Large stochastic systems of interacting particles.

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From very small to very large "particles"

Many-particle or multi-agent systems are used in a widespread range of applications

- Plasmas: Particles are ions or electrons.
- Astrophysics: Particles are dark matter particles, galaxies or galaxy clusters...
- Fluids: Point vortices, suspensions...
- Bio-mechanics: Medical aerosols in the respiratory tract, suspensions in the blood...
- Bio-Sciences: Collective behaviors of animals, swarming or flocking, but also dynamics of micro-organisms, chemotaxis, cell migration, neural networks...
- Social Sciences: Opinion dynamics, consensus formation...
- Economics: Mean-field games...

Very large particles: Galaxies

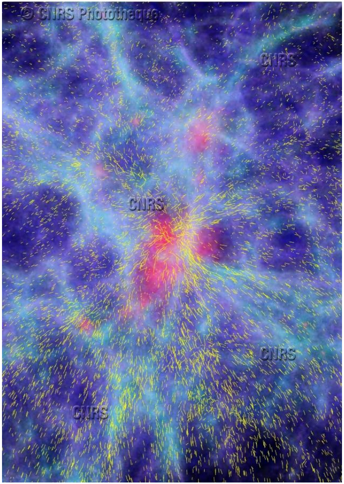

Figure: Credits: CNRS, France; Numerical simulation of the formation of large scale structures in the universe: Dynamics of galaxies moving to the central concentration.

Or very small: Biological neurons

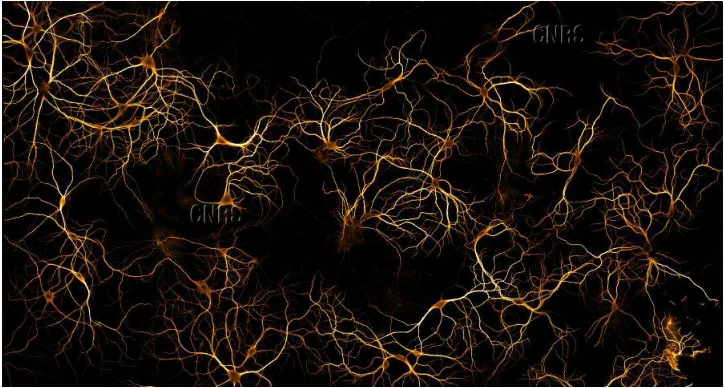

Figure: Credits: CNRS Bordeaux, France; 2D reconstruction of rat hippocampus, marked for cytoskeleton protein.

Interacting particles

Consider N particles, identical and interacting two by two through the kernel K. For $X_i(t) \in \Pi^d$ the position of the i-th particle,

$$d\,X_i = \frac{1}{N} \sum_{j=1}^{N} K(X_i - X_j)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

with the mean field scaling and for N independent Brownian motions W_i^t. For simplicity take $K(0) = 0$: No self-interaction.

- Main question: Behavior of the system for $N >> 1$.
- For simplicity in the talk, σ is fixed but the case $\sigma = \sigma_N$ is also of interest, with for example $\sigma \sim \frac{1}{N}$ playing a special role for Coulomb gases.

Some of the Existing Literature: The deterministic setting

The previous ideas have been considerably extended with some success in the deterministic case $\sigma = 0$:

- The Lipschitz case is still important to further understand the framework. See for example Golse 16, Golse-Mouhot-Ricci 13, Hauray-Mischler 14, Mischler-Mouhot 13...
- 2d incompressible Euler system in Goodman-Hou-Lowengrub 90, Schochet, with a general result by Hauray 09.
- Deterministic Riesz kernels recently in Duerinckx 16, Duerinckx-Serfaty 18 and Serfaty 19.
- 2nd order systems are less well understood: Hauray-Jabin 09 and 15 for $K(x) << |x|^{-1}$, Lazarovici and Pickl 17, Pickl 19.
- Singularity not at the origin: Carrillo-Choi-Hauray-Salem 18 for swarming models.
- Collisional models (Boltzmann) are hard: Lanford 75, and Bodineau-Gallagher-Saint-Raymond-Texier.

