Compressible Euler limit from Boltzmann equation with boundary

Ning Jiang
Wuhan University

Kinetic Equations: from Modeling, Computation to Analysis,
CIRM (Marseille Luminy, France), March 22nd, 2021

Collaborators

- Yi-Long Luo (South China University of Technology)
- Shaojun Tang (Wuhan University of Technology)

Boltzmann equation with Euler scaling

- Scaled Boltzmann equation

$$\begin{cases} \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} B(F_\epsilon, F_\epsilon) & \text{on } \mathbb{R}_+ \times \mathbb{R}^3_+ \times \mathbb{R}^3, \\ F_\epsilon(0, x, v) = F_\epsilon^{in}(x, v) \geq 0 & \text{on } \mathbb{R}^3_+ \times \mathbb{R}^3. \end{cases} \tag{0.1}$$

- $\epsilon > 0$: Knudsen number
- collision operator with hard potential

$$\begin{aligned} B(F_1, F_2)(v) = & \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma_0} F_1(u') F_2(v') b(\theta) \mathrm{d}\omega \mathrm{d}u \\ & - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma_0} F_1(u) F_2(v) b(\theta) \mathrm{d}\omega \mathrm{d}u \end{aligned} \tag{0.2}$$

  - $u' = u + [(v - u) \cdot \omega]\omega$, $v' = u + [(v - u) \cdot \omega]\omega$
  - $\cos\theta = (v - u) \cdot \omega / |v - u|$, $0 < b(\theta) \leq C|\cos\theta|$
  - $0 \leq \gamma_0 \leq 1$
  - Fluid limit: $F_\epsilon \to ?$ as $\epsilon \to 0$.

Fluid limits of Boltzmann equation

- Formally, as $\epsilon \to 0$, $F_\epsilon \to \mathfrak{M}$, the local Maxwellian,

$$\mathfrak{M}(t,x,v) := \frac{\rho(t,x)}{(2\pi T(t,x))^{\frac{3}{2}}} \exp\Big(- \frac{|v - \mathfrak{u}(t,x)|^2}{2T(t,x)} \Big),$$

  where $(\rho, \mathfrak{u}, T)$ satisfies the compressible Euler system. This is the most important and hardest problem in the field of hydrodynamic limit of Boltzmann equation.
- Mainly due to our poor understanding of compressible Euler system, this limit is far from well-understood.
- Classic result: R. Caflisch in early 80's, using Hilbert expansion method, short time (before singularities).

- Compared to the compressible Euler limit, the limits from Boltzmann equation to incompressible Navier-Stokes equations are much better understood. The most successful problem might be the BGL program: initialized from Bardos-Golse-Levermore (late 80's), and eventually completed by Golse-Saint-Raymond (2004, 2009). They worked in the context of DiPerna-Lions renormalized solutions.
- In the context of smooth solutions, results are even more: Bardos-Ukai, Guo, J-Xu-Zhao, M. Briant, Briant-Merino-Aceituno-Mouhot, Gallagher-Tristani...

Fluid limits with boundary

- In the domain with boundary, the fluid limits usually involve boundary layers: kinetic layer (and) fluid viscous layer, or sometimes called Knudsen layer and Prandtl layer.
- When do we need boundary layers: if the boundary condition of original equations and the limiting equations do not match.
- The boundary conditions of fluid equations usually are derived from the solvability conditions of the kinetic boundary layer equations.
- Kinetic boundary layer equations: Bardos, Caflisch, Nicolenako, Coron, Golse, Sulem, Perthame, Poupaud, Liu-Yu-Yang...
- From Boltzmann to incompressible Navier-Stokes: Masmoudi-Saint-Raymond (no boundary layer), J-Masmoudi (Knudsen+Pranttl layers)

Compressible limit with boundary

- Recently, two works:
  - Guo-Huang-Wang: Boltzmann equation with specular reflection boundary condition;
  - Jiang-Luo-Tang: Boltzmann equation with Maxwell reflection boundary condition.
- This is the topic of this talk.
- General ideas: Hilbert expansion, three terms: interior terms + viscous boundary layer terms + kinetic boundary layers terms.
- This idea was already used in Jiang-Masmoudi (CPAM 2017), but different format.
- This is another related work of Aoki-Golse-Jiang on acoustic limit from Boltzmann equation with incoming boundary condition.

Boltzmann equation with Euler scaling

- Maxwell reflection boundary condition

$$\gamma_- F_\epsilon = (1-\alpha_\epsilon) L\gamma_+ F_\epsilon + \alpha_\epsilon K\gamma_+ F_\epsilon \quad \text{on } \mathbb{R}_+ \times \Sigma_- \,. \qquad (0.3)$$

  - $\alpha_\epsilon \in [0,1]$: accommodation coefficient. Here we take $\alpha_\epsilon = \sqrt{2\pi}\epsilon^{\frac{1}{2}}$ as $\epsilon \to 0$.
  - specular-reflection
    $L\gamma_+ F_\epsilon(t,x,v) = F_\epsilon(t,x,R_x v)\,,\ R_x v = v - 2(v\cdot n)n = (\bar{v}, -v_3)\,.$
  - diffuse-reflection
    $K\gamma_+ F_\epsilon(t,x,v) = \sqrt{2\pi} M_w(t,\bar{x},v) \int_{v\cdot n>0} \gamma_+ F_\epsilon (v\cdot n)\mathrm{d}v,$
    where $M_w(t,\bar{x},v)$ is the local Maxwellian of the wall:

$$M_w(t,\bar{x},v) = \frac{\rho_w(t,\bar{x})}{[2\pi T_w(t,\bar{x})]^{\frac{3}{2}}} \exp\left\{ -\frac{|v-u_w(t,\bar{x})|^2}{2T_w(t,\bar{x})} \right\}. \qquad (0.4)$$

ρ_w, u_w, T_w are density, velocity and temperature of the boundary. We also assume $u_{w,3} = 0$, which denotes the boundary wall is fixed.

Notations (I)

- $n = (0, 0, -1)$: the outward normal of $\mathbb{R}^3_+$
- $\Sigma = \partial\mathbb{R}^3_+ \times \mathbb{R}^3$: the phase space boundary of $\mathbb{R}^3_+ \times \mathbb{R}^3$
- $\Sigma_+ = \{(x, v) : x \in \partial\mathbb{R}^3_+, v \cdot n = -v_3 > 0\}$: the outgoing boundary
- $\Sigma_- = \{(x, v) : x \in \partial\mathbb{R}^3_+, v \cdot n = -v_3 < 0\}$: the incoming boundary
- $\Sigma_0 = \{(x, v) : x \in \partial\mathbb{R}^3_+, v \cdot n = -v_3 = 0\}$: the grazing boundary
- $\gamma_\pm F = \mathbf{1}_{\Sigma_\pm} F$: the trace operator
- $\bar{u} = (u_1, u_2), \quad \forall\, u = (u_1, u_2, u_3) \in \mathbb{R}^3$
- $V^0 = V|_{x_3=0}, \quad \forall V = V(x)$
- $D_x V^0 = (D_x V)^0, \quad \forall$ derivative operator D_x
- For $G = G(t, \bar{x}, y, v)$, the Taylor expansion at $y = 0$

$$G = G^0 + \sum_{1 \le l \le N} \frac{y^l}{l!} G^{(l)} + \frac{y^{N+1}}{(N+1)!} \widetilde{G}^{(N+1)}, \tag{0.5}$$

$$G^{(l)} = \partial_y^l G^0, \; \widetilde{G}^{(N+1)} = (\partial_y^{N+1} G)(t, \bar{x}, \eta, v) \text{ for } \eta \in (0, y).$$

Notations (II)

- Local Maxwellian

$$\mathfrak{M}(t,x,v) := \frac{\rho(t,x)}{(2\pi T(t,x))^{\frac{3}{2}}} \exp\Big(- \frac{|v-\mathfrak{u}(t,x)|^2}{2T(t,x)} \Big). \tag{0.6}$$

- The linearized collision operator $\mathcal{L}$ is defined as

$$\mathcal{L}g = -\frac{1}{\sqrt{\mathfrak{M}}}\Big\{ B(\mathfrak{M}, \sqrt{\mathfrak{M}}g) + B(\sqrt{\mathfrak{M}}g, \mathfrak{M}) \Big\}.$$

- The null space $\mathcal{N}$ of $\mathcal{L}$ is spanned by

$$\frac{1}{\sqrt{\rho}}\sqrt{\mathfrak{M}}, \; \frac{v_i-\mathfrak{u}_i}{\sqrt{\rho T}}\sqrt{\mathfrak{M}}\;(i=1,2,3), \; \frac{1}{\sqrt{6\rho}}\Big\{\frac{|v-\mathfrak{u}|^2}{T}-3\Big\}\sqrt{\mathfrak{M}}.$$

Notations (III)

- The collision frequency $\nu(v) \equiv \nu(\mathfrak{M})(v)$:

$$\nu(\mathfrak{M}) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\theta)|v - v'|^{\gamma_0}\mathfrak{M}(v')\mathrm{d}v'\mathrm{d}\omega\,.$$

- For $0 \leq \gamma_0 \leq 1$,

$$\nu(\mathfrak{M}) \sim \rho\langle v\rangle^{\gamma_0}\,, \quad \langle v\rangle = \sqrt{1 + |v|^2}\,. \tag{0.7}$$

- $\mathcal{P} : L^2_v \mapsto \mathcal{N}$
- $\exists c_0 > 0$, s.t.

