Small Knudsen rate of convergence to rarefaction wave for the Landau equation

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$$Q(G, F)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v - v_*) \{ G(v_*) \nabla_v F(v) - \nabla_{v_*} G(v_*) F(v) \} \, dv_*,$$

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• $\gamma = -3$ corresponds to the original (Fokker-Planck)-Landau collision operator for Coulomb potentials. To the end, let $-3 \leq \gamma < -2$.

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$$D(F) \geq 0$$

with "$=$" iff $F = M$ where

$$M = M_{[\rho,u,\theta](t,x)}(v) = \frac{\rho(t,x)}{(2\pi R\theta(t,x))^{3/2}} \exp\left(-\frac{|v - u(t,x)|^2}{2R\theta(t,x)} \right).$$

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A fundamental question is to determine the global solution as well as its large time behavior for the Cauchy problem of the Landau equation with initial data

$$F(0, x, v) = F_0(x, v) \to M_\pm \text{ as } x \to \pm\infty,$$

with $M_- \neq M_+$. The similar happens in the context of the Boltzmann equation.

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with the fluid quantities determined by 1D compressible Euler system

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2)_x + p_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2,3, \\ \left\{\rho(e + \frac{|u|^2}{2})\right\}_t + \left\{\rho u_1(e + \frac{|u|^2}{2}) + p u_1\right\}_x = 0, \end{cases}$$

where

$$\begin{cases} \rho(t,x) = \int_{\mathbb{R}^3} \psi_0(v) F \, dv, \\ \rho u_i(t,x) = \int_{\mathbb{R}^3} \psi_i(v) F \, dv, \quad \text{for } i = 1,2,3, \\ \rho(e + \frac{1}{2}|u|^2)(t,x) = \int_{\mathbb{R}^3} \psi_4(v) F \, dv. \end{cases}$$

Remark: There exist global entropic solutions for initial data with small total variation due to pioneering works by Glimm, Bressan-Colombo, Liu,...

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- Under the scaling $(t,x) \to (\epsilon^{-a}t, \epsilon^{-a}x)$ with $\frac{2}{3} \le a \le 1$, for suitable initial data, to show

$$\sup_{t\ge l} \Big\| \frac{F(t,x,v) - M_{[\rho^R,u^R,\theta^R](x/t)}(v)}{\sqrt{\mu}} \Big\|_{L_x^\infty L_v^2} \le C_l \epsilon^{\frac{3}{5}-\frac{2}{5}a} |\ln \epsilon|,$$

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Remark: Applicable for $\gamma = -3$ Coulomb potentials.

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For well prepared initial data,

  - Xin (for NS; CPAM 93): $a = \frac{3}{4}$, Error $\lesssim \epsilon^{\frac{1}{4}} |\ln \epsilon|$.

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Micro (kinetic) equation —

$$\partial_t G + P_1(v_1 \partial_x G) + P_1(v_1 \partial_x M) = \frac{1}{\epsilon} L_M G + \frac{1}{\epsilon} Q(G, G).$$

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$$\hat{A}_j(v) = \frac{|v|^2 - 5}{2} v_j, \quad \hat{B}_{ij}(v) = v_i v_j - \frac{1}{3}\delta_{ij}|v|^2 \quad \text{for} \quad i, j = 1, 2, 3,$$

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$$A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big) = L_M^{-1}\Big[\hat{A}_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M\Big], \quad B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big) = L_M^{-1}\Big[\hat{B}_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M\Big].$$

Macro (fluid) equations —

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2)_x + p_x = \frac{4}{3}\epsilon(\mu(\theta) u_{1x})_x - (\int_{\mathbb{R}^3} v_1^2 L_M^{-1} \Theta \, dv)_x, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = \epsilon(\mu(\theta) u_{ix})_x - (\int_{\mathbb{R}^3} v_1 v_i L_M^{-1} \Theta \, dv)_x, \quad i = 2, 3, \\ \left\{\rho(\theta + \frac{|u|^2}{2})\right\}_t + \left\{\rho u_1(\theta + \frac{|u|^2}{2}) + p u_1\right\}_x = \epsilon(\kappa(\theta)\theta_x)_x + \frac{4}{3}\epsilon(\mu(\theta) u_1 u_{1x})_x \\ \quad + \epsilon(\mu(\theta) u_2 u_{2x})_x + \epsilon(\mu(\theta) u_3 u_{3x})_x - \frac{1}{2}(\int_{\mathbb{R}^3} v_1 |v|^2 L_M^{-1} \Theta \, dv)_x. \end{cases}$$

$$\Theta := \epsilon \partial_t G + \epsilon P_1(v_1 \partial_x G) - Q(G, G).$$

$\kappa(\theta) > 0$: viscosity coefficient, $\mu(\theta) > 0$: heat-conductivity,

$$\mu(\theta) = -R\theta \int_{\mathbb{R}^3} \hat{B}_{ij}(\frac{v-u}{\sqrt{R\theta}}) B_{ij}(\frac{v-u}{\sqrt{R\theta}}) dv > 0, \quad i \neq j,$$