Some of the Existing Literature: The stochastic setting

In contrast, the stochastic case with $\sigma_N > 0$ is much less well understood

- Locally Lipschitz interactions in Bolley-Cañizo-Carrillo 11, Bossy-Faugeras-Talay 15.
- For 2d Navier-Stokes, if $K = \nabla^\perp V$, only qualitative convergence by Osada 85, Fournier-Hauray-Mischler 16.
- For the Patlak-Keller-Segel system, various attempts by Cattiaux-Pédèches 16, Godinh-Quininao 15, Haskovec-Schmeiser 11... Recently Fournier-Jourdain 17 proved some limit for $\lambda < 1$ but no propagation of chaos. See also Bolley-Chafaï-Fontbona 18 for the repulsive Keller-Segel.
- Recent result by Rosenzweig extending the Serfaty method to some stochastic settings.

Cells dynamics under chemotaxis

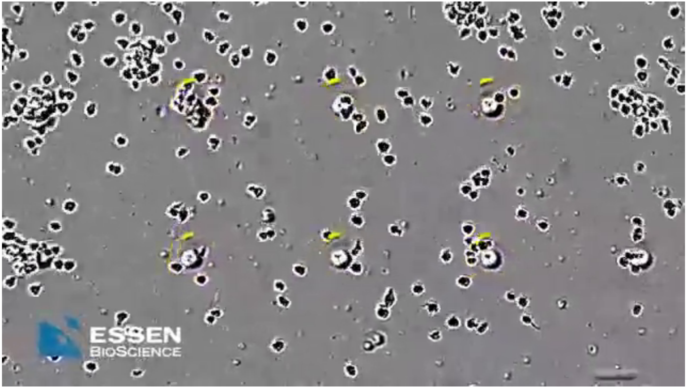

Figure: Credits: Essen Bio-Science from Labtube; Directional migration of Jurkat cells toward the chemo-attractant SDF1a, visualized on an IncuCyte ClearView 96-well Cell Migration Plate.

Elementary chemotaxis models for micro-organisms

Consider N micro-organisms following the gradient of the concentration $c(t,x)$ of some chemical. In the simplest model, their velocities solve

$$dX_i = \nabla c(t, X_i(t))\, dt + \sqrt{2\,\sigma}\, dW_i,$$

where the independent Wiener processes W_i may represent random changes in direction.
Assume now that the chemical is also produced by the organisms and diffuses fast:

$$-\Delta c = \frac{\alpha}{N} \sum_i \delta(x - X_i) + \text{possible source.}$$

$\longrightarrow$ Toy model from the biological point of view but captures the singularity of the interaction.

Elementary chemotaxis models for micro-organisms

Consider N micro-organisms following the gradient of the concentration $c(t,x)$ of some chemical. In the simplest model, their velocities solve

$$dX_i = -\frac{\lambda}{N}\sum_{j\neq i}\frac{X_i - X_j}{|X_i - X_j|^2}\,dt + \sqrt{2\,\sigma}\,dW_i,$$

where the independent Wiener processes W_i may represent random changes in direction.

Assume now that the chemical is also produced by the organisms and diffuses fast: In dimension 2

$$c(t,x) = -\frac{\lambda}{N}\sum_i \log|x - X_i| + S(t,x).$$

$\longrightarrow$ Toy model from the biological point of view but captures the singularity of the interaction.

The Patlak-Keller-Segel system

The mean-field limit is the well known Patlak-Keller-Segel system (1953 and 1970)

$$\begin{cases} \partial_t \bar{\rho} + \mathrm{div}\,(\bar{\rho}\, u) = \sigma\,\Delta\,\bar{\rho}, \\ u = \nabla c, \quad -\Delta c = 2\,\pi\,\lambda\,\bar{\rho}. \end{cases}$$

Again not a very accurate model of chemotaxis but a good prototype of what relevant models may look like.
Similar to the so-called Smoluchowski-Poisson equation in astrophysics, cf. Chandrasekhar 1943.

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Kernel with the same singularity as the Biot-Savart law but very different structure: Hamiltonian for Navier-Stokes vs singular attractive gradient flow.