$$\langle \mathcal{L}g, g\rangle \geq c_0\|(\mathcal{I} - \mathcal{P})g\|^2_\nu\,. \tag{0.8}$$

Notations (IV)

- Norms for hyperbolic system:

$$\partial^{\alpha}_{t,\bar{x}} = \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\,,\ \forall \alpha = (\alpha_0,\alpha_1,\alpha_2) \in \mathbb{N}^3\,,$$

$$\|f(t)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} = \sum_{|\alpha|+i\le k} \|\partial^{\alpha}_{t,\bar{x}}\partial^i_{x_3} f(t)\|^2_{L^2(\mathbb{R}^3_+)}\,,$$

$$\|g(t)\|^2_{\mathcal{H}^k(\mathbb{R}^2)} = \sum_{|\alpha|\le k} \|\partial^{\alpha}_{t,\bar{x}} g(t)\|^2_{L^2(\mathbb{R}^2)}$$

- Norms for linear compressible Prandtl type equation (1): for $f = f(t,\bar{x},\zeta)$

$$\|f\|^2_{L^2_l} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^+} (1+\zeta)^l |f(t,\bar{x},\zeta)|^2 \mathrm{d}\bar{x}\mathrm{d}\zeta\,,$$

$$l_j = l + 2(r-j)\,,\ 0\le j\le r\,,\quad \beta = (\beta_1,\beta_2)\in\mathbb{N}^2\,,$$

$$\|f(t)\|^2_{l,r,n} = \sum_{2\gamma+|\beta|=r-n} \|\partial_t^{\gamma}\partial_{\bar{x}}^{\beta}\partial_{\zeta}^{n} f(t)\|^2_{L^2_{l_r}}\ (0\le n\le r)\,,$$

$$\|f(t)\|^2_{l,r} = \sum_{n=0}^{r} \|f(t)\|^2_{l,r,n} = \sum_{2\gamma+|\beta|+n=r} \|\partial_t^{\gamma}\partial_{\bar{x}}^{\beta}\partial_{\zeta}^{n} f(t)\|^2_{L^2_{l_r}}\,,$$

$$\|f(t)\|^2_{\mathbb{H}^r_{l,n}(\mathbb{R}^3_+)} = \sum_{j=0}^{r} \|f(t)\|^2_{l,j,n}\ (n=0,1,\cdots,r)\,,$$

$$\|f(t)\|^2_{\mathbb{H}^r_l(\mathbb{R}^3_+)} = \sum_{j=0}^{r} \|f(t)\|^2_{l,j} = \sum_{n=0}^{r} \|f(t)\|^2_{\mathbb{H}^r_{l,n}(\mathbb{R}^3_+)}\,.$$

Notations (V)

- Norms for linear compressible Prandtl type equation (2):

$$\|g\|^2_{\mathbb{H}^r_l(\mathbb{R}^3_+)} = \sum_{j=0}^{r} \sum_{|\beta|+n=j} \|\partial^\beta_{\bar{x}} \partial^n_\zeta g\|^2_{L^2_{l_j}},$$

$$\|h(t)\|^2_{\mathbb{H}^r(\mathbb{R}^2)} = \sum_{j=0}^{r} \|h(t)\|^2_{\Gamma,j} = \sum_{j=0}^{r} \sum_{2\gamma+|\beta|=j} \|\partial^\gamma_t \partial^\beta_{\bar{x}} h(t)\|^2_{L^2(\mathbb{R}^2)}$$

for $g = g(\bar{x}, \zeta)$ and $h = h(t, \bar{x})$.

- Define

$$f_{R,\epsilon} = \frac{F_{R,\epsilon}}{\sqrt{\mathfrak{M}}}, \quad h^\ell_{R,\epsilon} = \langle v \rangle^\ell \frac{F_{R,\epsilon}}{\sqrt{\mathfrak{M}_M}} \tag{0.9}$$

for $\ell \geq 9 - 2\gamma_0$, where the global Maxwellian $\mathfrak{M}_M = \mathfrak{M}_M(v)$ is

$$\mathfrak{M}_M = \frac{1}{(2\pi T_M)^{\frac{3}{2}}} \exp\left\{ -\frac{|v|^2}{2T_M} \right\}. \tag{0.10}$$

Here the constant T_M satisfies

$$T_M < \max_{t\in[0,\tau], x\in\mathbb{R}^3_+} T(t,x) < 2T_M. \tag{0.11}$$

Then there exists constants C_1, C_2 such that for some $\frac{1}{2} < z < 1$ and for each $(t, x, v) \in [0, \tau] \times \mathbb{R}^3_+ \times \mathbb{R}^3$, the following inequality holds:

$$C_1 \mathfrak{M}_M \leq \mathfrak{M} \leq C_2 (\mathfrak{M}_M)^z. \tag{0.12}$$

Main Results (I)

- Well-prepared initial data of scaled Boltzmann equation (0.1)

$$F_\epsilon(0,x,v) = \mathfrak{M}^{in}(x,v) + \sum_{k=1}^{5} \sqrt{\epsilon}^k \Big\{ F_k^{in}(x,v) + F_k^{b,in}(\bar{x}, \tfrac{x_3}{\sqrt{\epsilon}}, v) + F_k^{bb,in}(\bar{x}, \tfrac{x_3}{\epsilon}, v) \Big\} + \sqrt{\epsilon}^4 F_{R,\epsilon}^{in}(x,v) \geq 0\,, \tag{0.13}$$

$$\mathfrak{M}^{in}(x,v) = \frac{\rho^{in}(x)}{[2\pi T^{in}(x)]^{\frac{3}{2}}} \exp\Big\{ -\frac{|v-\mathfrak{u}^{in}(x)|^2}{2T^{in}(x)} \Big\}.$$

- Special form of local Maxwellian distribution $M_w(t,x)$

$$M_w(t,\bar{x},v) = \mathfrak{M}^0(t,\bar{x},v)\,. \tag{0.14}$$

Main Results (II)

Theorem

Consider the hard potential interaction $(0 \le \gamma_0 \le 1)$ *Boltzmann collision kernel* B *with an angular cutoff* (*see* (0.2)). *Let* $\ell \ge 9 - 2\gamma_0$, *and integers* s_0, s_k, s_k^b, s_k^{bb}, l_k^b $(1 \le k \le 3)$ *be sufficiently large. Assume that* $\|(\rho^{in}, \mathfrak{u}^{in}, T^{in})\|_{H^{s_0}(\mathbb{R}^3_+)} < \infty$ *and*

$$\mathcal{E}^{in} := \sum_{k=1}^{3} \Big\{ \|(\rho_k^{in}, u_k^{in}, \theta_k^{in})\|_{\mathcal{H}^{s_k}(\mathbb{R}^3_+)} + \|(\bar{u}_k^{b,in}, \theta_k^{b,in})\|_{\mathbb{H}^{s_k^b}_{l_k^b}(\mathbb{R}^3_+)} \Big\} < \infty . \qquad (0.15)$$

Let $(\rho, \mathfrak{u}, T)$ *be the solution to the compressible Euler equations* (0.18) *over the time interval* $t \in [0, \tau]$, *which determines the local Maxwellian* $\mathfrak{M}$ *defined in* (0.6). *For* $1 \le k \le 5$, *let* $F_k(t, x, v)$, $F_k^b(t, \bar{x}, \frac{x_3}{\sqrt{\epsilon}}, v)$ *and* $F_k^{bb}(t, \bar{x}, \frac{x_3}{\epsilon}, v)$ *be the known functions solved by the formal analysis. The local Maxwellian* $M_w(t, \bar{x}, v)$ *of the boundary is assumed as in* (0.14).

Main Results (III)

Theorem (Continued)

There is a small constant $\epsilon_0 > 0$ *such that if for* $\epsilon \in (0, \epsilon_0)$

$$\|\frac{F_{R,\epsilon}^{in}}{\sqrt{\mathfrak{M}^{in}}}\|_2 + \sqrt{\epsilon}^3 \|\langle v\rangle^\ell \frac{F_{R,\epsilon}^{in}}{\sqrt{\mathfrak{M}_M}}\|_\infty < \infty \,,$$

then the scaled Boltzmann equation (0.1) *with Maxwell reflection boundary condition* (0.3) *and well-prepared initial data* (0.13) *admits a unique solution for* $\epsilon \in (0, \epsilon_0)$ *over the time interval* $t \in [0, \tau]$ *with the expanded form* (0.47)*, i.e.,*

$$\begin{aligned} F_\epsilon(t,x,v) = \mathfrak{M}(t,x,v) + \sum_{k=1}^{5} \Big\{ & F_k(t,x,v) + F_k^b(t, \bar{x}, \tfrac{x_3}{\sqrt{\epsilon}}, v) \\ & + F_k^{bb}(t, \bar{x}, \tfrac{x_3}{\epsilon}, v) \Big\} + \sqrt{\epsilon}^4 F_{R,\epsilon}(t,x,v) \geq 0 \,, \end{aligned}$$

where the remainder $F_{R,\epsilon}(t,x,v)$ *satisfies*

$$\sup_{t\in[0,\tau]} \Big\{ \|\frac{F_{R,\epsilon}(t)}{\sqrt{\mathfrak{M}}}\|_2 + \sqrt{\epsilon}^3 \|\langle v\rangle^\ell \frac{F_{R,\epsilon}(t)}{\sqrt{\mathfrak{M}_M}}\|_\infty \Big\} \leq C(\tau, \|(\rho^{in}, \mathfrak{u}^{in}, T^{in})\|_{H^{s_0}(\mathbb{R}^3_+)}, \mathcal{E}^{in}) < \infty \,.$$

