Classical and quantum particles coupled to a vibrational environment

with :
A. Vavasseur (Univ. Côte d'Azur, now BCAM)
L. Vivion (Univ. Côte d'Azur, now Ec. Polytechnique)
S. De Bièvre (Lille)

Thierry GOUDON
Team COFFEE, Inria

Motivation: description of dissipation in (quantum-)mechanical systems

Caldeira-Legget (Ann. Phys. 83)

Dissipation arising on a physical system might come from a coupling with a complex environment: energy is evacuated into the environment and does not come back to the system.
Dissipation = transfer of energy from the single degree of freedom characterising the system to the set of degrees of freedom describing the environment.

Motivation: description of dissipation in (quantum-)mechanical systems

$$\ddot{q} = -\lambda \dot{q} + F_{\mathrm{ext}}$$

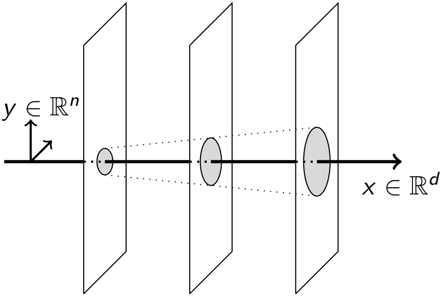
Where does the friction λ come from ?

Many approaches, depending on the model of the degrees of freedom for the environment:
Komech, Spohn '90-00
Jaksic-Pillet Acta. Math. 98

A series of works by **S. De Bièvre** (Lille)
with L. Bruneau (CMP'02),
and P. Parris, P. Lafitte, B. Aguer, E. Soret...
"Dynamical Lorentz gas": interaction of a particle with a vibrational field
"a particle in a pinball machine"

The BdB model for a single particle

A particle moving through a *transverse* continuum of vibrating membranes



$$\ddot{q}(t) = -\nabla_q V(q(t)) - \int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(q(t) - z)\ \sigma_2(y)\ \nabla_z \Psi(t, z, y)\, \mathrm{d}y\, \mathrm{d}z,$$

$$\partial_{tt}^2 \Psi(t, x, y) - c^2 \Delta_y \Psi(t, x, y) = -\sigma_2(y)\sigma_1(x - q(t))$$

A particle coupled to its environment

$$\ddot{q}(t) = -\nabla_q V(q(t)) - \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-z)\ \sigma_2(y)\ \nabla_z\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z,$$
$$\partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\sigma_1(x-q(t))$$

Modeling parameters:

form functions σ_1,σ_2 (C_c^∞, non negative...), c: wave speed

Hamiltonian structure:

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}(|\partial_t\Psi|^2 + c^2|\nabla_y\Psi|^2)(t,x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_1(x-q(t))\sigma_2(y)\Psi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x \text{ is conserved}$$

Starting point: findings of De Bièvre's group

Assume $n = 3$ (wave dimension) and $c \gg 1$ large enough

• If V is a confining potential, as time becomes large, $q(t)$ converges to the potential well and the velocity $\dot{q}(t)$ tends to 0.

• If $V(q) = \mathcal{F} \cdot q$, there exists a limiting velocity $v(\mathcal{F})$ and, for $\mathcal{F}$ small enough, $q(t) \sim q_\infty + v(\mathcal{F})t$ and $\dot{q}(t) \to v(\mathcal{F})$.

It is remarkable that $v(\mathcal{F}) \sim \mu\mathcal{F}$ for small $\mathcal{F}$'s: linear response.

• Typical convergence rate $e^{-\gamma t/c^3}$.

These results bring out **dissipative effects** of the interaction with medium: asymptotically it acts like a friction force and energy is evacuated in the membranes.

Many particles: mean–field interpretation

- Consider a set of P particles

$$\begin{cases} \ddot{q}_j(t) = -\nabla_q V(q_j(t)) - \nabla_q \Phi(t, q_j(t)), \\ \\ \partial_{tt}^2 \Psi(t,x,y) - c^2 \Delta_y \Psi(t,x,y) = -\sigma_2(y) \displaystyle\sum_{k=1}^{P} \sigma_1(x - q_k(t)). \end{cases}$$

- Rescale the self–consistent potential
$$\Phi(t,x) = \left(\sigma_1 * \int_{\mathbb{R}^n} \sigma_2(y)\Psi(t,\cdot,y)\,\mathrm{d}y\right)(x) \to \frac{1}{P}\nabla\Phi(t,x).$$

- Consider $F_P(t,x,v) = \frac{1}{P}\sum_{j=1}^{P} \delta(x = q_j(t)) \otimes \delta(v = \dot{q}_j(t))$ and let $P \to \infty$

Further results on the P particles case and the behavior as $t \to \infty$, $P \to \infty$: A. Vavasseur '20

But the large time behavior is much more complex, with several scenario, than for a single particle.