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To treat the integral terms, change $v_i v_j$ and $|v|^2 v_i$ to Burnett functions $m = m(v) \in \ker^{\perp}$, then

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Set

$$\begin{cases}\widetilde{\rho}=\rho(t,x)-\bar{\rho}(t,x),\\ \widetilde{u}=u(t,x)-\bar{u}(t,x),\\ \widetilde{\theta}=\theta(t,x)-\bar{\theta}(t,x),\\ \widetilde{G}=G(t,x,v)-\overline{G}(t,x,v),\quad \widetilde{G}=\sqrt{\mu}f(\tau,y,v)\ \text{(purely micro)}.\end{cases}$$

$$\overline{G}=\epsilon^{1-a}L_M^{-1}P_1v_1M\{\frac{|v-u|^2\bar{\theta}_y}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_y}{R\theta}\}.$$

$\mu=M_{[1,0,\frac{3}{2}]}(v)$ is the reference global Maxwellian such that

$$\begin{cases}\eta_0:=\sup\limits_{t\geq0,x\in\mathbb{R}}\{|\bar{\rho}(t,x)-1|+|\bar{u}(t,x)|+|\bar{\theta}(t,x)-\frac{3}{2}|\}\ \text{is small},\\ \frac{1}{2}\sup\limits_{t\geq0,x\in\mathbb{R}}\bar{\theta}(t,x)<\frac{3}{2}<\inf\limits_{t\geq0,x\in\mathbb{R}}\bar{\theta}(t,x).\end{cases}$$

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where $\delta > 0$ is a small constant depending on the Knudsen number ϵ. In fact, we will choose

$$\delta = \frac{1}{k}\epsilon^{\frac{3}{5} - \frac{2}{5}a}$$

for a suitably small constant $k > 0$ independent of ϵ. □

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\end{aligned}$$

To close the a priori assumption,

$$\epsilon^{1-\frac{2}{3}a+\frac{1}{6}q} \leq \epsilon^q, \quad \text{that is} \quad q \leq \frac{6}{5} - \frac{4}{5}a.$$

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1st key point —
The zero-order energy estimate on fluid part is based on the entropy and entropy flux, but the estimate related to source terms includes

$$\int_{\mathbb{R}} H_3 dy = \int_{\mathbb{R}^3} \Big\{ \frac{3}{2} (\frac{\widetilde{\theta}}{\theta})_y \int_{\mathbb{R}^3} (\frac{1}{2} v_1 |v|^2 - \sum_{i=1}^{3} u_i v_1 v_i) L_M^{-1} \Theta dv \Big\} dy$$
$$= \int_{\mathbb{R}^3} \Big\{ \frac{3}{2} (\frac{\widetilde{\theta}}{\theta})_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 (\frac{v-u}{\sqrt{R\theta}}) \frac{\Theta}{M} dv \Big\} dy.$$

Recall

$$\Theta = \epsilon^{1-a} \partial_\tau G + \epsilon^{1-a} P_1 (v_1 \partial_y G) - Q(G,G),$$

and

$$G = \overline{G} + \sqrt{\mu} f.$$

Notice that for any multi-index β and $m \geq 0$, we can prove the fast decay property that

$$\int_{\mathbb{R}^3} \frac{|\langle v \rangle^m \sqrt{\mu} \partial_\beta A_1 (\frac{v-u}{\sqrt{R\theta}})|^2}{M^2} dv \leq C.$$