Formal Analysis

As inspired in [Caflisch-1980-CPAM], the **Hilbert expansion approach** is employed:

- Due to the thickness of viscous boundary layer is $\sqrt{\epsilon}$, and the accommodation coefficient $\alpha_\epsilon = O(\sqrt{\epsilon})$, we expand $F_\epsilon(t,x,v)$ by order $\sqrt{\epsilon}$.
- We remark that if $\alpha_\epsilon = O(\epsilon^\beta)$ with $\beta \neq \frac{1}{2}, 0$, the expansion will be more involved.
- Interior expansion:

$$F_\epsilon(t,x,v) \sim \sum_{k\geq 0} \sqrt{\epsilon}^k F_k(t,x,v) .$$

- Viscous boundary layer expansion:

$$F_\epsilon^b(t,\bar{x},\zeta) \sim \sum_{k\geq 1} \sqrt{\epsilon}^k F_k^b(t,\bar{x},\zeta,v) , \ \zeta = \frac{x_3}{\sqrt{\epsilon}} .$$

- Knudsen boundary layer expansion:

$$F_\epsilon^{bb}(t,\bar{x},\xi,v) \sim \sum_{k\geq 1} \sqrt{\epsilon}^k F_k^{bb}(t,\bar{x},\xi,v) , \ \xi = \frac{x_3}{\epsilon} .$$

Formal Analysis: Interior expansion (I)

Plug $F_\epsilon(t,x,v) \sim \sum_{k\geq 0} \sqrt{\epsilon}^k F_k(t,x,v)$ into (0.1):

$$\begin{aligned}
\sqrt{\epsilon}^{-2}: &\quad 0 = B(F_0,F_0)\,, \\
\sqrt{\epsilon}^{-1}: &\quad 0 = B(F_0,F_1) + B(F_1,F_0)\,, \\
\sqrt{\epsilon}^{0}: &\quad (\partial_t + v\cdot\nabla_x)F_0 = B(F_0,F_2) + B(F_2,F_0) + B(F_1,F_1)\,, \\
\sqrt{\epsilon}^{1}: &\quad (\partial_t + v\cdot\nabla_x)F_1 = B(F_0,F_3) + B(F_3,F_0) + B(F_1,F_2) + B(F_2,F_1)\,, \\
\cdots\cdots & \\
\sqrt{\epsilon}^{k}: &\quad (\partial_t + v\cdot\nabla_x)F_k = B(F_0,F_{k+2}) + B(F_{k+2},F_0) + \sum_{\substack{i+j=k+2,\\ i,j\geq 1}} B(F_i,F_j)\,.
\end{aligned} \tag{0.16}$$

H-theorem implies that F_0 must be a local Maxwellian:

$$F_0(t,x,v) = \mathfrak{M}(t,x,v)\,. \tag{0.17}$$

Formal Analysis: Interior expansion (III)

Split $O(\sqrt{\epsilon}^k)$ $(k \geq 0)$ of (0.16) into fluid part $\mathcal{N}$ and kinetic part $\mathcal{N}^\perp$:

- $k = 0$: $(\rho, \mathfrak{u}, T)$ satisfy the compressible Euler system

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho \mathfrak{u}) = 0\,, \\ \partial_t(\rho \mathfrak{u}) + \mathrm{div}_x(\rho \mathfrak{u} \otimes \mathfrak{u}) + \nabla p = 0\,, \\ \partial_t\Big[\rho\Big(\frac{3}{2}T + \frac{1}{2}|\mathfrak{u}|^2\Big)\Big] + \mathrm{div}_x\Big[\rho \mathfrak{u}\Big(\frac{3}{2}T + \frac{1}{2}|\mathfrak{u}|^2\Big)\Big] + \mathrm{div}_x(p\mathfrak{u}) = 0\,, \\ p = \rho T\,. \end{cases} \tag{0.18}$$

  - Impermeable boundary condition

$$\mathfrak{u} \cdot n|_{x_3=0} = -\mathfrak{u}_3|_{x_3=0} = -\mathfrak{u}_3^0 = 0\,. \tag{0.19}$$

    It will be formally derived latter by Maxwell reflection boundary condition.
  - Initial data

$$(\rho, \mathfrak{u}, T)(0, x) = (\rho^{in}, \mathfrak{u}^{in}, T^{in})(x) \tag{0.20}$$

    with compatibility condition $\mathfrak{u}^{in} \cdot n|_{x_3} = 0$.

Proposition (Local well-posedness of compressible Euler system: Schochet-1986-CMP)

Let $s_0 \geq 3$. Assume $(\rho^{in}, \mathfrak{u}^{in}, T^{in}) \in H^{s_0}(\mathbb{R}^3_+)$ satisfies

$$0 < \rho_\# \leq \rho^{in}(x) \leq \rho^\# \,, \quad 0 < T_\# \leq T^{in}(x) \leq T^\#$$

for some constants $\rho_\#, \rho^\#$, $T_\#$ and $T^\#$ with $\rho^\# \sqrt{T^\#} < \frac{1}{2\sqrt{2}}$. Then there is a $\tau > 0$ such that the compressible Euler system (0.18)-(0.19)-(0.20) admits a unique solution

$$(\rho, \mathfrak{u}, T) \in C\Big([0,\tau]; H^{s_0}(\mathbb{R}^3_+)\Big) \cap C^1\Big([0,\tau]; H^{s_0-1}(\mathbb{R}^3_+)\Big)$$

such that

$$0 < \tfrac{1}{2}\rho_\# \leq \rho(t,x) \leq 2\rho^\# \,, \quad 0 < \tfrac{1}{2} T_\# \leq T(t,x) \leq 2T^\#$$

hold for any $(t,x) \in [0,\tau] \times \mathbb{R}^3_+$. Moreover, the following estimate holds:

$$\|(\rho, \mathfrak{u}, T)\|_{C\left([0,\tau]; H^{s_0}(\mathbb{R}^3_+)\right) \cap C^1\left([0,\tau]; H^{s_0-1}(\mathbb{R}^3_+)\right)} \leq C_0 \,. \tag{0.21}$$

Here the constants τ, $C_0 > 0$ depend only on the H^{s_0}-norm of $(\rho^{in}, \mathfrak{u}^{in}, T^{in})$.

Formal Analysis: Interior expansion (IV)

- $k \geq 1$: $\frac{F_k}{\sqrt{\mathfrak{M}}} = \left\{ \frac{\rho_k}{\rho} + u_k \cdot \frac{v-\mathfrak{u}}{T} + \frac{\theta_k}{6T}\left(\frac{|v-\mathfrak{u}|^2}{T} - 3 \right) \right\} \sqrt{\mathfrak{M}} + (\mathcal{I} - \mathcal{P})\left(\frac{F_k}{\sqrt{\mathfrak{M}}} \right).$
  - Kinetic part
$$(\mathcal{I} - \mathcal{P})\Big(\frac{F_k}{\sqrt{\mathfrak{M}}}\Big) = \mathcal{L}^{-1}\Bigg(- \frac{(\partial_t + v\cdot\nabla_x)F_{k-2} - \sum_{\substack{i+j=k,\\ i,j\geq 1}} B(F_i,F_j)}{\sqrt{\mathfrak{M}}} \Bigg). \tag{0.22}$$
  - Fluid variables (ρ_k, u_k, θ_k) obey the following linear hyperbolic system
$$\begin{cases} \partial_t \rho_k + \mathrm{div}_x(\rho u_k + \rho_k \mathfrak{u}) = 0\,, \\ \rho\Big(\partial_t u_k + u_k \cdot \nabla_x \mathfrak{u} + \mathfrak{u}\cdot\nabla_x u_k\Big) \\ \qquad\qquad - \frac{\nabla_x(\rho T)}{\rho}\rho_k + \nabla_x\Big(\frac{\rho\theta_k + 3T\rho_k}{3}\Big) = \mathcal{F}_u^{\perp}(F_k)\,, \\ \rho\Big[\partial_t\theta_k + \mathfrak{u}\cdot\nabla_x\theta_k + \frac{2}{3}\Big(\theta_k \mathrm{div}_x \mathfrak{u} + 3T\mathrm{div}_x u_k\Big) \\ \qquad\qquad + 3u_k\cdot\nabla_x T\Big] = \mathcal{G}_\theta^{\perp}(F_k)\,, \end{cases} \tag{0.23}$$

    where $\mathcal{F}_u^{\perp}(F_k)$ and $\mathcal{G}_\theta^{\perp}(F_k)$ explicitly depend on $(\mathcal{I} - \mathcal{P})\Big(\frac{F_k}{\sqrt{\mathfrak{M}}}\Big)$.
  - The initial data of (0.23) are imposed on
$$(\rho_k, u_k, \theta_k)(0,x) = (\rho_k^{in}, u_k^{in}, \theta_k^{in})(x) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}\,. \tag{0.24}$$

Formal Analysis: Viscous boundary layer expansion (I)

Viscous boundary layer

$$F^b_\epsilon(t,\bar{x},\zeta) \sim \sum_{k\geq 1} \sqrt{\epsilon}^k F^b_k(t,\bar{x},\zeta,v)\,, \tag{0.25}$$

with the far field condition

$$F^b_k(t,\bar{x},\zeta,v) \to 0\,, \quad \text{as}\,\zeta \to +\infty\,. \tag{0.26}$$

