Karlsruhe Institute of Technology

# Dynamical low-rank approximation for radiation transport

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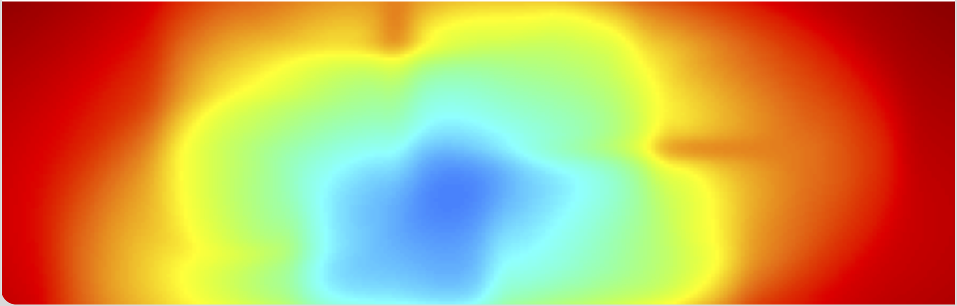


# What is the Dynamical Low-Rank Approximation?

# What is the Dynamical Low-Rank Approximation (DLRA)?

- 1D+1D kinetic equation

$$\partial_t f(t,x,\mu) = -\mu\partial_x f(t,x,\mu) + \sigma_s(x)\phi(t,x) - \sigma_t(x) f(t,x,\mu) + \frac{1}{2}Q(x),$$

- Notation: $\phi(t,x) = \frac{1}{2}\langle f\rangle_\mu$, $\langle g\rangle_x = \int_{\mathbb{R}} g dx$, $\langle h\rangle_\mu = \int_{-1}^{1} h d\mu$
- Imagine we discretize $f$ in space and angle in some way, e.g.

$$f_{ij}(t) = f(t, x_i, \mu_j)$$

- Matrix differential equation, written as

$$\frac{d}{dt}Y(t) = F(t, Y(t))$$

# Low-Rank Matrices & Projection

- Rank-$r$ approximation of $m \times n$ matrix $A$ by truncated SVD-like decomposition
$$Y_r(t) = U(t)S(t)V(t)^T$$
- Orthogonal matrices $U(t)$ $(m \times r)$, $V(t)$ $(n \times r)$ and $r \times r$ matrix $S(t)$ (not necessarily diagonal)
- Different concept than sparsity
- Insert low-rank ansatz into matrix differential equation
$$\frac{d}{dt}Y_r(t) = F(t, Y_r(t)) + \text{residual}$$
- Project r.h.s. onto the tangent space of rank $r$ matrices (nonlinear Galerkin-like projection)
$$\frac{d}{dt}Y_r(t) = \mathcal{P}_{T_r(Y_r(t))}(F(t, Y_r(t))$$

# Characterizing the Tangent Space

- SVD-like decomposition
$$Y = USV^T$$
- Element in the tangent space
$$\delta Y = \delta U S V^T + U \delta S V^T + U S \delta V^T$$
- Orthogonality $U^T U = I$ implies $\delta U^T U + U^T \delta U = 0$; same for $V$
- Factors can be retrieved by imposing $U^T \delta U = 0$, $V^T \delta V = 0$:
$$\begin{aligned}\delta S &= U^T \delta Y V,\\ \delta U &= (I - UU^T)\delta Y V S^{-1},\\ \delta V &= (I - VV^T)\delta Y^T U S^{-T}\end{aligned}$$

# Projection onto Tangent Space

- Matrix differential equation

$$\frac{d}{dt}Y(t) = F(t, Y(t))$$

- Low-rank ansatz

$$Y_r(t) = U(t)S(t)V(t)^T$$

- Projection onto tangent space leads to ODEs for the factors

$$\dot{S} = U^T F(t, Y_r)V$$
$$\dot{U} = (I - UU^T)F(t, Y_r)VS^{-1}$$
$$\dot{V} = (I - VV^T)F(t, Y_r)^\top US^{-T}$$

- Memory footprint / computational effort

$$\mathcal{O}(mr + nr + r^2) \text{ instead of } \mathcal{O}(mn)$$

# Application to Radiation Transport

- Turns out this integrator is unstable, but a variant is stable ...
- 1D+1D kinetic equation

$$\partial_t f = -\mu \partial_x f + \sigma_s \frac{1}{2} \langle f \rangle_\mu - \sigma_t f + \frac{1}{2} Q$$

- Low-rank representation of $f$

$$f(t, x, \mu) \approx \sum_{i,j=1}^{r} X_i(t, x) S_{ij}(t) W_j(t, \mu),$$

- Orthogonal projections (nonlinear)

$$P_{\bar{X}} g = \sum_{i=1}^{r} X_i \langle X_i g \rangle_x , \quad P_{\bar{W}} g = \sum_{j=1}^{r} W_j \langle W_j g \rangle_\mu$$

- Projection onto tangent space of low-rank manifold

$$P_{\bar{X}} - P_{\bar{X}} P_{\bar{W}} + P_{\bar{W}}$$

# Dynamical low-rank algorithm

- Step 1: $f_I(t,x,\mu) = \sum_{j=1}^{r} K_j(t,x)W_j(\mu)$ and

$$\partial_t K_j = -\sum_l \langle \mu W_l W_j \rangle_\mu \partial_x K_l + \frac{\sigma_s}{2} \sum_l \langle W_j \rangle_\mu \langle W_l \rangle_\mu K_l - \sigma_t K_j + \frac{\langle W_j \rangle_\mu}{2} Q$$