2nd key point —

Micro dissipation estimate has to be based on the entropy inequality. We turn to the formulation around global Maxwellians by Guo:

$$
\begin{aligned}
&\partial_\tau f + v_1 \partial_y f - \boxed{\epsilon^{a-1}\mathcal{L} f} \\
&= \epsilon^{a-1}\Gamma(f, \frac{M-\mu}{\sqrt{\mu}}) + \epsilon^{a-1}\Gamma(\frac{M-\mu}{\sqrt{\mu}}, f) + \epsilon^{a-1}\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}) \\
&\quad + \frac{P_0(v_1\sqrt{\mu}\partial_y f)}{\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} P_1 v_1 M \{ \frac{|v-u|^2 \widetilde{\theta}_y}{2R\theta^2} + \frac{(v-u)\cdot \widetilde{u}_y}{R\theta} \} \\
&\quad - \frac{P_1(v_1 \partial_y \overline{G})}{\sqrt{\mu}} - \frac{\partial_\tau \overline{G}}{\sqrt{\mu}},
\end{aligned}
$$

where

$$
\Gamma(h, g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} h, \sqrt{\mu} g), \quad \mathcal{L} h = \Gamma(h, \sqrt{\mu}) + \Gamma(\sqrt{\mu}, h).
$$

Note: $\int (1, v, |v|^2)\sqrt{\mu} f = 0$, so $P_\mu f = 0$, i.e., f is purely kinetic (micro).

Notations and norms by Guo can be carried over to here:

To treat the degeneracy of dissipation in large velocity (no spectral gap), introduce

$$w = w(v) \equiv \langle v \rangle^{\gamma+2}, \quad \langle v \rangle = \sqrt{1+|v|^2}.$$

The Landau collision frequency is

$$\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v - v_*)\mu(v_*)\, dv_*, \quad 1 \leq i,j \leq 3, \ \gamma + 2 < 0.$$

Define the weighted dissipation norms as

$$|g|^2_{\sigma,\ell} \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w^{2\ell}\{\sigma^{ij}\partial_i g \partial_j g + \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2\}\, dv, \quad \|g\|^2_{\sigma,\ell} \equiv \int_{\mathbb{R}} |g|^2_{\sigma,\ell}\, dy.$$

Guo showed that

$$|g|_\sigma \approx \left|\langle v\rangle^{\frac{\gamma+2}{2}} g\right|_2 + \left|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v g \cdot \frac{v}{|v|}\right|_2 + \left|\langle v\rangle^{\frac{\gamma+2}{2}} \nabla_v g \times \frac{v}{|v|}\right|_2,$$

and $-\langle \mathcal{L}g, g\rangle \geq \sigma_1 |g|^2_\sigma$. Moreover, Guo showed:

Lemma

For any $\eta > 0$, there exists $C_\eta > 0$ such that

$$-\langle \partial_\beta \mathcal{L} g, w^{2|\beta|} \partial_\beta g \rangle \geq |\partial_\beta g|^2_{\sigma,|\beta|} - \eta \sum_{|\beta_1| \leq |\beta|} |\partial_{\beta_1} g|^2_{\sigma,|\beta_1|} - C_\eta |g|^2_\sigma.$$

Lemma

Let $\ell \geq 0$, then for any arbitrarily large constant $b > 0$,

$$|\langle \partial^\alpha \Gamma(g_1, g_2), \partial^\alpha g_3 \rangle| \lesssim \sum_{|\alpha_1| \leq |\alpha|} |\langle v \rangle^{-b} \partial^{\alpha_1} g_1|_2 |\partial^{\alpha-\alpha_1} g_2|_\sigma |\partial^\alpha g_3|_\sigma,$$

and

$$|\langle \partial^\alpha_\beta \Gamma(g_1, g_2), w^{2\ell} \partial^\alpha_\beta g_3 \rangle| \lesssim \sum_{|\alpha_1| \leq |\alpha|} \sum_{|\beta'| \leq |\beta_1| \leq |\beta|} |\langle v \rangle^{-b} \partial^{\alpha_1}_{\beta'} g_1|_2 |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\sigma,\ell} |\partial^\alpha_\beta g_3|_{\sigma,\ell}.$$

Moreover, the transport is well controlled: For $|\alpha| + |\beta| \leq N$ with $|\beta| \geq 1$,

$$\iint dvdy\, \partial^\alpha_\beta (v_1 \partial_y f) \cdot \partial^\alpha_\beta f \langle v \rangle^{2(\gamma+2)|\beta|}$$