Plug $F_\epsilon + F^b_\epsilon$ into (0.1):

$$\sqrt{\epsilon}^{-1}: \quad 0 = B(\mathfrak{M}^0, F^b_1) + B(F^b_1, \mathfrak{M}^0)\,,$$

$$\begin{aligned}\sqrt{\epsilon}^{0}: \quad v_3\cdot\partial_\zeta F^b_1 = &\Big[B(\mathfrak{M}^0, F^b_2) + B(F^b_2, \mathfrak{M}^0)\Big] + [B(F^0_1, F^b_1) + B(F^b_1, F^0_1)] \\ &+ B(F^b_1, F^b_1) + \zeta\Big[B(\mathfrak{M}^{(1)}, F^b_1) + B(F^b_1, \mathfrak{M}^{(1)})\Big],\end{aligned}$$

$$\begin{aligned}\sqrt{\epsilon}: \quad \partial_t F^b_1 + \bar{v}\cdot\nabla_{\bar{x}} F^b_1 + v_3\cdot\partial_\zeta F^b_2 = &\Big[B(\mathfrak{M}^0, F^b_3) + B(F^b_3, \mathfrak{M}^0)\Big] \\ &+ \tfrac{\zeta}{1!}\Big[B(\mathfrak{M}^{(1)}, F^b_2) + B(F^b_2, \mathfrak{M}^{(1)})\Big] + \tfrac{\zeta^2}{2!}\Big[B(\mathfrak{M}^{(2)}, F^b_1) + B(F^b_1, \mathfrak{M}^{(2)})\Big] \\ &+ \Big[B(F^0_1, F^b_2) + B(F^b_2, F^0_1)\Big] + \Big[B(F^0_2, F^b_1) + B(F^b_1, F^0_2)\Big] \\ &+ \tfrac{\zeta}{1!}\Big[B(F^{(1)}_1, F^b_1) + B(F^b_1, F^{(1)}_1)\Big] + \Big[B(F^b_1, F^b_2) + B(F^b_2, F^b_1)\Big],\end{aligned}$$

$\cdots\cdots$

Formal Analysis: Viscous boundary layer expansion (II)

$$\begin{aligned}\sqrt{\epsilon}^k: \quad & \partial_t F_k^b + \bar{v}\cdot\nabla_{\bar{x}} F_k^b + v_3\cdot\partial_\zeta F_{k+1}^b \\ &= \sum_{\substack{i+j=k+2,\\ i,j\geq 1}} B(F_i^b, F_j^b) + \Big[B(\mathfrak{M}^0, F_{k+2}^b) + B(F_{k+2}^b, \mathfrak{M}^0)\Big] \\ &\quad + \sum_{\substack{l+j=k+2,\\ 1\leq l\leq N, j\geq 1}} \frac{\zeta^l}{l!}\Big[B(\mathfrak{M}^{(l)}, F_j^b) + B(F_j^b, \mathfrak{M}^{(l)})\Big] \\ &\quad + \sum_{\substack{i+j=k+2,\\ i,j\geq 1}} \Big[B(F_i^0, F_j^b) + B(F_j^b, F_i^0)\Big] \\ &\quad + \sum_{\substack{i+j+l=k+2,\\ 1\leq l\leq N, i,j\geq 1}} \frac{\zeta^l}{l!}\Big[B(F_i^{(l)}, F_j^b) + B(F_j^b, F_i^{(l)})\Big] \quad \text{for } k\geq 1, \end{aligned} \tag{0.27}$$

where the following Taylor expansions at $x_3 = 0$ are used:

$$\mathfrak{M} = \mathfrak{M}^0 + \sum_{1\leq l\leq N} \frac{\zeta^l}{l!}\mathfrak{M}^{(l)} + \frac{\zeta^{N+1}}{(N+1)!}\widetilde{\mathfrak{M}}^{(N+1)},$$

$$F_i = F_i^0 + \sum_{1\leq l\leq N} \frac{\zeta^l}{l!} F_i^{(l)} + \frac{\zeta^{N+1}}{(N+1)!}\widetilde{F}_i^{(N+1)}.$$

Here the number $N \in \mathbb{N}_+$ will be chosen later while truncating the expansion.

Let

$$f_k^b = \frac{F_k^b}{\sqrt{\mathfrak{M}^0}}\,, \tag{0.28}$$

which can be decomposed as

$$\begin{aligned} f_k^b =& \mathcal{P}^0 f_k^b + (\mathcal{I} - \mathcal{P}^0) f_k^b \\ =& \Big\{ \frac{\rho_k^b}{\rho^0} + u_k^b \cdot \frac{v - \mathfrak{u}^0}{T^0} + \frac{\theta_k^b}{6T^0} \Big(\frac{|v - \mathfrak{u}^0|^2}{T^0} - 3 \Big) \Big\} \sqrt{\mathfrak{M}^0} + (\mathcal{I} - \mathcal{P}^0) f_k^b \,. \end{aligned}$$

Here $u_k^b = (u_{k,1}^b, u_{k,2}^b, u_{k,3}^b) \in \mathbb{R}^3$. Furthermore, let

$$p_k^b = \frac{\rho^0 \theta_k^b + 3T^0 \rho_k^b}{3}\,. \tag{0.29}$$

Formal Analysis: Viscous boundary layer expansion (IV)

- $u^b_{1,3}(t,\bar{x},\zeta) \equiv 0\,,\ p^b_1(t,\bar{x},\zeta) \equiv 0\,,\ \forall (t,\bar{x},\zeta) \in [0,\tau] \times \mathbb{R}^2 \times \mathbb{R}_+$
- $(\mathcal{I}-\mathcal{P}^0) f^b_{k+1}$ $(k \geq 1)$ satisfies

$$(\mathcal{I}-\mathcal{P}^0) f^b_{k+1} = (\mathcal{L}^0)^{-1}\Big\{ -(\mathcal{I}-\mathcal{P}^0)(v_3 \partial_\zeta \mathcal{P}^0 f^b_k) + \tfrac{\zeta}{\sqrt{\mathfrak{M}^0}}\Big[B(\mathfrak{M}^{(1)},\ \sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k) + B(\sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k, \mathfrak{M}^{(1)})\Big] + \tfrac{1}{\sqrt{\mathfrak{M}^0}}\Big[B(F^0_1,\ \sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k) + B(\sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k, F^0_1)\Big] + \tfrac{1}{\sqrt{\mathfrak{M}^0}}\Big[B(\sqrt{\mathfrak{M}^0} f^b_1,\ \sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k) + B(\sqrt{\mathfrak{M}^0}\mathcal{P}^0 f^b_k,\ \sqrt{\mathfrak{M}^0} f^b_1)\Big]\Big\} + J^b_{k-1} \tag{0.30}$$

- $u^b_{k+1,3}$ $(k \geq 1)$ satisfy

$$\begin{cases} \partial_\zeta u^b_{k+1,3} = -\dfrac{1}{\rho^0}(\partial_t \rho^b_k + \mathrm{div}_{\bar{x}}(\rho^0 \bar{u}^b_k + \rho^b_k \bar{\mathfrak{u}}^0))\,, \\ \lim\limits_{\zeta\to\infty} u^b_{k+1,3}(t,\bar{x},\zeta) = 0\,, \end{cases} \tag{0.31}$$

- p^b_{k+1} $(k \geq 1)$ satisfy

$$\begin{cases} \partial_\zeta p^b_{k+1} = -\rho^0 \partial_t u^b_{k,3} - \rho^0 \bar{\mathfrak{u}}^0 \cdot \nabla_{\bar{x}} u^b_{k,3} + \rho^0 \partial_{x_3} \mathfrak{u}^0_3 u^b_{k,3} + \tfrac{4}{3}\mu(T^0)\partial_{\zeta\zeta} u^b_{k,3} \\ \qquad -\tfrac{4}{3}\rho^0 \partial_\zeta\Big[\Big(\partial_{x_3}\mathfrak{u}^0 \zeta + u^0_{1,3}\Big) u^b_{k,3}\Big] - \partial_\zeta \langle T^0 \mathrm{A}^0_{33}, J^b_{k-1}\rangle + W^b_{k-1,3}\,, \\ \lim\limits_{\zeta\to\infty} p^b_{k+1}(t,\bar{x},\zeta) = 0\,, \end{cases} \tag{0.32}$$

Formal Analysis: Viscous boundary layer expansion (V)

- $(u^b_{k,1}, u^b_{k,2}, \theta^b_k)$ $(k \geq 1)$ satisfy the following *linear compressible Prandtl-type* equations

$$\begin{cases} \rho^0(\partial_t + \bar{\mathfrak{u}}^0 \cdot \nabla_{\bar{x}})u^b_{k,i} + \rho^0(\partial_{x_3}\mathfrak{u}^0_3\zeta + u^0_{1,3})\partial_\zeta u^b_{k,i} + \rho^0\bar{u}^b_k \cdot \nabla_{\bar{x}}\mathfrak{u}^0_i + \frac{\partial_{x_3}p^0}{3T^0}\theta^b_k \\ \qquad = \mu(T^0)\partial^2_\zeta u^b_{k,i} + \mathfrak{f}^b_{k-1,i}\ (i=1,2)\,, \\ \rho^0\partial_t\theta^b_k + \rho^0\bar{\mathfrak{u}}^0 \cdot \nabla_{\bar{x}}\theta^b_k + \rho^0\Big(\partial_{x_3}\mathfrak{u}^0\zeta + \mathfrak{u}^0_{1,3}\Big)\partial_\zeta\theta^b_k + \frac{2}{3}\rho^0 \mathrm{div}_x\mathfrak{u}^0\theta^b_k \\ \qquad = \frac{3}{5}\kappa(T^0)\partial_{\zeta\zeta}\theta^b_k + \mathfrak{g}^b_{k-1}\,, \\ \lim_{\zeta\to\infty}(\bar{u}^b_k, \theta^b_k)(t,\bar{x},\zeta) = 0\,, \end{cases} \tag{0.33}$$

- The all symbols $\mathfrak{f}^b_{k-1,i}$ $(i=1,2)$, $\mathfrak{g}^b_{k-1}$, $W^b_{k-1,i}$ $(i=1,2,3)$, H^b_{k-1} and J^b_{k-1} before explicitly depend on $f_i = \frac{F_i}{\sqrt{\mathfrak{M}}}$ and f^b_i $(i \leq k-1)$.
- The initial conditions of (0.33) are imposed on

$$(\bar{u}^b_k, \theta^b_k)(0,\bar{x},\zeta) = (\bar{u}^{b,in}_k, \theta^{b,in}_k)(\bar{x},\zeta) \in \mathbb{R}^2 \times \mathbb{R}, \quad k = 1,2,3,\cdots \tag{0.34}$$

with $\lim_{\zeta\to\infty}(\bar{u}^{b,in}_k, \theta^{b,in}_k)(\bar{x},\zeta) = 0$.