$\longrightarrow$ Solutions may not exist for all times as the singular attractive interactions can lead to concentration: From Dolbeault-Perthame 2004 for example,

- Global existence if $\lambda \leq 4\,\sigma$ (or $\lambda \leq 2\,d\,\sigma$).
- Always blow-up if $\lambda > 4\,\sigma$.

A new statistical approach

Instead of looking at trajectories, we consider

$$\rho_N(t, x_1, \ldots, x_N) : \text{joint law of the positions } X_1(t), \ldots, X_N(t) \text{ at time } t.$$

It contains most of the statistical information on the system but not all the information: Correlations in time are lost and it may be difficult to reconstruct trajectories of the system.

Aim: Compare ρ_N with the tensorized

$$\bar{\rho}_N = \Pi_{i=1}^N \bar{\rho}(t, x_i) = \bar{\rho}^{\otimes N},$$

which is the joint law of the i.i.d. sequence $\bar{X}_i$, in terms of their observables or marginals:

$$\rho_{N,k} = \int_{\Pi^{d\,(N-k)}} \rho_N \, dx_{k+1} \ldots dx_N \longleftrightarrow \bar{\rho}_{N,k} = \bar{\rho}^{\otimes k}.$$

The Gibbs entropy is critical

We based our method on the scaled relative entropy

$$H_N(\rho_N|\bar{\rho}_N)(t) = \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N \log\frac{\rho_N}{\bar{\rho}_N}.$$

Thanks to the sub-additive nature of the entropy, it controls the marginals

$$\frac{1}{k}\int_{\Pi^{k\,d}} \rho_{N,k} \log\frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}} \leq \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N \log\frac{\rho_N}{\bar{\rho}_N}.$$

For fixed k, the Csiszár-Kullback-Pinsker inequality bounds

$$\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^1} \leq C\sqrt{k\,H_N(\rho_N|\bar{\rho}_N)(t)}.$$

It has the right initial scaling: If the X_i^0 are i.i.d. with law ρ^0 then

$$H_N(\rho_N|\bar{\rho}_N)(t=0) = \int_{\Pi^d} \rho^0 \log\frac{\rho^0}{\bar{\rho}^0}.$$

The relative entropy for many-particle systems

- Uses of the full relative entropy between trajectories: Ben Arous-Zeitouni 99 for smooth Langevin dynamics, and Ben Arous-Tannenbaum-Zeitouni 03, Fontbona-Jourdain 16.
- Some connections with Random Matrix Theory, Erdös-Yau 17.
- Closest to the method here is Yau 91 concerning the hydrodynamics of Ginzburg-Landau models.

How to modify the relative entropy approach

For gradient flows where $K = -\nabla\Phi$ such as the Patlak-Keller-Segel, we need to find the right object that still sees the advection part of the operator L_N in a anti-symmetrical manner.

We introduce a weighted relative entropy

$$E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{G_N(X^N)}\frac{G_{\bar\rho_N}(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N,$$

through the Gibbs equilibrium of the system, and its equivalent mean-field representation

$$G_N(t,X^N) = \exp\left(-\frac{1}{2N\sigma}\sum_{i\neq j}\Phi(x_i - x_j)\right),$$

$$G_{\bar\rho_N}(t,X^N) = \exp\left(-\frac{1}{\sigma}\sum_{i=1}^N \Phi\star\bar\rho(x_i) + \frac{N}{2\sigma}\int_{\Pi^d}\Phi\star\bar\rho\,\bar\rho\right).$$

Our new result

Consider even potentials $\Phi(-x) = \Phi(x)$, s.t.

- Any possibly singular potential $\Phi \in L^1(\Pi^d)$ with at most a mildly singular attractive part

$$\Phi(x) \geq -C - \lambda \log \frac{1}{|x|} \quad \text{for } \lambda < 2\, d\, \sigma, \tag{1}$$

 and some structure on the repulsive and potentially very singular part such as $\Phi \sim |x|^{-k}$.

