Stabilization of random kinetic equations

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Scope of the Talk

- General interest in feedback boundary control of kinetic and hyperbolic equations
- Techniques typically require design and decay of suitable Lyapunov function
- **This talk:** Linear kinetic equations with relaxation term
- No decay in deterministic limit, but decay in expectation for uncertain relaxation rate

Lyapunov function for linear transport problems [1]

Toy problem

$$\partial_t u(t,x) + \partial_x a\, u(t,x) = 0,\ u(t,x) \in \mathbb{R}, a > 0, x \in [0,1]$$

- Idea:

$$\mathcal{L}(t) = \int_0^1 \exp(-\mu x) u^2(t,x) dx$$

fulfills

$$\mathcal{L}(t) \leq \exp(-\mu a t)\mathcal{L}(0)$$

under dissipative ($\kappa < 1$) boundary condition $u(t,0) = \kappa u(t,1)$ implies exponential decay towards a steady state $u \equiv 0$

- Extension to systems and *dissipative* source terms , to the nonlinear case (through linearization), networks, feedback design, uncertainty quantification, ...

[1] Many references available, e.g. *Control and Nonlinearity* by J.-M. Coron

Two-velocity relaxation model

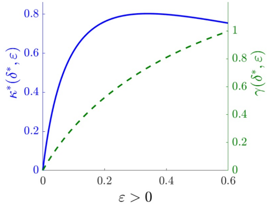
$$\vec{f} = (f_1, f_2),\ T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } L_\epsilon = \frac{1}{2\epsilon}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\partial_t \vec{f}(t,x) + T\vec{f} = L_\epsilon \vec{f}$$

- Global steady state $F = \frac{1}{2}(1,1)^T$
- Previous Lyapunov function $\mathcal{L}$ does **not** decay exponentially towards F since source is not dissipative w.r.t. to $\mathcal{L}$
- Hypocoercivity framework utilize a functional, s.t.

$$\|\vec{f} - F\| \leq C(\epsilon)\ \exp(-k(\epsilon)t)\ \|\vec{f}(0) - F\|$$

but $k(\epsilon) \to 0$ for $\epsilon \to 0$



Notation and framework

$$\partial_t f(t,x,v) + \frac{1}{\epsilon^\alpha} Tf(t,x,v) = \frac{1}{\epsilon^{\alpha+1}} Lf(t,x,v),\ f(0,x,v) = f_0(x,v)$$

- Linear kinetic equation $f = f(t,x,v)$ with transport operator T and collision L, e.g. BGK type
- Hypocoercivity[2] give conditions on T, L to ensure convergence towards local equilibrium F in the sense

$$\|f(t) - F\|^2 \leq C(\epsilon)\exp(-k(\epsilon)t)\|f_0 - F\|^2$$

- Modified entropy functional for $\gamma \geq 0$

$$H[f] = \frac{1}{2}\|f\|^2 + \gamma < Af, f >,\ A = (1 + (T\Pi)^*(T\Pi))^{-1}\ (T\Pi)^*$$

- Π is projection onto null space of L
- $\|\cdot\|$ is weighted L^2 with weight $\frac{1}{F}$

[2] Many references, e.g. Dolbeaut, Mouhot, Schmeiser 2015

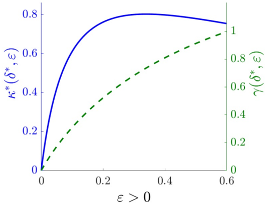
Conditions on operators for hypocoercivity

$$\|f(t) - F\|^2 \leq C(\epsilon) \exp(-k(\epsilon)t) \|f_0 - F\|^2$$

(H1) Microscopic coercivity $-< L_\epsilon f, f > \geq \lambda_m \|(1 - \Pi)f\|^2$
(H2) Macroscopic coercivity $\|T_\epsilon \Pi f\| \geq \lambda_M \|\Pi f\|^2$
(H3) Projection of Transport $\Pi T_\epsilon \Pi = 0$
(H4) Boundedness of $AT_\epsilon(1 - \Pi), AL_\epsilon$ and

$$\|AT_\epsilon(1 - \Pi)f\| + \|AL_\epsilon f\| \leq C_M \|(1 - \Pi)f\|$$

Conditions ensure strictly positive decay rate $k(\epsilon)$ for $\epsilon > 0$



Scaling and existing results on decay rate

$$\partial_t f(t,x,v) + \frac{1}{\epsilon^\alpha} Tf(t,x,v) = \frac{1}{\epsilon^{\alpha+1}} Lf(t,x,v)$$