Kinetic version of the BdB model

We are finally led to

$$
\begin{aligned}
&\partial_t F + v \cdot \nabla_x F - \nabla_x (V + \Phi) \cdot \nabla_v F = 0,\\
&\Phi(t,x) = \int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x-z)\sigma_2(y)\Psi(t,z,y)\,\mathrm{d}z\,\mathrm{d}y,\\
&\big(\partial_{tt}^2 \Psi - c^2 \Delta_y \Psi\big)(t,x,y) = -\sigma_2(y) \int_{\mathbb{R}^d} \sigma_1(x-z)\rho(t,z)\,\mathrm{d}z,\\
&\rho(t,x) = \int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v.
\end{aligned}
$$

Bear in mind that $y \in \mathbb{R}^n$ is a *transverse* variable: the model differs from the coupling Vlasov-Wave (coming from Vlasov-Maxwell) dealt with e. g. by Bouchut-Golse-Pallard'04.

Overview of the results

Energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\Psi(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y+\frac{c^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_y\Psi(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y\right.$$
$$\left.+\int_{\mathbb{R}^d\times\mathbb{R}^d}F(t,x,v)\left(\frac{|v|^2}{2}+V(x)+\Phi(t,x)\right)\mathrm{d}v\,\mathrm{d}x\right\}=0.$$

"Vibrational+ Kinetic & Potential particle energy + Coupling"

- Existence–uniqueness
  with general external potential $x \mapsto V(x)$ and data
  (it slightly generalizes BdB's analysis: $n \in \mathbb{N}\setminus\{0\}$, $P > 1$ particles...)
- mean-field asymptotics
- asymptotic analysis
- equilibrium states and stability (by a variational approach)
- Landau damping

A (quite surprising) connection to the Vlasov–Poisson equation

- Rescale the equation: large wave speed and strong coupling

$$\partial^2_{tt}\Psi_\epsilon - \frac{1}{\epsilon}\Delta_y\Psi_\epsilon = -\frac{1}{\epsilon}\sigma_2(y)\int \sigma_1(x-z)\rho_\epsilon(t,z)\,\mathrm{d}z.$$

- Formally as $\epsilon \to 0$: $-\Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \sigma_1 * \rho(t,x)$:

$$\Psi(t,x,y) = \Upsilon(y)\sigma_1 * \rho(t,x) \text{ with } \Delta_y\Upsilon(y) = \sigma_2(y).$$

  Therefore, we get $\Phi(t,x) = -\Lambda\sigma_1 * \sigma_1 * \rho(t,x)$, with
  $\Lambda = -\int \Upsilon\sigma_2(y)\,\mathrm{d}y\Big(= \int |\nabla_y\Upsilon(y)|^2\,\mathrm{d}y > 0\Big)$.

- We can rescale as well $\sigma_1 \to \sigma_{1,\epsilon}$ so that $|\widehat{\sigma_{1,\epsilon}}(\xi)|^2 \sim \frac{1}{|\xi|^2}$. It yields the *ATTRACTIVE* Vlasov–Poisson system.

From VW to attractive VP

- $\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon - \nabla_x \Phi_\epsilon \cdot \nabla_v F_\epsilon = 0$
- $\Phi_\epsilon = \sigma_{1,\epsilon} * \int \sigma_2 \Psi_\epsilon \, \mathrm{d}y$
- $\Psi_\epsilon(t,x,y) \simeq \Upsilon(y)\sigma_{1,\epsilon} * \rho_\epsilon(t,x)$ with $\Delta_y \Upsilon = \sigma_2$
- $\Phi_\epsilon \simeq -\Lambda \sigma_{1,\epsilon} * \sigma_{1,\epsilon} * \rho_\epsilon$ with $\Lambda = -\int \Upsilon \sigma_2 \, \mathrm{d}y > 0$,
- $\widehat{\Phi}_\epsilon \to -\frac{\Lambda}{\xi^2}\widehat{\rho}$

