Derivation of the BGK equation from a stochastic particle system

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Cirm, Marseille, 2021

The model

The BGK model is a kinetic equation for the one-particle distribution function $f = f(x, v, t)$, $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ (position and velocity of the particle) and $t \in \mathbb{R}_+$ is the time.

$$(\partial_t f + v \cdot \nabla_x f)(x, v, t) = \varrho(x, t) M_f(x, v, t) - f(x, v, t),$$

$\lambda > 0$,

$$M_f(x, v, t) = \frac{1}{(2\pi T(x, t))^{d/2}} \exp\left(-\frac{|v - u(x, t)|^2}{2T(x, t)}\right),$$

$$\varrho(x, t) = \int dv\, f(x, v, t), \quad \varrho u(x, t) = \int dv\, f(x, v, t) v$$

and

$$\varrho(u^2 + Td)(x, t) = \int dv\, f(x, v, t) |v|^2 .$$

Comments

The BGK model describes the dynamics of a tagged particle which thermalizes instantaneously at Poisson random time of intensity 1. The Maxwellian M_f has mean velocity and temperature given by f itself. Same hydrodynamics as the Boltzmann eq.n (assumed as the basic microscopic law) because the same conservation laws. And also the H-thm.

P.L. Bhatnagar, E.P. Gross, and M. Krook (1954) introduced BGK to handle situations where the mean free path is very small (but positive) so that the hydrodynamic picture is inadequate.

Consider a stochastic particle system like the DSMCM (the Bird Montecarlo method), equivalent to the Boltzmann evolution. The collision intensity is very large. Then it is useless to compute in detail the very many interactions taking place locally, since we know a priori that the system is locally thermalizing. Then replace the true dynamics with a jump process in which position and velocity of a given particle are instantaneously distributed according to the local equilibrium.

Preliminaries

We we want introduce a particle system in which a particle thermalize locally via a jump process, which replace the DSMCM. Then prove that in a suitable scaling limit, the one-particle distribution, does converge to the solution to the BGK model. Before let us see which kind of control we have over the BGK sol.ns.
First note that in the original BGK work , the jump rate is chosen ϱ, namely the jumps are favorite whenever the spatial density is high. This case is mathematically much more involved compared with the case in which the rate is constant (we assume for simplicity to be 1). In this case we have a constructive existence and uniqueness theorem, Perthame and P. (1993).

Preliminaries

We also consider the solution $g = g(t) = g(x, v, t)$ of the "regularized" BGK equation,

$$\partial_t g + v \cdot \nabla_x g = \varrho_g^\varphi M_g^\varphi - g,$$

where

$$M_g^\varphi(x, v, t) = M_{u_g^\varphi(x,t), T_g^\varphi(x,t)}(v)\,,$$

$$\varrho_g^\varphi(x, t) = (\varphi * \varrho_g)(x, t) = \int dy\, \varphi(x - y)\varrho_g(y, t),$$

$$\varrho_g^\varphi(x, t) u_g^\varphi(x, t) = \int dy\, dv\, \varphi(x - y) g(y, v, t)\, v\,,$$

$$\varrho_g^\varphi(x, t) T_g^\varphi(x, t) = \frac{1}{d} \int dy\, dv\, \varphi(x - y) g(y, v, t)\, |v - u_g^\varphi(x, t)|^2,$$

where $\varrho_g(x, t) = \int dv\, g(x, v, t)$. φ is a strictly positive, even, and smooth smearing function.

Particle System

N particles moving in the d-dimensional torus $\mathbb{T}^d$. $Z_N = (X_N, V_N)$ the state of the system, where $X_N \in (\mathbb{T}^d)^N$ and $V_N \in (\mathbb{R}^d)^N$ are the positions and velocities of particles, respectively.
$X_N = (x_1 \cdots x_N)$, $V_N = (v_1 \cdots v_N)$.
The (smeared) empirical hydrodynamical fields ϱ_N^φ, u_N^φ, and T_N^φ (depending on Z_N) are defined by

$$\varrho_N^\varphi(x) = \frac{1}{N}\sum_{j=1}^N \varphi(x - x_j)\,, \quad \varrho_N^\varphi u_N^\varphi(x) = \frac{1}{N}\sum_{j=1}^N \varphi(x - x_j) v_j,$$

$$\varrho_N^\varphi T_N^\varphi(x) = \frac{1}{Nd}\sum_{j=1}^N \varphi(x - x_j)|v_j - u_N^\varphi(x)|^2\,.$$

Particle System

The generator $\mathcal{L}_N$ is defined as

$$\mathcal{L}_N G(Z_N) = [(V_N \cdot \nabla_{X_N} - N)G](Z_N) \\ + \sum_{i=1}^{N} \int d\tilde{x}_i \, d\tilde{v}_i \, \varphi(\tilde{x}_i - x_i) M^{\varphi}_{Z_N}(\tilde{x}_i, \tilde{v}_i) G(Z_N^{i,(\tilde{x}_i,\tilde{v}_i)}),$$

$Z_N^{i,(y,w)} = (X_N^{i,y}, V_N^{i,w})$ is obtained from $Z_N = (X_N, V_N)$ by replacing $(x_i v_i) \to (y, w)$. G is a test function and

$$M^{\varphi}_{Z_N}(x, v) = M_{u^{\varphi}_N(x), T^{\varphi}_N(x)}(v) \, .$$

is the Maxwellian constructed via the empirical fields.