Formal Analysis: Knudsen boundary layer expansion (I)

Knudsen boundary layer

$$F_\epsilon^{bb}(t,\bar{x},\xi,v) \sim \sum_{k\geq 1} \sqrt{\epsilon}^k F_k^{bb}(t,\bar{x},\xi,v) \tag{0.35}$$

with far field condition

$$\lim_{\xi\to\infty} F_k^{bb}(t,\bar{x},\xi,v) = 0\,.$$

Let $f_k^{bb} = \frac{F_k^{bb}}{\sqrt{\mathfrak{M}^0}}$.

Plug $F_\epsilon + F_\epsilon^b + F_\epsilon^{bb}$ in (0.1):

$$v_3 \partial_\xi f_k^{bb} + \mathcal{L}^0 f_k^{bb} = S_k^{bb}\,, \quad k \geq 1\,, \tag{0.36}$$

where $S_k^{bb} = S_{k,1}^{bb} + S_{k,2}^{bb}$ with

$$\begin{aligned} S_{k,1}^{bb} &= -\mathcal{P}^0 \left\{ \frac{(\partial_t + \bar{v}\cdot\nabla_{\bar{x}}) F_{k-2}^{bb}}{\sqrt{\mathfrak{M}^0}} \right\} \in \mathcal{N}^0\,, \\ S_{k,2}^{bb} &= S_{k,2}^{bb}(f_i, f_i^b, f_i^{bb}; i \leq k-1) \in (\mathcal{N}^0)^\perp\,. \end{aligned} \tag{0.37}$$

Lemma (Bardos-Caflisch-Nicolaenko-1986-CPAM)

Assume that

$$S^{bb}_{k,1} = \left\{a_k + b_k \cdot (v - \mathfrak{u}^0) + c_k |v - \mathfrak{u}^0|^2\right\} \sqrt{\mathfrak{M}^0} \in \mathcal{N}^0 \,.$$

Then there exists a function

$$f^{bb}_{k,1} = \left\{\Psi_k v_3 + \Phi_{k,1} v_3 (v_1 - \mathfrak{u}^0_1) + \Phi_{k,2} v_3 (v_2 - \mathfrak{u}^0_2) + \Phi_{k,3} + \Theta_k v_3 |v - \mathfrak{u}^0|^2\right\} \sqrt{\mathfrak{M}^0}$$

such that

$$v_3 \partial_\xi f^{bb}_{k,1} - S^{bb}_{k,1} \in (\mathcal{N}^0)^\perp \,,$$

where

$$\begin{aligned} \Psi_k(t,\bar{x},\xi) &= -\int_\xi^{+\infty} \Big(\tfrac{2}{T^0} a_k + 3c_k\Big)(t,\bar{x},s) \mathrm{d}s \,, \\ \Phi_{k,i}(t,\bar{x},\xi) &= -\int_\xi^{+\infty} \tfrac{1}{T^0} b_{k,i}(t,\bar{x},s) \mathrm{d}s \,, \ i = 1,2 \,, \\ \Phi_{k,3}(t,\bar{x},\xi) &= -\int_\xi^{+\infty} b_{k,3}(t,\bar{x},s) \mathrm{d}s \,, \\ \Theta_k(t,\bar{x},\xi) &= \tfrac{1}{5(T^0)^2} \int_\xi^{+\infty} a_k(t,\bar{x},s) \mathrm{d}s \,. \end{aligned} \tag{0.38}$$

Formal Analysis: Knudsen boundary layer expansion (II)

Let $f_{k,2}^{bb} = f_k^{bb} - f_{k,1}^{bb}$. Then

$$\begin{aligned} v_3\partial_\xi f_{k,2}^{bb} + \mathcal{L}^0 f_{k,2}^{bb} = S_{k,2}^{bb} - (v_3\partial_\xi f_{k,1}^{bb} + \mathcal{L}^0 f_{k,1}^{bb} - S_{k,1}^{bb}) \in (\mathcal{N}^0)^\perp \,, \\ \lim_{\xi\to\infty} f_{k,2}^{bb}(t,\bar{x},\xi,v) = 0 \,. \end{aligned} \tag{0.39}$$

Once we impose the following boundary condition on (0.39):

$$f_{k,2}^{bb}(t,\bar{x},0,\bar{v},v_3)|_{v_3>0} = f_{k,2}^{bb}(t,\bar{x},0,\bar{v},-v_3) + \mathbf{f}_k(t,\bar{x},\bar{v},-v_3) \tag{0.40}$$

for some function $\mathbf{f}_k(t,\bar{x},\bar{v},v_3)$ only defined for $v_3 < 0$ and extended to be 0 for $v_3 > 0$, [Golse-Perthame-Sulem-1988-ARMA] proved that the solvability conditions of (0.39)-(0.40) were

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \bar{v} - \bar{\mathfrak{u}}^0 \\ |v - \mathfrak{u}^0|^2 \end{pmatrix} v_3 \mathbf{f}_k(t,\bar{x},v)\sqrt{\mathfrak{M}^0}\,\mathrm{d}v \equiv 0 \,. \tag{0.41}$$

Formal Analysis: Expansion of Maxwell BC (I)

- Plug $F_\epsilon + F_\epsilon^b + F_\epsilon^{bb}$ into Maxwell BC (0.3) on Σ_-,

$$
\begin{aligned}
&\sqrt{\epsilon}^0 : \quad L^R \mathfrak{M} = 0\,,\\
&\sqrt{\epsilon}^1 : \quad L^R(F_1 + F_1^b + F_1^{bb}) = L^D \mathfrak{M}\,,\\
&\quad \cdots\cdots\\
&\sqrt{\epsilon}^k : \quad L^R(F_k + F_k^b + F_k^{bb}) = L^D(F_{k-1} + F_{k-1}^b + F_{k-1}^{bb})\ (k \geq 2)\,,
\end{aligned}
\tag{0.42}
$$

where the operators L^R and L^D are defined as

$$
L^R F = (\gamma_- - L\gamma_+)F\,, \quad L^D F = \sqrt{2\pi}(K\gamma_+ - L\gamma_+)F\,.
$$

- $O(\sqrt{\epsilon}^0)$ implies impermeable BC (0.19), i.e., $\mathfrak{u}_3^0 = 0$.
- $O(\sqrt{\epsilon}^k)$ $(k \geq 1)$: The functions $\mathbf{f}_k(t, \bar{x}, \bar{v}, v_3)$ in BC (0.40) of Knudsen boundary layer equation (0.39) are

$$
\mathbf{f}_k(t,\bar{x},\bar{v},v_3) = \begin{cases} 0\,, \quad \text{if } v_3 > 0\,,\\ (f_k + f_k^b + f_{k,1}^{bb})(t,\bar{x},0,\bar{v},v_3) - (f_k + f_k^b + f_{k,1}^{bb})(t,\bar{x},0,\bar{v},-v_3)\\ \qquad + \sqrt{2\pi}\Big\{\Big[\langle\gamma_+(f_{k-1} + f_{k-1}^b + f_{k-1}^{bb})\rangle_{\partial\mathbb{R}^3_+}\sqrt{\mathfrak{M}^0}\Big]\\ \qquad\qquad -(f_{k-1} + f_{k-1}^b + f_{k-1}^{bb})\Big\}(t,\bar{x},0,\bar{v},v_3)\,, \text{ if } v_3 < 0\,, \end{cases}
\tag{0.43}
$$

$$
\langle\gamma_+ f\rangle_{\partial\mathbb{R}^3_+} := \sqrt{2\pi}\frac{M_w(v)}{\mathfrak{M}^0}\int_{v\cdot n(x)>0} v\cdot n(x)(\gamma_+ f)\sqrt{\mathfrak{M}^0}\,\mathrm{d}v\,.
$$

Formal Analysis: Expansion of Maxwell BC (II)