- $\alpha = 1$ is the parabolic scaling and it has been shown $\lim_{\epsilon \to 0} k(\epsilon) > 0$
- $\alpha = 1$ and random[3] relaxation parameter $\epsilon = \epsilon(\omega)$ gives exponential decay of mean squared deviations from *deterministic* equilibrium F with $\kappa > 0$

$$\mathbb{E}\left(\|f(t) - F\|\right) \leq C \exp(-\kappa t) \|f_0 - F\|$$

- $\alpha = 0$ is acoustic scaling

$$\lim_{\epsilon} k(\epsilon) \to 0$$

(as for toy example)

[3] Li,Wang SIAM UQ 2017

Random kinetic equations with acoustic scaling

Replace the deterministic system

$$\partial_t f(t,x,v) + Tf(t,x,v) = \frac{1}{\epsilon} Lf(t,x,v)$$

by system with parametric uncertainty ξ and deterministic $\eta > 0$

$$\partial_t f(t,x,v,\xi) + Tf(t,x,v,\xi) = (\xi + \eta) Lf(t,x,v,\xi)$$

- ξ is a non–negative random variable with **unbounded** support
- Realization of $\xi \to \infty$ correspond to previous $\epsilon \to 0$ limit
- Introduction of ξ leads to f that is uncertain $f = f(t,x,v,\xi)$
- **Assume:** equilibrium F is deterministic
- Many references for uncertainty quantification of kinetic equations [4]

[4] e.g. by S. Jin and collaborators

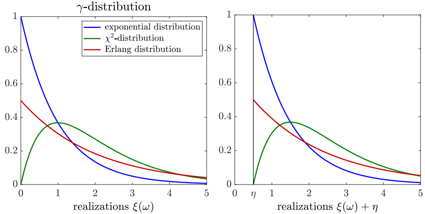
Preliminary discussion on the setting

$$\partial_t f(t,x,v,\xi) + Tf(t,x,v,\xi) = (\xi+\eta)Lf(t,x,v,\xi)$$

- η can be arbitrary small but positive to prevent degeneration of the system for realizations $\xi(\omega)=0$
- ξ is distributed with probability density

  $$p(\xi) = \frac{\beta^{\bar{\alpha}+1}}{\Gamma(\bar{\alpha}+1)}\xi^{\bar{\alpha}}\exp(-\beta\xi)$$

  for parameters $\bar{\alpha} \geq 0, \beta > 0$
- Orthogonal polynomials to this density are Laguerre polynomials



Non–intrusive vs. intrusive approach

$$\partial_t f(t,x,v,\xi) + Tf(t,x,v,\xi) = (\xi+\eta)Lf(t,x,v,\xi)$$

Goal: Exponential decay of the expected (weighted) mean square error

$$\int_0^\infty \|f(t,\cdot,\cdot,\xi) - F\|^2 p(\xi) d\xi$$

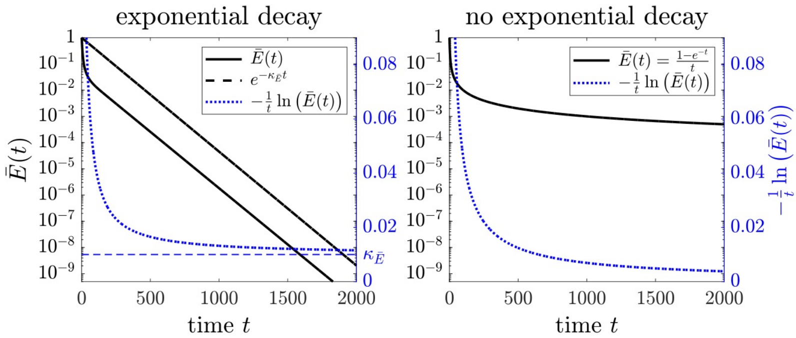
- Intrusive (gPC): Consider a series expansion wrt to polynomials ϕ_i

