T_1}^{T_2} \left\| (1-\Delta_x)^{\frac{s''}{2}} \left(f_{K,+}^{(\ell)} \right)^2 \right\|_{L^p_{x,v}}^p \mathrm{d}\tau \right)^{\frac{1}{p}}.$$

The parameters $s'' > 0$, $p > 1$, $c_0 > 0$, and $C_0 > 0$ will be suitably chosen.

(Key estimate on functional)

For $M < K$, it holds that

$$\left\| f_{K,+}^{(\ell)}(T_2) \right\|_{L^2_{x,v}}^2 + c_0 \int_{T_1}^{T_2} \| \langle v \rangle^{\gamma/2} (1-\Delta_v)^{\frac{s}{2}} f_{K,+}^{(\ell)}(\tau) \|_{L^2_{x,v}}^2 \, \mathrm{d}\tau$$
$$+ \frac{1}{C_0} \left(\int_{T_1}^{T_2} \| (1-\Delta_x)^{\frac{s''}{2}} \left(f_{K,+}^{(\ell)} \right)^2 \|_{L^p_{x,v}}^p \, \mathrm{d}\tau \right)^{\frac{1}{p}}$$
$$\leq C \| \langle v \rangle^2 f_{K,+}^{(\ell)}(T_1) \|_{L^2_{x,v}}^2 + C \| \langle v \rangle^2 f_{K,+}^{(\ell)}(T_1) \|_{L^{2p}_{x,v}}^2$$
$$+ \frac{CK}{K-M} \sum_{i=1}^{4} \frac{\mathscr{E}_p(M, T_1, T_2)^{\beta_i}}{(K-M)^{a_i}},$$

where $\beta_i > 1$ and $a_i > 0$ and C is independent of K, M, f, T_1, T_2. Furthermore, the estimate holds for $f_{K,-}^{(\ell)}$.

Set

$$M_k := K_0(1 - 1/2^k), \qquad k = 0, 1, 2, \cdots$$

Take $T_2 \in (0, T)$ with $T > 0$ fixed

$$f_k := f^{(\ell)}_{M_k,+} \quad \text{and} \quad \mathscr{E}_k := \mathscr{E}_p(M_k, 0, T), \qquad k = 0, 1, 2, \cdots.$$

Then

$$\mathscr{E}_p(M_{k-1}, 0, T_2) \le \mathscr{E}_p(M_{k-1}, 0, T) = \mathscr{E}_{k-1}, \qquad k = 1, 2, \cdots$$

and

$$\mathscr{E}_k \le C \left\| \langle v \rangle^2 f_k(0) \right\|^2_{L^2_{x,v}} + C \left\| \langle v \rangle^{\frac{1}{2}} f_k(0) \right\|^2_{L^{2p}_{x,v}} + C \sum_{i=1}^{4} \frac{2^{k(a_i+1)} \, \mathscr{E}_{k-1}^{\beta_i}}{K_0^{a_i}}.$$

Terms related to the initial data will vanish by setting

$$K_0 \geq 2\left\|\langle v\rangle^{\ell} f_0\right\|_{L^\infty_{x,v}}.$$

Then

$$\mathscr{E}_k \leq C \sum_{i=1}^{4} \frac{2^{k(a_i+1)}\,\mathscr{E}_{k-1}^{\beta_i}}{K_0^{a_i}}, \qquad K_0 \geq 2\left\|\langle v\rangle^{\ell} f_0\right\|_{\infty}.$$

Let

$$Q_0 = \max_{1\leq i\leq 4}\left\{2^{\frac{a_i+1}{\beta_i-1}}\right\}, \qquad \mathscr{E}_k^* = \mathscr{E}_0 (1/Q_0)^k \qquad \text{for } k = 0,1,2,\cdots,$$

and

$$K_0 \geq K_0(\mathscr{E}_0) := \max_{1\leq i\leq 4}\left\{4\,C^{\frac{1}{a_i}}\,\mathscr{E}_0^{\frac{\beta_i-1}{a_i}}\,Q_0^{\frac{\beta_i}{a_i}}\right\}.$$

Then one can check via a direct computation that $\mathscr{E}_k^*$ satisfies

$$\mathscr{E}_0^* = \mathscr{E}_0, \qquad \mathscr{E}_k^* \geq C \sum_{i=1}^{4} \frac{2^{k(a_i+1)}\left(\mathscr{E}_{k-1}^*\right)^{\beta_i}}{K_0^{a_i}}, \qquad k = 0,1,2,\cdots.$$

By a comparison principle (since $\mathscr{E}_0 = \mathscr{E}_0^*$) one obtains that

$$\mathscr{E}_k \leq \mathscr{E}_k^* \to 0 \quad \text{as} \quad k \to \infty,$$

since $\beta_i > 1$ (so that $Q_0 > 1$). In particular, we can infer that

$$\sup_{t\in[0,T)} \left\| f_{K_0,+}^{(\ell)}(t,\cdot,\cdot) \right\|_{L^2_{x,v}} = 0 \quad \text{for} \quad K_0 = \max\left\{ 2\left\| \langle v\rangle^{\ell} f_0 \right\|_{L^\infty_{x,v}}, K_0(\mathscr{E}_0) \right\},$$

which implies that

$$\sup_{t\in[0,T)} \left\| \langle v\rangle^{\ell} f_+(t,\cdot,\cdot) \right\|_{L^\infty_{x,v}} \leq K_0.$$

The spectrum gap is used to obtain a uniform bound on $\mathscr{E}_0$ so that the L^∞ bound on the solution is global in time. In addition, the exponential decay of the perturbation in both L^∞ and L^2 norm follows.

THANK YOU!