Lemma

Let the local Maxwellian of the boundary $M_w = \mathfrak{M}^0$ *in* (0.3) *and* $\mathbf{f}_k(t, \bar{x}, \bar{v}, v_3)$ *be given in* (0.43). *Then the solvability conditions* (0.41) *of the Knudsen boundary layer problem* (0.39)-(0.40) *imply that for* $k \geq 1$, *the linear hyperbolic system* (0.23) *has the following slip boundary condition*

$$u_{k,3}(t, \bar{x}, 0) = d_k(t, \bar{x}) , \tag{0.44}$$

and for $k \geq 2$, *the linear Prandtl-type equations* (0.33) *are of the Robin-type boundary conditions*

$$\begin{cases} \Big(\partial_\zeta u^b_{k-1,i} - \frac{\rho^0 \sqrt{T^0}(2+\rho^0 \sqrt{T^0})}{\mu(T^0)} u^b_{k-1,i}\Big)(t, \bar{x}, 0) = \Lambda^b_{k-1,i}(t, \bar{x}), i = 1, 2 , \\ \Big(\partial_\zeta \theta^b_{k-1} - \frac{\rho^0 \sqrt{T^0}}{\kappa(T^0)} (2\rho^0 \sqrt{T^0} + \frac{\sqrt{2\pi}}{3}\rho^0 + \frac{2}{3})\theta^b_{k-1}\Big)(t, \bar{x}, 0) = \Lambda^b_{k-1,\theta}(t, \bar{x}) , \end{cases} \tag{0.45}$$

where

$$\begin{aligned} d_k(t, \bar{x}) &= d_k(f_j, f^b_j, f^{bb}_j; j \leq k-1)(t, \bar{x}, 0) , \\ \Lambda^b_{k-1,i}(t, \bar{x}) &= \Lambda^b_{k-1,i}(f_j, f^b_j, f^{bb}_j; j \leq k-2)(t, \bar{x}, 0) \ (i = 1, 2) , \\ \Lambda^b_{k-1,\theta}(t, \bar{x}) &= \Lambda^b_{k-1,\theta}(f_j, f^b_j, f^{bb}_j; j \leq k-2)(t, \bar{x}, 0) \end{aligned}$$

are all explicitly computable. In particular, if $k = 1$ *in* (0.44), *we have*

$$u^0_{1,3} = u_{1,3}(t, \bar{x}, 0) = \sqrt{T^0}(\rho^0 \sqrt{T^0} + 1) . \tag{0.46}$$

Formal Analysis: Truncation of expansion (I)

- Truncated form:

$$F_\epsilon(t,x,v) = \mathfrak{M}(t,x,v) + \sum_{k=1}^{5} \Big\{ F_k(t,x,v) + F_k^b(t,\bar{x},\tfrac{x_3}{\sqrt{\epsilon}},v) + F_k^{bb}(t,\bar{x},\tfrac{x_3}{\epsilon},v) \Big\} + \sqrt{\epsilon}^4 F_{R,\epsilon}(t,x,v) \geq 0\,. \tag{0.47}$$

- $\mathfrak{M}, F_k, F_k^b, F_k^{bb}$ $(k = 1, 2, 3)$ are completely determined
- F_4, F_4^b, F_4^{bb}: only completely solve the kinetic part and fluid parts vanish
- F_5, F_5^b, F_5^{bb}: only partially solve the kinetic part and fluid parts vanish

Formal Analysis: Truncation of expansion (II)

- Remainder equation

$$\partial_t F_{R,\epsilon} + v \cdot \nabla_x F_{R,\epsilon} - \frac{1}{\epsilon}[B(\mathfrak{M}, F_{R,\epsilon}) + B(F_{R,\epsilon}, \mathfrak{M})]$$
$$= \sqrt{\epsilon}^2 B(F_{R,\epsilon}, F_{R,\epsilon}) + R_\epsilon + R_\epsilon^b + R_\epsilon^{bb}$$
$$+ \sum_{i=1}^{5} \sqrt{\epsilon}^{i-2}[B(F_i + F_i^b + F_i^{bb}, F_{R,\epsilon}) + B(F_{R,\epsilon}, F_i + F_i^b + F_i^{bb})] \tag{0.48}$$

with Maxwell reflection type boundary condition

$$\gamma_- F_{R,\epsilon} = (1 - \alpha_\epsilon) L\gamma_+ F_{R,\epsilon} + \alpha_\epsilon K \gamma_+ F_{R,\epsilon} + \sqrt{\epsilon}^2 \Gamma_\epsilon \quad \text{on } \Sigma_- , \tag{0.49}$$

where

$$\Gamma_\epsilon = \frac{\alpha_\epsilon}{\sqrt{\epsilon}}(K\gamma_+ - L\gamma_+)(F_5 + F_5^b + F_5^{bb}) , \tag{0.50}$$

and $(R_\epsilon, R_\epsilon^b, R_\epsilon^{bb}) = (R_\epsilon, R_\epsilon^b, R_\epsilon^{bb})(\mathfrak{M}, F_k, F_k^b, F_k^{bb}; 1 \le k \le 5)$ are explicitly computable.

- Initial data of (0.48):

$$F_{R,\epsilon}(0, x, v) = F_{R,\epsilon}^{in}(x, v) , \tag{0.51}$$

which satisfies the compatibility condition

$$\gamma_- F_{R,\epsilon}^{in} = (1 - \alpha_\epsilon) L\gamma_+ F_{R,\epsilon}^{in} + \alpha_\epsilon K \gamma_+ F_{R,\epsilon}^{in} + \sqrt{\epsilon}^2 \Gamma_\epsilon|_{t=0}$$

on Σ_-.

- Let $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t,x)$ satisfy the following linear hyperbolic system:

$$\begin{cases} \partial_t \tilde{\rho} + \mathrm{div}_x(\rho \tilde{u} + \tilde{\rho} \mathfrak{u}) = 0\,, \\ \rho(\partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \mathfrak{u} + \mathfrak{u} \cdot \nabla_x \tilde{u}) - \frac{\nabla_x p}{\rho} \tilde{\rho} + \nabla\Big(\frac{\rho\tilde{\theta} + 3T\tilde{\rho}}{3}\Big) = f\,, \\ \rho\Big[\partial_t \tilde{\theta} + \frac{2}{3}\Big(\tilde{\theta}\mathrm{div}_x \mathfrak{u} + 3T \mathrm{div}_x \tilde{u}\Big) + \mathfrak{u} \cdot \nabla_x \tilde{\theta} + 3\tilde{u} \cdot \nabla_x T\Big] = g\,, \end{cases} \tag{0.52}$$

with $(t,x) \in (0,\tau) \times \mathbb{R}^3_+$. We impose the boundary condition

$$\tilde{u}_3(t, \bar{x}, 0) = d(t,x)\,, \forall (t, \bar{x}) \in (0,\tau) \times \mathbb{R}^2 \tag{0.53}$$

and initial condition

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(0,x) = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(x)\,. \tag{0.54}$$

Define

$$E_k := \sup_{t\in[0,\tau]} \|(\rho, \mathfrak{u}, T)(t)\|_{H^k(\mathbb{R}^3_+)} \quad \text{for } k \geq 3\,. \tag{0.55}$$

Lemma (Guo-Huang-Wang-2020-arXiv)

Assume that

$$\mathbb{E}_0 = \|(\tilde{\rho}^0, \tilde{u}^0, \tilde{\theta}^0)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} + \sup_{t\in(0,\tau)} \left[\|(f,g)(t)\|^2_{\mathcal{H}^{k+1}(\mathbb{R}^3_+)} + \|d(t)\|^2_{\mathcal{H}^{k+2}(\mathbb{R}^2)}\right] < +\infty$$

with $k \geq 3$*, and the compatibility condition is satisfied for the initial data. Then there exists a unique smooth solution to* (0.52)-(0.54) *with boundary condition* (0.53) *for* $t \in [0, \tau]$*, such that*

$$\sup_{t\in[0,\tau]} \|(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t)\|^2_{\mathcal{H}^k(\mathbb{R}^3_+)} \leq C(\tau, E_{k+2})\mathbb{E}_0\,. \tag{0.56}$$

Ideas: Control F_k, F_k^b, F_k^{bb} $(1 \le k \le 5)$ (II)

- $(u, \theta) = (u_1, u_2, \theta)(t, \bar{x}, \zeta)$ obeys linear compressible Prandtl type system: for $i = 1, 2$,

$$\begin{cases} \rho^0 \partial_t u_i + \rho^0 \bar{\mathrm{u}}^0 \cdot \nabla_{\bar{x}} u_i + \rho^0 (\partial_{x_3} \mathrm{u}_3^0 \zeta + u_{1,3}^0) \partial_\zeta u_i + \rho^0 u \cdot \nabla_{\bar{x}} \mathrm{u}_i^0 + \frac{\partial_{x_i} p^0}{3T^0} \theta = \mu(T^0) \partial_\zeta^2 u_i + f_i \,, \\ \rho^0 \partial_t \theta + \rho^0 \bar{\mathrm{u}}^0 \cdot \nabla_{\bar{x}} \theta + \rho^0 (\partial_{x_3} \mathrm{u}_3^0 \zeta + u_{1,3}^0) \partial_\zeta \theta + \frac{2}{3} \rho^0 \mathrm{div}_x \mathrm{u}^0 \theta = \frac{3}{5} \kappa(T^0) \partial_\zeta^2 \theta + g \,, \end{cases} \tag{0.57}$$

- We assume

$$\mu(T^0) \ge \mu_0 > 0 \,, \; \frac{3}{5} \kappa(T^0) \ge \kappa_0 > 0 \tag{0.58}$$

  for some constants $\mu_0, \kappa_0 > 0$.
- BC of (0.57):

$$\begin{cases} \left(\partial_\zeta u_i - R_u u_i\right)\Big|_{\zeta=0} = b_i(t, \bar{x}) \,, \; \left(\partial_\zeta \theta - R_\theta \theta\right)\Big|_{\zeta=0} = a(t, \bar{x}) \,, \\ \lim_{\zeta \to \infty} (u, \theta)(t, \bar{x}, \zeta) = 0 \,, \end{cases} \tag{0.59}$$

  where

$$R_u \ge R_u^0 > 0 \,, R_\theta \ge R_\theta^0 > 0 \,. \tag{0.60}$$

- Initial data of (0.57):

$$u(t, \bar{x}, \zeta)|_{t=0} = u_0(\bar{x}, \zeta) \,, \; \theta(t, \bar{x}, \zeta)|_{t=0} = \theta_0(\bar{x}, \zeta) \,, \tag{0.61}$$

  which satisfy the corresponding compatibility conditions.