$$f(t,x,,v,\xi) = \sum_{i=1}^{\infty} f_i(t,x,v)\phi_i(\xi)$$

  and solve the obtained enlarged system for $\vec{f} = (f_i)_i$.

- Non-intrusive: Apply hypocoercivity framework for each realization ξ using e.g. Monte-Carlo. This leads for each sample to vanishing decay rate

Illustration of non-intrusive and intrusive approach



Intrusive (left): exponential decay

Non-intrusive (right): vanishing decay rate

$$E(t) = \mathbb{E}\left(C(\frac{1}{\xi + \eta} \exp(-k\frac{1}{\xi+\eta} t)) \right)$$

Steps showing exponential decay for gPC (intrusive)

$$\partial_t f(t,x,v,\xi) + Tf(t,x,v,\xi) = (\xi+\eta)Lf(t,x,v,\xi), \qquad f(0,x,v,\xi) = f_0(x,v)$$
$$\partial_t \vec{f}(t,x,v) + \mathbf{T}\vec{f}(t,x,v) = \mathbf{L}\vec{f}(t,x,v), \qquad \vec{f}_k(0,x,v) = f_0(x,v)\delta_{k,0}$$

- Deterministic initial data, $\mathbf{T} = T\,1$, $\mathbf{L} = L(P+\eta 1)$, $P_{k,i} = \int_0^\infty \xi\phi_k\phi_i p d\xi$
- Solution $\vec{f} = (f_i)_{i\in\mathbb{N}_0}$ belongs to **weighted** space: for $\sigma_k = k + \frac{\sqrt{\bar{\alpha}+1}}{2\beta\eta}$

$$\ell^2_\sigma := \{\ \vec{f} = \vec{f}(x,v) : \sum_{k=0}^\infty \sigma_k \|f_k\|^2 < \infty\},$$

- Bound on solution $\|\vec{f}\|^2_{\ell^2_\sigma} \le \frac{\sqrt{\bar{\alpha}+1}}{2\beta\eta}\|f_0\|^2$ uses explicitly properties of P linked to the particular orthogonal polynomials
- $\vec{f} \in \ell^2_\sigma \implies \vec{f} \in \ell^2$ and for $\vec{f} \in \ell^2_\sigma : \|\vec{f}\|^2_{\ell^2_\sigma} \le C\|\vec{f}\|^2_{\ell^2}$

Basic steps (cont'd)

$$\partial_t f(t,x,v,\xi) + Tf(t,x,v,\xi) = (\xi+\eta)Lf(t,x,v,\xi), \qquad f(0,x,v,\xi) = f_0(x,v)$$
$$\partial_t \vec{f}(t,x,v) + \mathbf{T}\vec{f}(t,x,v) = \mathbf{L}\vec{f}(t,x,v), \qquad \vec{f}_k(0,x,v) = f_0(x,v)\delta_{k,0}$$

- Assume $(H1)-(H4)$ for the deterministic system, then $(H1)-(H4)$ hold in ℓ^2 (no truncation required)
- Hypocoercivity result gives exponential decay of expectation

$$\int_0^\infty \|f(t,\cdot,\cdot,\xi) - F\|^2 p(\xi)d\xi \leq C\exp(-\kappa t)\|f_0 - F\|^2$$

- Remark 1: solution is in ℓ^2_σ, but no exponential decay in ℓ^2_σ
- Remark 2: results does not include truncation error due to (numerically) finite gPC series