It yields the *ATTRACTIVE* Vlasov–Poisson system:

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \Phi \cdot \nabla_v F = 0,$$
$$\Delta \Phi = \Lambda \rho, \qquad \Lambda > 0.$$

Intuition that VW inherits some features of the attractive VP system

A modified formulation

We rewrite the pb. as

$$\partial_t F + v \cdot \nabla_x F = \nabla_v F \cdot \nabla_x (\Phi_I + \Phi_S)$$

where we set

$$\Phi_I(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \sigma_1(x-z) \times \left(\widehat{\Psi_0}(z,\xi) \cos(c|\xi|t) + \widehat{\Psi_1}(z,\xi) \frac{\sin(c|\xi|t)}{c|\xi|} \right) \widehat{\sigma_2}(\xi)\, \mathrm{d}z\, \mathrm{d}\xi$$

and

$$\Phi_S(t,x) = -\int_{\mathbb{R}^d} \Sigma(x-z) \left(\int_0^t p_c(t-s)\rho(s,z)\, \mathrm{d}s \right) \mathrm{d}z$$

with $\rho(t,x) = \int F(t,x,v)\, \mathrm{d}v, \qquad \Sigma = \sigma_1 * \sigma_1,$

and $t \mapsto p_c(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(c|\xi|t)}{c|\xi|} |\widehat{\sigma_2}(\xi)|^2\, \mathrm{d}\xi.$

The kernel p_c drives the dissipation mechanism $(\sigma_2, c$ and $n)$

Landau Damping

Given $M : v \mapsto M(v) > 0$, with $\int M \, \mathrm{d}v = 1$, $\rho_0 M(v)$ defines a homogeneous solution, with mass ρ_0 of the VW system.

"Theorem". Initial data $F_0(x, v) = \rho_0 M(v) + f_0(x, v)$. Assume

- smooth data $\sigma_1, \sigma_2, M, f_0$,
- $n \geq 3$ odd
- the (**L**) stability criterion

There exists $\epsilon_0 > 0$ such that if $\|f_0\| \leq \epsilon_0$, then

$$\rho - \int F_0 \, \mathrm{d}v \, \mathrm{d}x \text{ and } \nabla_x \Phi \text{ tend to } 0 \text{ as } t \to \infty.$$

Landau '46 (linearized pb.)
Mouhot-Villani '11 (torus)
Faou-Rousset '16 (finite regularity)
Bedrossian-Masmoudi-Mouhot '16-'18 (torus, whole space)
Han-Kwan-Nguyen-Rousset '19 (whole space)

Quantum version: Schrödinger-Wave

$$
\begin{aligned}
&i\partial_t u + \frac{1}{2}\Delta_x u = \left(\sigma_1 \star \int_{\mathbb{R}^n} \sigma_2(y)\psi(t,x,y)\,\mathrm{d}y\right)\ u,\\
&\partial^2_{tt}\psi - c^2\Delta_y\psi = -c^2\sigma_2(y)\sigma_1 \star |u|^2
\end{aligned}
$$

- Energy conservation

$$
\begin{aligned}
\frac{1}{2}\int |\nabla_x u|^2\,\mathrm{d}x + \frac{1}{2}\int \Big(\frac{|\partial_t\psi|^2}{c^2} + |\nabla_y\psi|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\
+ \int \sigma_1(x-x')\sigma_2(y)\psi(x',y)|u(x)|^2\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}x.
\end{aligned}
$$

- Well-posedness in the energy space
- Semi-classical analysis *à la* Lions-Paul makes the connection with the kinetic model.

Connection to other models: preliminary observations

- The limit $c \to \infty$ leads to the Hartree model

$$i\partial_t u + \frac{1}{2}\Delta_x u = -\big(\Lambda\Sigma \star |u|^2\big)u.$$

  Many results known when Σ is replaced by δ_0 (focusing NLS) and in dimension 3 when Σ is replaced by $\frac{1}{|x|}$ (Newton-Hartree: Lieb, Lenzmann...)
  Intuition that the space dimension d might be important...
- But SW has no scale invariance, and it does not satisfy Galilean invariance.
- Both systems admit **Solitary Wave** solutions:
  $(u, \psi) = (e^{i\omega t}Q(x), \Psi(x, y))$

$$-\frac{1}{2}\Delta_x Q + \omega Q + \Lambda\Sigma \star |Q|^2 Q = 0, \qquad -\Delta_y \Psi = -\sigma_2(y)\sigma_1 \star |Q|^2$$

  Choquard's equation... which has infinitely many solutions.