Lemma (Well-posedness of Prandtl type system)

Let $k \geq 3$, $l \geq 0$, *and the compatibility conditions for the initial data* (0.61) *be satisfied. Assume*

$$\begin{aligned} \mathbb{E}_1 :=& \|u_0\|^2_{\mathbb{H}^k_l(\mathbb{R}^3_+)} + \sup_{t\in[0,\tau]} \Big(\|b(t)\|^2_{\mathbb{H}^{k+3}(\mathbb{R}^2)} + \|f(t)\|^2_{\mathbb{H}^{k+1}_l(\mathbb{R}^3_+)} \Big) \\ & + \|\theta_0\|^2_{\mathbb{H}^k_l(\mathbb{R}^3_+)} + \sup_{t\in[0,\tau]} \Big(\|a(t)\|^2_{\mathbb{H}^{k+3}(\mathbb{R}^2)} + \|g(t)\|^2_{\mathbb{H}^{k+1}_l(\mathbb{R}^3_+)} \Big) < \infty . \end{aligned} \tag{0.62}$$

Then there exists a unique smooth solution $(u, \theta)(t, \bar{x}, \zeta)$ *to* (0.57)-(0.61) *over* $t \in [0, \tau]$ *satisfying*

$$\begin{aligned} \sup_{t\in[0,\tau]} \Big(& \|(u,\theta)(t)\|^2_{\mathbb{H}^k_l(\mathbb{R}^3_+)} + \|B_c(u,\theta)(t)\|^2_{\mathbb{H}^{k-1}(\mathbb{R}^2)} \\ & + \int_0^t \|\partial_\zeta(u,\theta)(s)\|^2_{\mathbb{H}^k_l(\mathbb{R}^3_+)} + \|B_c(u,\theta)(s)\|^2_{\mathbb{H}^k(\mathbb{R}^2)} \mathrm{d}s \Big) \leq C(\tau, E_{k+1}) \mathbb{E}_1 , \end{aligned} \tag{0.63}$$

where $B_c(u,\theta) = (u,\theta)|_{\zeta=0} - (\frac{1}{R_u} b, \frac{1}{R_\theta} a)$.

Ideas: Control F_k, F_k^b, F_k^{bb} $(1 \le k \le 5)$ (III)

- Main ingredients of proof of Lemma (Well-posedness of Prandtl type system):
  - the boundary values of the higher order normal ζ-derivatives on $\{\zeta = 0\}$: employing the structures of equations to convert the higher normal ζ-derivatives to the values of tangential $\bar{x}$-derivatives and time derivatives on $\{\zeta = 0\}$
  - find a key boundary energy

$$\sup_{t\in[0,\tau]} \|B_c(u,\theta)(t)\|^2_{\mathbb{H}^{k-1}(\mathbb{R}^2)} + \int_0^\tau \|B_c(u,\theta)(s)\|^2_{\mathbb{H}^k(\mathbb{R}^2)} \mathrm{d}s$$

  - loss of derivatives with respect to the boundary values and source terms

Ideas: Control F_k, F_k^b, F_k^{bb} $(1 \leq k \leq 5)$ (IV)

- Knudsen boundary layer problem in half-space for $f(t, \bar{x}, \xi, v)$ over $(t, \bar{x}, \xi, v) \in [0, \tau] \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^3$:

$$\begin{cases} v_3 \partial_\xi f + \mathcal{L}^0 f = S(t, \bar{x}, \xi, v)\,, \\ f(t, \bar{x}, 0, \bar{v}, v_3)|_{v_3>0} = f(t, \bar{x}, 0, \bar{v}, -v_3) + \mathbf{f}_k(t, \bar{x}, \bar{v}, -v_3)\,, \\ \lim\limits_{\xi \to \infty} f(t, \bar{x}, \xi, v) = 0\,. \end{cases} \tag{0.64}$$

[Golse-Perthame-Sulem-1988-ARMA] has proved an existence result in the norm $\int_{\mathbb{R}_+ \times \mathbb{R}^3} \langle v \rangle e^{2\eta\xi} f^2 \mathrm{d}v \mathrm{d}\xi + \int_{\mathbb{R}^3} \|e^{\eta\xi} f\|^2_{L^\infty_\xi} \mathrm{d}v$. Due to the higher order derivatives are required,
[Guo-Huang-Wang-2020-arXiv] similarly gave a modified version of existence result.

Lemma (Guo-Huang-Wang-2020-arXiv)

Let $0 < a < \frac{1}{2}$, $r \geq 0$ *and* $l \geq 3$. *For each* $(t, \bar{x}) \in [0, \tau] \times \mathbb{R}^2$, *we assume that* $S \in (\mathcal{N}^0)^{\perp}$, $\mathbf{f}_k(t, \bar{x}, v)$ *satisfies the solvability condition* (0.41), *and*

$$\mathbb{E}_2 := \sup_{t\in[0,\tau]} \sum_{|\alpha|\leq r} \Big\{ \|\langle v\rangle^l (\mathfrak{M}^0)^{-a} \partial^{\alpha}_{t,\bar{x}} \mathbf{f}_k(t)\|_{L^{\infty}_{\bar{x},v} \cap L^2_{\bar{x}} L^{\infty}_v} \\ + \|\langle v\rangle^l (\mathfrak{M}^0)^{-a} e^{\lambda_0 \xi} \partial^{\alpha}_{t,\bar{x}} S(t)\|_{L^{\infty}_{\bar{x},\xi,v} \cap L^2_{\bar{x}} L^{\infty}_{\xi,v}} \Big\} < +\infty$$

for some positive constant $\lambda_0 > 0$. *Then the Knudsen boundary layer equation* (0.64) *has a unique solution* $f(t, \bar{x}, \xi, v)$ *satisfying*

$$\sum_{|\alpha|\leq r} \sup_{t\in[0,\tau]} \Big\{ \|\langle v\rangle^l (\mathfrak{M}^0)^{-a} \partial^{\alpha}_{t,\bar{x}} f(t, \cdot, 0, \cdot)\|_{L^{\infty}_{\bar{x},v} \cap L^2_{\bar{x}} L^{\infty}_v} \\ + \|\langle v\rangle^l (\mathfrak{M}^0)^{-a} e^{\lambda \xi} \partial^{\alpha}_{t,\bar{x}} f(t)\|_{L^{\infty}_{\bar{x},\xi,v} \cap L^2_{\bar{x}} L^{\infty}_{\xi,v}} \Big\} \leq \frac{C}{\lambda_0 - \lambda} \mathbb{E}_2 \qquad (0.65)$$

for all $\lambda \in (0, \lambda_0)$, *where* $C > 0$ *is independence of* $(t, \bar{x})$. *Moreover, if* S *is in* $C\Big([0, \tau] \times \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^3\Big)$ *and* $\mathbf{f}_k(t, \bar{x}, \bar{v}, -v_3)$ *is in* $C\Big([0, \tau] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+\Big)$, *then the solution* $f(t, \bar{x}, \xi, v)$ *is continuous away from the grazing set* $[0, \tau] \times \Sigma_0$.

Based on the previous three lemmas, we have

Proposition (Control F_k, F_k^b, F_k^{bb})

Let $0 < \frac{1}{2z}(1-z) < a < \frac{1}{2}$, where $z \in (\frac{1}{2}, 1)$ is given in (0.12), and sufficiently large $s_0 \in \mathbb{N}_+$ be in Proposition (Local well-posedness of compressible Euler system: Schochet-1986-CMP). There are $s_k, s_k^b, s_k^{bb}, l_k^b \in \mathbb{N}_+$, $p_k, p_k^b, p_k^{bb} \in \mathbb{R}_+$ $(1 \le k \le 5)$ satisfying

$$s_0 \ge s_1 + 10\,,\ s_1 = s_1^b = s_1^{bb}\,,\ s_{k-1}^{bb} \gg s_k \gg s_k^b \gg s_k^{bb} \gg 1\ (2 \le k \le 5)\,,$$
$$p_k \gg p_k^b \gg p_k^{bb} \gg p_{k+1} \gg 1\ (1 \le k \le 4)\,,$$

and $l_j^k = l_k^b + 2(s_k^b - j)$ $(0 \le j \le s_k^b, 1 \le k \le 5)$ with

$$l_j^5 \ge 8\,,\ l_j^k \ge 2l_j^{k+1} + 26\ (1 \le k \le 4)\,,$$

such that if the initial data $(\rho_k^{in}, u_k^{in}, \theta_k^{in})$ $(1 \le k \le 3)$ in (0.24) and $(\bar{u}_k^{b,in}, \theta_k^{b,in})$ $(1 \le k \le 3)$ in (0.34) satisfy (0.15), i.e., $\mathcal{E}^{in} < \infty$, then there are solutions

Proposition (Control F_k, F_k^b, F_k^{bb} (Continued))