- We can be more precise by asking $\Phi = \Phi_a + \Phi_r$ with

$$\hat{\Phi}_r \geq 0, \quad |\nabla_\xi \hat{\Phi}_r(\xi)| \leq C\, \frac{\hat{\Phi}_r(\xi)}{1+|\xi|} + \frac{C}{1+|\xi|^{d+1}}, \qquad |\nabla \Phi_a(x)| \leq \frac{C}{|x|}. \tag{2}$$

Our new result

Theorem

Assume $K = -\nabla\Phi$ *with* Φ *as above. Consider* $\bar\rho$ *a smooth enough solution with* $\inf \bar\rho > 0$*. There exists* $C > 0$ *and* $\theta > 0$ *s.t. for* $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$*, and for the joint law* ρ_N *on* Π^{dN} *of any entropy solution to the SDE system, for* σ *fixed*

$$H_N(t) + |E_N(t)| \le e^{C_{\bar\rho}\,\|K\|\,t} \left(H_N(t=0) + |E_N(t=0)| + \frac{C}{N^\theta} \right).$$

Hence if $H_N^0 + |E_N^0| \le C\,N^{-\theta}$*, for any fixed marginal* $\rho_{N,k}$

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i)\|_{L^1(\Pi^{k\,d})} \le C_{T,\bar\rho,k}\, N^{-\theta/2}.$$

A modified free energy

One may also write

$$E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \mathcal{H}_N(\rho_N|\bar\rho_N) + \mathcal{K}_N(G_N|G_{\bar\rho_N}),$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\right) dX^N$$

is exactly the relative entropy introduced in J.-Wang and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{G_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\right) dX^N$$

is the expectation of the modulated potential energy from Duerincx-Serfaty.

$\longrightarrow$ E_N is a modulated free energy for the system.

Propagating E_N

Because it is based on the free energy, E_N has the right algebraic structure with for any Φ even that

$$\frac{d}{dt}E_N\left(\frac{\rho_N}{G_N}\,\Big|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)\leq-\frac{\sigma}{N}\int_{\Pi^{d\,N}}d\rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N}-\nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2$$
$$-\frac{1}{2}\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}}\nabla\Phi(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x)-\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right)(d\mu_N-d\bar\rho)^{\otimes 2}d\rho_N,$$

where $\mu_N=\frac{1}{N}\sum_{i=1}^N\delta(x-x_i)$ is as before the empirical measure and where we denote

$$G_{\bar\rho}(t,x)=\exp\left(-\frac{1}{\sigma}\,V\star\bar\rho(x)+\frac{1}{2\,\sigma}\int_{\Pi^d}V\star\bar\rho\,\bar\rho\right).$$

Another large deviation inequality

Theorem

For $\Phi \geq -C - \lambda \log \frac{1}{|x|}$ with $\lambda < 2\,d\,\sigma$, define the functional

$$F_\eta(\mu_N) = -\frac{1}{2\,\sigma} \int_{\Pi^{2d} \cap \{x \neq y\}} \Phi(x-y)\, \mathbb{I}_{|x-y| \leq \eta}\, (d\bar\rho - d\mu_N)^{\otimes 2},$$

then there exists $\eta > 0$ s.t.

$$\frac{1}{N} \log Z_N = \frac{1}{N} \log \int_{\Pi^{dN}} \bar\rho_N\, e^{N\,\gamma\, F(\mu_N)}\, dX^N \leq \frac{C}{N^{\frac{1}{2\,(2d+1)}}}.$$

$\longrightarrow$ A delicate extension of the logarithmic Hardy, Littlewood, Sobolev inequality to remove the singular parts and then use a large deviation control of the type: For any $\lambda < 2\,d\,\sigma$

$$\int d\mu \log \frac{\mu}{\bar\rho} + \frac{\lambda}{2\,\sigma} \int_{0<|x-y|<\eta} \log|x-y|\, (d\mu - d\bar\rho)^{\otimes 2} \geq 0.$$

Conclusions

- Using the right physics is the key...
- The method provides a statistical control on large systems with a large class of attractive-repulsive interactions.
- We obtain explicit rates of convergence, which are optimal for point vortices but may not be for Keller-Segel.

Many open questions

- Systems with different structures: Non Hamiltonian, non gradient flows?
- Non-exchangeable systems, such as neuron networks?