Ground states

Minimization of the energy

$$\frac{1}{2}\int |\nabla_x u|^2 \,\mathrm{d}x + \frac{1}{2}\int \Big(\frac{|\chi|^2}{c^2} + |\nabla_y \psi|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\ + \int \sigma_1(x-x')\sigma_2(y)\psi(x',y)|u(x)|^2 \,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}x,$$

with a mass constraint $\|u\|_{L^2}^2 = M$.

- Mass threshold:
  the minimal energy is 0 for $M < M_0$; it is negative when $M > M_0$ and it is reached at $(Q, \Psi, 0)$, solution of Choquard's equation for a certain ω.
- Regularity and radial symmetry... but uniqueness is open !

Orbital stability

- P-L. Lions' concentration-compactness approach leads to a weak stability statement: starting close to a ground state, the solutions remains close to the manifold on all ground states.
- Strengthened results can be obtained by linearization and spectral analysis:

$$\|u(t)-e^{i\gamma(t)}Q(\cdot-x(t))\|_{H^1}^2+\|\nabla_y\psi(t)-\nabla_y\Psi(\cdot-x(t))\|_{L^2}^2+\frac{1}{c^2}\|\partial_t\psi\|_{L^2}^2\leq\epsilon.$$

  The proof relies on a perturbation argument from Lenzmann's analysis of the case where $d=3$ and $\Sigma\to\frac{1}{|x|}$.
  Critical step: coercivity estimate.
- The stability is shown for an admissible class of relevant form functions σ_1 such that by rescaling $\sigma_1\star\sigma_1$ approaches $\frac{1}{|x|}$.

Surprising fact: the convolution with the smooth kernel Σ is much harder than the case with $\frac{1}{|x|}$!

Numerical experiments

Initial data $u_0(x) = Q(x)e^{ip_0/M^2}$ with $|p_0| \ll 1$.
Simplified dynamics:

$$M\dot{q} = p,$$
$$\dot{p} = -\nabla_x \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sigma_1(x - x') \int_{\mathbb{R}^d} \sigma_2(y)\psi(t, x', y)\,\mathrm{d}y\,\mathrm{d}x' \right) Q^2(x - q(t))\,\mathrm{d}x$$
$$\partial_{tt}^2\psi - c^2\Delta_y\psi = -c^2\sigma_2(y) \int_{\mathbb{R}^d} \sigma_1 \star (x - x')Q^2(x' - q)\,\mathrm{d}x'.$$

- p, q close to the center of mass and the impulsion defined from the SW system
- analogy with the classical dynamics
- $|q(t) - q_\infty| + |p(t)| \leq Ce^{-\lambda t/c}$

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We rewrite the pb. as

$$\partial_t F + v \cdot \nabla_x F = \nabla_v F \cdot \nabla_x \left(\Phi_I + \Phi_S\right)$$

where we set De1

$$\Phi_I(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \sigma_1(x-z) \\ \times \left(\widehat{\Psi_0}(z,\xi) \cos(c|\xi|t) + \widehat{\Psi_1}(z,\xi) \frac{\sin(c|\xi|t)}{c|\xi|} \right) \widehat{\sigma_2}(\xi) \, \mathrm{d}z \, \mathrm{d}\xi$$

and

$$\Phi_S(t,x) = -\int_{\mathbb{R}^d} \Sigma(x-z) \left(\int_0^t p_c(t-s)\rho(s,z) \, \mathrm{d}s \right) \, \mathrm{d}z$$

with $\rho(t,x) = \int F(t,x,v) \, \mathrm{d}v, \qquad \Sigma = \sigma_1 * \sigma_1,$

and $t \mapsto p_c(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(c|\xi|t)}{c|\xi|} |\widehat{\sigma_2}(\xi)|^2 \, \mathrm{d}\xi.$

VP vs VW

Vlasov eq.