Moreover, the transport is well controlled: For $|\alpha| + |\beta| \leq N$ with $|\beta| \geq 1$,

$$
\begin{aligned}
&\iint dvdy\, \partial_\beta^\alpha(v_1\partial_y f)\cdot \partial_\beta^\alpha f \langle v\rangle^{2(\gamma+2)|\beta|} \\
&= \iint dvdy\, \chi_{\beta_1\geq 1}\partial_y\partial_{\beta-e_1}^\alpha f\cdot \partial_\beta^\alpha f\langle v\rangle^{2(\gamma+2)|\beta|} \\
&= \iint dvdy\, \chi_{\beta_1\geq 1}\partial_y\partial_{\beta-e_1}^\alpha f\langle v\rangle^{(\gamma+2)|\beta-e_1|}\cdot \partial_\beta^\alpha f\langle v\rangle^{(\gamma+2)|\beta|}\cdot \langle v\rangle^{\gamma+2} \\
&\leq \eta\|\partial_\beta^\alpha f\|_{\sigma,|\beta|}^2 + \frac{C}{\eta}\|\partial_{\beta-e_1}^{\alpha+e_1} f\|_{\sigma,|\beta-e_1|}^2,
\end{aligned}
$$

thus, iteratively one need to obtain the dissipation

$$
\sum_{1\leq|\alpha|\leq N} \|\partial^\alpha f\|_{\sigma,0}^2.
$$

These motivate the energy functional

$$\begin{aligned}\mathcal{E}_2(\tau) = &\sum_{|\alpha|\le 1} \|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(\tau)\|^2 + \sum_{|\alpha|\le 1} \|\partial^\alpha f(\tau)\|^2 \\ &+ \epsilon^{2-2a} \sum_{|\alpha|=2} \{\|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(\tau)\|^2 + \|\partial^\alpha f(\tau)\|^2\} \\ &+ \sum_{|\alpha|+|\beta|\le 2, |\beta|\ge 1} \|\partial^\alpha_\beta f(\tau)\|^2_{2,|\beta|}.\end{aligned}$$

and the corresponding dissipation functional

$$\begin{aligned}\mathcal{D}_2(\tau) = &\epsilon^{1-a} \sum_{1\le|\alpha|\le 2} \|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(\tau)\|^2 + \epsilon^{1-a} \sum_{|\alpha|=2} \|\partial^\alpha f(\tau)\|^2_\sigma \\ &+ \epsilon^{a-1} \sum_{|\alpha|\le 1} \|\partial^\alpha f(\tau)\|^2_\sigma + \epsilon^{a-1} \sum_{|\alpha|+|\beta|\le 2, |\beta|\ge 1} \|\partial^\alpha_\beta f(\tau)\|^2_{\sigma,|\beta|}.\end{aligned}$$

Note: $f = \frac{1}{\sqrt{\mu}}(F - M - \bar{G})$ is purely micro.

Summary of energy estimates:

- macro part: Based on the fluid equations, use the Burnett functions.
- micro part: ∂^α_β, $|\alpha| + |\beta| \le 2$,
  - $|\beta| = 0$, $|\alpha| \le 1$: apply ∂^α to

$$\partial_\tau f + v_1 \partial_y f - \epsilon^{a-1} \mathcal{L} f = \cdots$$

    test it by $\partial^\alpha f$.
  - $|\beta| = 0$, $|\alpha| = 2$: apply ∂^α to

$$\partial_\tau (\frac{F}{\sqrt{\mu}}) + v_1 \partial_y (\frac{F}{\sqrt{\mu}}) - \epsilon^{a-1} \mathcal{L} f = \cdots$$

    test it by $\frac{\partial^\alpha F}{\sqrt{\mu}}$.
  - $|\beta| \ge 1$: Guo's velocity-derivative iteration technique... □

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- The proof of the dynamical stability is applicable to the two-species VPL system; see D.-Yu $(\mathrm{AdvM}\ 20)$,

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- Exploring a new weight function (more robust than $\exp\{q\langle v\rangle^2/(1+t)^\vartheta\}$ in D.-Yang-Zhao; explicit time-decay is not necessarily needed), we are able to prove the dynamical stability of viscous contact wave (cf. Huang-Xin-Yang for Boltzmann equation in cutoff hard potentials) for the Landau equation with Coulomb potentials; see D.-Yang-Yu (submitted), energy estimates can be closed without turning to time-decay of solutions.

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- Rate of convergence in ϵ can be obtained for the VPL system of ions flow where the electric potential is non-trivial in large time and can also have the high-oscillation stronger than grazing collisions (cf.: a recent work "*Long Wave Asymptotics for the Vlasov-Poisson-Landau Kinetic Equation*" by Bobylev-Potapenko); see D.-Yang-Yu (in progress).

Thank you!