Particle System

The process $Z_N(t) = (X_N(t), V_N(t))$ is s.t. at each Poisson time, of intensity N, a particle chosen with probability $1/N$ performs a jump from its actual position and velocity (x_i, v_i) to the new ones $(\tilde{x}_i, \tilde{v}_i)$, extracted according to the distribution $\varphi(\cdot - x_i)$ for the position and then to the empirical Maxwellian $M^{\varphi}_{Z_N}(\tilde{x}_i, \cdot)$ for the velocity. $F_N(t) = F_N(Z_N, t)$ the density of the law of $Z_N(t)$.

$$\partial_t F_N = \mathcal{L}^*_N F_N$$

Results

Two steps.
1) Fix φ and show the convergence of the particle system (at fixed φ) to g^φ.
2) Let $\varphi \to \delta$ and take a diagonal limit.

Theorem (P. Buttà, M. Hauray, M. P. 2021)
Under suitable hypotheses, let f_j^N be the j-particle marginals of the law F_N

$$\mathcal{W}_2\big(f_j^N(t), g(t)^{\otimes j}\big)^2 \leq \frac{j}{N} L_t \Gamma_\varphi \exp(L_t \Gamma_\varphi)\,,$$

for some constant Γ_φ and a nondecreasing function L_t, where $\mathcal{W}_2$ is the 2-Wasserstein distance. In particular, the one particle marginal $f_1^N(t)$ converges to $g(t)$ weakly , as $N \to +\infty$ for any $t \geq 0$.

Results

Step 2) is just analysis.

$$\mathcal{W}_2\big(f_j^N(t), f(t)^{\otimes j}\big) \leq \mathcal{W}_2\big(f_j^N(t), g(t)^{\otimes j}\big) + \mathcal{W}_2\big(f(t)^{\otimes j}, g(t)^{\otimes j}\big).$$

Last term is controlled in a stronger (weighed L^1) norm, via PP paper.

Heuristics

The interaction is mean-field, thus we expect that propagation of chaos does hold. If this is true

$$\frac{d}{dt}\int dx_1\, dv_1\, f_1^N(x_1,v_1,t)\psi(x_1,v_1) = \frac{d}{dt}\int dZ_N\, F_N(Z_N,t)\psi(x_1,v_1)$$
$$= \int dx_1\, dv_1\, f_1^N(x_1,v_1,t)(v_1\cdot\nabla_{x_1}-1)\psi(x_1,v_1)$$
$$+\int dZ_N\, F_N(Z_N,t)\int d\tilde{x}_1\, d\tilde{v}_1\,\varphi(\tilde{x}_1-x_1)M^{\varphi}_{Z_N}(\tilde{x}_1,\tilde{v}_1)\psi(\tilde{x}_1,\tilde{v}_1)\,,$$

where ψ is a test function on the one-particle state space. Now, due to the law of large numbers, if Z_N is distributed according to $F_N(t)\approx f_1^N(t)^{\otimes N}$ then

Heuristics

$$\frac{1}{N}\sum_i \delta(x-x_i)\delta(v-v_i) \approx f_1^N(x,v) \quad \text{(weakly)}\,,$$

and

$$M^{\varphi}_{Z_N}(\tilde{x}_1,\tilde{v}_1) \approx M^{\varphi}_{f_1^N}(\tilde{x}_1,\tilde{v}_1)\,.$$

Therefore,

$$\int dZ_N\, F_N(Z_N,t) \int d\tilde{x}_1\, d\tilde{v}_1\, \varphi(\tilde{x}_1 - x_1) M^{\varphi}_{Z_N}(\tilde{x}_1,\tilde{v}_1)\psi(\tilde{x}_1,\tilde{v}_1)$$
$$\approx \int dx_1\, dv_1\, f_1^N(x_1,v_1,t) \int d\tilde{x}_1\, d\tilde{v}_1\, \varphi(\tilde{x}_1 - x_1) M^{\varphi}_{f_1^N}(\tilde{x}_1,\tilde{v}_1)\psi(\tilde{x}_1,\tilde{v}_1)$$
$$= \int d\tilde{x}_1\, d\tilde{v}_1\, \varrho^{\varphi}_{f_1^N}(\tilde{x}_1) M^{\varphi}_{f_1^N}(\tilde{x}_1,\tilde{v}_1)\psi(\tilde{x}_1,\tilde{v}_1)\,,$$

and the claim follows.

Conclusions

Propagation of chaos is obtained by coupling techniques.

Thank you