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \Phi \cdot \nabla_v F = 0.$$

with either

$$\text{VP case: } \Phi(t,x) = \Sigma \star \rho(t,x)$$

$$\text{VW case: } \Phi(t,x) = \Phi_I(t,x) + \int_0^t p_c(t-s)\Sigma \star \rho(s,x)\,\mathrm{d}s$$

- Effect of the initial data for the vibrational field: decay related to the dispersion of the wave eq.
- "Memory effect" through p_c interplay between dispersion (depends on n and c) and regularity of the coefficients σ_1, σ_2

The kernel p_c

Let W solution of $\Box_{t,z} W = 0$, $(W, \partial_t W)\big|_{t=0} = (0, \sigma_2)$

$$p_c(t) = \frac{1}{c} \int_{\mathbb{R}^n} \sigma_2(z) W(ct, z)\, \mathrm{d}z = \int_{\mathbb{R}^n} \frac{\sin(c|\zeta| t)}{c|\zeta|}\ |\widehat{\sigma}_2(\zeta)|^2 \frac{\mathrm{d}\zeta}{(2\pi)^n}.$$

Energy dissipation mechanisms $\longleftrightarrow$ decay of p_c.

When $n \geq 3$, p_c is integrable and satisfies

$$\int_0^\infty p_c(t)\, \mathrm{d}t = \frac{\Lambda}{c^2}, \qquad \text{with} \qquad \Lambda = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}_2(\zeta)|^2}{|\zeta|^2}\, \mathrm{d}\zeta < \infty,$$

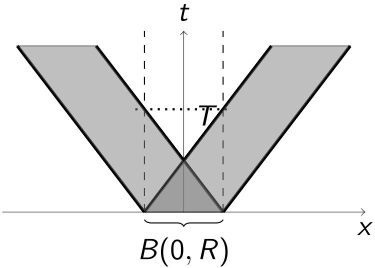
In dimension $n = 1$, a direct computation by means of D'Alembert formula shows that

$$p_c(t) = \frac{1}{2c} \int_{-\infty}^{+\infty} \sigma_2(z) \left(\int_{z-ct}^{z+ct} \sigma_2(s)\, \mathrm{d}s \right) \mathrm{d}z \xrightarrow[t\to\infty]{} \frac{1}{2c} \|\sigma_2\|^2_{L^1_z} > 0.$$

The kernel p_c, ctn'd

$$p_c(t) = \frac{1}{c}\int_{\mathbb{R}^n} \sigma_2(z) W(ct,z)\,\mathrm{d}z, \quad \Box_{t,z} W = 0 \quad (W, \partial_t W)\big|_{t=0} = (0, \sigma_2)$$

If $n \geq 3$ odd, $\sigma_2 \in C_c^0(\mathbb{R}^n)$ with $\mathrm{supp}(\sigma_2) \subset B(0, R_2)$, then p_c has a compact support included in $[0, \frac{2R_2}{c}]$ and $|p_c(t)| \lesssim 1/c$.



Assumptions on σ_2 can be relaxed, and including for even dimension $n \geq 4$ we can obtain algebraic decay of p_c

A modified formulation ▸ Bk1

The splitting of $\Phi(t,x) = \left(\sigma_1 * \int_{\mathbb{R}^n} \sigma_2(y)\Psi(t,\cdot,y)\,\mathrm{d}y\right)(x)$ with

$$(\partial_{tt}^2 - c^2\Delta_y)\Psi(t,x,y) = -\sigma_2(y)\,\sigma_1 * \rho(t,x), \qquad (\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1)$$

relies on the *linearity* of the wave eq.:

$$\Phi = \Phi_I + \Phi_S$$

with Φ_I associated to the **free** wave equation

$$(\partial_{tt}^2 - c^2\Delta_y)\Upsilon(t,x,y) = 0, \qquad (\Upsilon, \partial_t\Upsilon)\Big|_{t=0} = (\Psi_0, \Psi_1)$$

(or with data $\sigma_1 * (\Psi_0, \Psi_1)$), and Φ_S associated to the sol. of

$$(\partial_{tt}^2 - c^2\Delta_y)\widetilde{\Psi}(t,x,y) = -\sigma_2(y)\,\sigma_1 * \rho(t,x), \qquad (\widetilde{\Psi}, \partial_t\widetilde{\Psi})\Big|_{t=0} = 0$$

Proceeding this way, we distinguish the influence of the initial data and the influence of the coupling.