- Factorization (QR or SVD) $K_j = \sum_{j=1}^{r} X_i S_{ij}$
- Step 2: $f_{II}(t,x,\mu) = \sum_{i,j=1}^{r} X_i(x) S_{ij}(t) W_j(\mu)$

$$\frac{d}{dt} S_{ij} = \sum_{k,l=1}^{r} \langle \mu W_j W_l \rangle_\mu \langle \partial_x X_i X_k \rangle_x S_{kl} + \sum_l \frac{1}{2} \langle \sigma_s X_i \rangle_x \langle W_j \rangle_\mu \langle W_l \rangle_\mu S_{il}$$
$$- \langle \sigma_t X_i \rangle_x S_{ij} + \frac{1}{2} \langle Q X_i \rangle_x \langle W_j \rangle_\mu$$

- Reduction $L_i = \sum_{j=1}^{r} S_{ij} W_j$ and
- Step 3: $f_{III}(t,x,\mu) = \sum_{i=1}^{r} X_i(x) L_i(t,\mu)$ and

$$\partial_t L_i = -\sum_k \langle X_i \partial_x X_k \rangle_x \mu L_k + \frac{1}{2} \langle \sigma_s X_i \rangle_x \langle L_i \rangle_\mu - \langle \sigma_t X_i \rangle_x L_i + \frac{1}{2} \langle Q X_i \rangle_x$$

- Factorization (QR or SVD) $L_i = \sum_{j=1}^{r} S_{ij} W_j$

# Challenges

- Numerically represent abstract basis functions
- Find efficient ways to compute $\langle \mu W_j W_l \rangle_\mu$, $\langle \partial_x X_i X_l \rangle_x$, $\langle \sigma_s X_i \rangle_x$, $\langle W_j \rangle_\mu$, …not to destroy overall effort
- Spatial discretization of $\langle X_i \partial_x X_k \rangle_x$ with finite volume discretization requires numerical stabilization (artificial diffusion $\alpha$)

- No positivity
- No conservation
- No entropy decay
- Asymptotic-preserving?

# Why Should the Solution have Low Rank?

# Why Should the Solution have Low Rank?

- Rank of the solution is hard to analyze
- Sparsity is easier because it is related to smoothness
- But ...

$$\varepsilon \partial_t f = -\mu \partial_x f(t,x,\mu) + \frac{1}{\varepsilon}\sigma_s \phi(t,x) - \frac{1}{\varepsilon}\sigma_t f(t,x,\mu) + \frac{\varepsilon}{2} Q(x)$$

- As $\varepsilon \to 0$

$$f(t,x,\mu) \to f_0(t,x) = X_1(t,x) \cdot S_{11}(t) \cdot 1$$

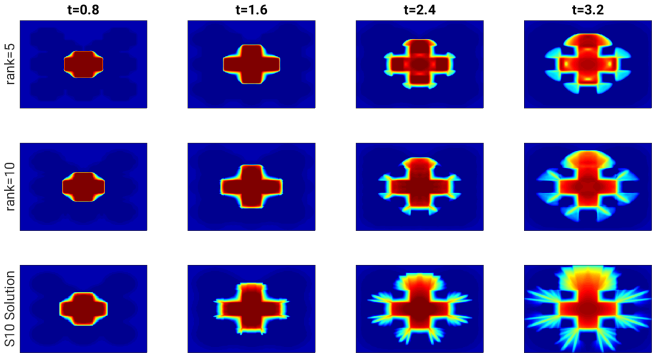
we get rank 1

# Literature Overview

- DLRA originates from the area of quantum dynamics (Dirac 1930)
- Stripped of the domain-specific language and mathematically analyzed (Lubich and co-workers, since 2007)
- Application to Vlasov-Poisson (Einkemmer & Lubich ’18)
- Low-rank for time-dependent transport (McClarren, Peng, F 2019)
- DLR for linear transport (Ding, Einkemmer, Li 2019)
- ...
- Conservation for Vlasov-Poisson (Einkemmer, Joseph 2021)

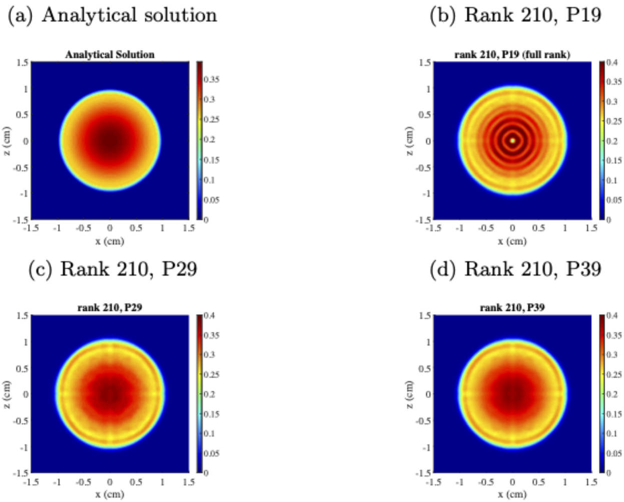
# Numerical Results

# Checkerboard Test




Computed with $m = 250 \times 250$ spatial grid points and $n = 820$ angular degrees of freedom (spherical harmonics $P_N$ discretization of order 39)

# Linesource Test



# Conclusion

- DLRA seems well-suited for kinetic equations, even further away from the asymptotic limit
- Generalized to 3D+3D by hierarchical tensors
- Significantly reduces memory footprint (and computational cost)

- Lots of questions open
- Can be combined with lots of well-established techniques

**Thank You!**

# Literature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