Illustration of remark 1

$$\partial_t \vec{f}(t,x,v) + \mathbf{T}\vec{f}(t,x,v) = \mathbf{L}\vec{f}(t,x,v), \qquad \vec{f}_k(0,x,v) = f_0(x,v)\delta_{k,0}$$

- Difference in non-intrusive and intrusive by using $\vec{f} \in \ell^2_\sigma$ and hypocoercivity on ℓ^2
- Exponential decay in ℓ^2_σ also not observed numerically

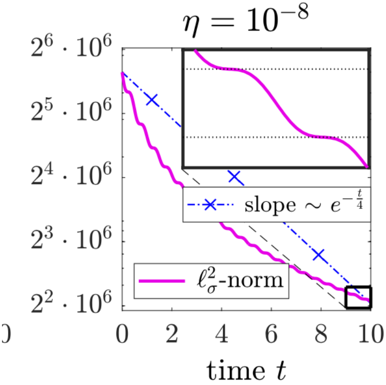
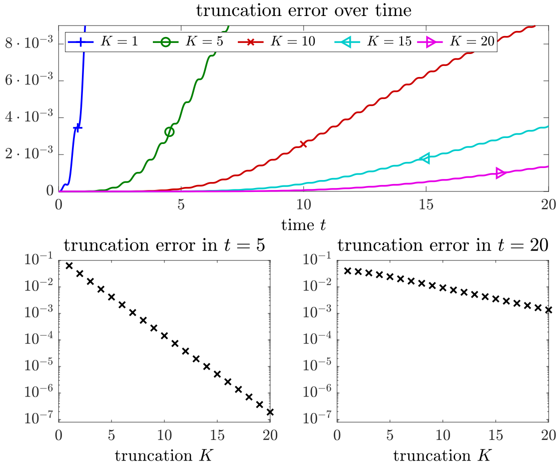


Illustration of remark 2

$$\partial_t \vec{f}(t,x,v) + \mathbf{T}\vec{f}(t,x,v) = \mathbf{L}\vec{f}(t,x,v), \qquad \vec{f}_k(0,x,v) = f_0(x,v)\delta_{k,0}$$

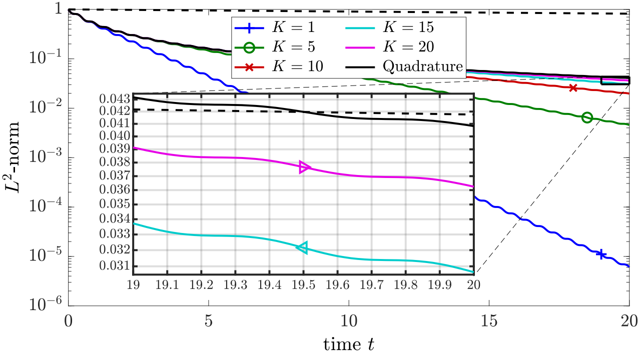
Results on truncation error: $\mathbb{E}(\|\vec{f}^K(t) - \vec{f}(t)\|^2) \approx C^{-r}t$



Decay of the mean squared error for truncated series

$$\partial_t \vec{f}(t,x,v) + \mathbf{T}\vec{f}(t,x,v) = \mathbf{L}\vec{f}(t,x,v), \qquad \vec{f}_k(0,x,v) = f_0(x,v)\delta_{k,0}$$

Expected decay rate of complete series is the black dashed line.
Deterministic toy example with two velocities leads to a system of $2K$ equations for $\vec{f}$



Summary and Outlook

- Extension of deterministic setting to parametric uncertainty in the acoustic scaling
- Intrusive approach allows to get exponential decay
- Estimates on the sequence space use the properties of probability density
- Decay rate is not explicit
- Derivation of boundary feedback control

Thank you for your attention!

Reference: S. Gerster, M. Herty and H. Yu, Hypocoercivity of stochastic Galerkin formulations for stabilization of kinetic equations, Comm. Math. Sci. 2021