$F_k = \sqrt{\mathfrak{M}} f_k$, $F_k^b = \sqrt{\mathfrak{M}^0} f_k^b$ and $F_k^{bb} = \sqrt{\mathfrak{M}^0} f_k^{bb}$ $(1 \leq k \leq 5)$ over the time interval $t \in [0, \tau]$ subjecting to the uniform bounds

$$\begin{aligned}
\sup_{t\in[0,\tau]} \sum_{k=1}^{5} \Bigg\{ & \sum_{\gamma+|\beta|\leq s_k} \|\langle v\rangle^{p_k} \mathfrak{M}^{-a} \partial_t^\gamma \partial_x^\beta f_k(t)\|_{L^2_x L^\infty_v} \\
& + \sum_{j=0}^{s_k^b} \sum_{2\gamma+|\bar\beta|=j} \|\langle v\rangle^{p_k^b} (\mathfrak{M}^0)^{-a} \partial_t^\gamma \partial_{\bar x}^{\bar\beta} f_k^b(t)\|_{L^2_{l_j^k} L^\infty_v} \\
& + \sum_{\gamma+|\bar\beta|\leq s_k^{bb}} \|e^{\frac{\xi}{2^{k-1}}} \langle v\rangle^{p_k^{bb}} (\mathfrak{M}^0)^{-a} \partial_t^\gamma \partial_{\bar x}^{\bar\beta} f_k^{bb}(t)\|_{L^\infty_{\bar x,\xi,v} \cap L^2_{\bar x} L^\infty_{\xi,v}} \Bigg\} \\
\leq & C\Big(\tau, \|(\rho^{in}, \mathfrak{u}^{in}, T^{in})\|_{H^{s_0}(\mathbb{R}^3_+)} + \mathcal{E}^{in}\Big). \qquad (0.66)
\end{aligned}$$

Ideas: Uniform bounds for remainder (I)

- $L^2 - L^\infty$ framework: interplay between L^2 norm $\|f_{R,\epsilon}(t)\|_2$ and L^∞ norm $\|h^\ell_{R,\epsilon}(t)\|_\infty$ for the Boltzmann equation

Lemma (L^2 Estimates)

Under the same assumptions in Proposition (Control F_k, F^b_k, F^{bb}_k) and $\ell \geq 9 - 2\gamma_0$, let $(\rho, \mathfrak{u}, T)$ be a smooth solution to the Euler equations over $t \in [0, \tau]$ obtained in Proposition (Local well-posedness of compressible Euler system: Schochet-1986-CMP). Let $c_0 > 0$ be mentioned in (0.8), *and the local Maxwellian $M_w(t, \bar{x}, v)$ of the boundary be assumed as in* (0.14). *Then there are constants $\epsilon'_0 > 0$ and $C = C\Big(\mathfrak{M}, F_k, F^b_k, F^{bb}_k; 1 \leq k \leq 5\Big) > 0$ such that for all $0 < \epsilon < \epsilon'_0$,*

Ideas: Uniform bounds for remainder (II)

Lemma (L^2 Estimates (Continued))

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_{R,\epsilon}\|_2^2 + \frac{c_0}{\epsilon}\|(\mathcal{I}-\mathcal{P})f_{R,\epsilon}\|_\nu^2 + c_1 \iint_{\Sigma_-} |v\cdot n(x)||L\gamma_+ f_{R,\epsilon}|^2 \mathrm{d}\sigma_{\Sigma_-}$$
$$\leq C\Big(1+\epsilon^2\|h^\ell_{R,\epsilon}\|_\infty\Big)\Big(\|f_{R,\epsilon}\|_2^2 + \|f_{R,\epsilon}\|_2\Big) + C\sqrt{\epsilon}^2 \tag{0.67}$$

over $t\in[0,\tau]$, where $c_1 = 1 - 2\rho^\# \sqrt{2T^\#} > 0$ and $\rho^\#, T^\# > 0$ are given in Proposition (Local well-posedness of compressible Euler system: Schochet-1986-CMP).

- Under assumption $M_w = \mathfrak{M}^0$, the boundary integral $-\frac{1}{2}\iint_{\partial\mathbb{R}^3_+\times\mathbb{R}^3} v_3|f_{R,\epsilon}(t,\bar{x},0,v)|^2\mathrm{d}v\mathrm{d}\bar{x}$ contains a key boundary energy dissipative rate $c_1\iint_{\Sigma_-}|v\cdot n(x)||L\gamma_+ f_{R,\epsilon}|^2\mathrm{d}\sigma_{\Sigma_-}$

Ideas: Uniform bounds for remainder (III)

Lemma (L^∞ Estimate)

Under the same assumptions in Lemma (L^2 Estimates), there are constants $\epsilon_0'' > 0$ *and* $C = C\Big(\mathfrak{M}, F_k, F_k^b, F_k^{bb}; 1 \le k \le 5\Big) > 0$ *such that for all* $0 < \epsilon < \epsilon_0''$,

$$\sup_{s\in[0,\tau]} \| \sqrt{\epsilon}^3 h^\ell_{R,\epsilon}(s)\|_\infty \le C\Big(\| \sqrt{\epsilon}^3 h^\ell_{R,\epsilon}(0)\|_\infty + \sup_{0\le s\le t} \|f_{R,\epsilon}(s)\|_2 + \sqrt{\epsilon}^2\Big). \tag{0.68}$$

- Disparity of order $\sqrt{\epsilon}^3$ between L^∞ and L^2 estimates is an essence resulted from the linear structure

$$\partial_t f_{R,\epsilon} + v\cdot\nabla_x f_{R,\epsilon} + \frac{1}{\epsilon}\mathcal{L} f_{R,\epsilon} = \text{some other terms}\,.$$

  - In L^2 estimates, the hypocoercivity of the operator $\frac{1}{\epsilon}\mathcal{L}$ produces a dissipative rate $\frac{c_0}{\epsilon}\|(\mathcal{I}-\mathcal{P})f_{R,\epsilon}\|_\nu^2$
  - In L^∞ estimates, such a hypocoercivity fails

Comparison of MRBC and SRBC (I)

Maxwell reflection BC: MRBC **Specular reflection BC: SRBC**

- BC of linear compressible Prandtl-type equations (viscous layers)
  - MRBC case: Robin-type BC
    - lead to a key boundary energy structure

$$\sup_{t\in[0,\tau]} \|B_c(u,\theta)(t)\|^2_{\mathbb{H}^{k-1}(\mathbb{R}^2)} + \int_0^\tau \|B_c(u,\theta)(s)\|^2_{\mathbb{H}^k(\mathbb{R}^2)} \mathrm{d}s\,,$$

    which cannot be dominated by interior energy.
  - SRBC case: Neumann-type BC (no such boundary energy)
- BC of linear hyperbolic system (interior): $k = 1$
  - MRBC case: $u^0_{1,3} = u_{1,3}(t,\bar{x},0) = \sqrt{T^0}(\rho^0\sqrt{T^0} + 1)$
  - SRBC case: $u^0_{1,3} = u_{1,3}(t,\bar{x},0) = 0$
- BC of linear hyperbolic system (interior) for $k \geq 2$: the form of $d_k(t,\bar{x})$ in the MRBC case are more complicated than that in the SRBC case

Comparison of MRBC and SRBC (II)

- the boundary integral $-\frac{1}{2}\iint_{\partial\mathbb{R}^3_+\times\mathbb{R}^3} v_3|f_{R,\epsilon}(t,\bar{x},0,v)|^2\mathrm{d}v\mathrm{d}\bar{x}$ in L^2 estimates of remainder:
  - MRBC case: leading to a key boundary energy dissipative rate $\iint_{\Sigma_-}|v\cdot n(x)||L\gamma_+ f_{R,\epsilon}|^2\mathrm{d}\sigma_{\Sigma_-}$ under assumption $M_w=\mathfrak{M}^0$
  - SRBC case: the boundary integral vanishes

The related results of SRBC case can be found in [Guo-Huang-Wang-2020-arXiv].

Optimal truncation in $L^2 - L^\infty$ framework

- We hope the number of expanded terms as small as possible.
- Our theorem has proved that the remainder $\sqrt{\epsilon}^4 F_{R,\epsilon}$ is bounded by $O(\sqrt{\epsilon})$ in $L^2 - L^\infty$ framework.
- Disparity of order $\sqrt{\epsilon}^3$ between L^∞ and L^2 estimates (see Lemma (L^2 estimates) and (L^∞ estimate)) is an essence resulted from the linear structure (no matter how many terms are expanded)

$$\partial_t f_{R,\epsilon} + v \cdot \nabla_x f_{R,\epsilon} + \frac{1}{\epsilon}\mathcal{L} f_{R,\epsilon} = \text{some other terms} \ :$$

  - In L^2 estimates, the hypocoercivity of the operator $\frac{1}{\epsilon}\mathcal{L}$ produces a dissipative rate $\frac{c_0}{\epsilon}\|(\mathcal{I}-\mathcal{P})f_{R,\epsilon}\|_\nu^2$.
  - In L^∞ estimates, such a hypocoercivity fails: transforming $f_{R,\epsilon}$ to $h^\ell_{R,\epsilon}$ and integrating along the trajectory $[X_{cl}(s;t,x,v), V_{cl}(s;t,x,v)]$, regarding $\frac{1}{\epsilon}\mathcal{L}$ term as a source term.
- Once the remainder were $\sqrt{\epsilon}^m F_{R,\epsilon}$ ($m \le 3$), it will only be bounded by $O(\sqrt{\epsilon}^{m-3})$, which fails to achieve our goal.

Thank you for your attention!