Asymptotic preserving scheme for Lévy Fokker Planck equation with fractional diffusion limit

Li Wang

School of Mathematics
University of Minnesota

Kinetic equations: from modeling computation to analysis

Centre International de Rencontres Mathématiques

March 22–26, 2021

Joint with **Wuzhe Xu**

Outline

Outline

Lévy Fokker Planck equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \underbrace{\nabla_v \cdot (vf) - (-\Delta_v)^s f}_{:=\mathcal{L}^s(f)}, \qquad s \in (0,1)\,, \\ f(0,x,v) = f_{in}(x,v)\,, \end{cases}$$

- fractional Laplacian:

$$(-\Delta_v)^s f(v) := C_{s,d} \text{P.V.} \int_{\mathbb{R}^d} \frac{f(v) - f(w)}{|v-w|^{d+2s}} \mathrm{d}w, \qquad C_{s,d} = \frac{4^s \Gamma(d/2+s)}{\pi^{d/2} |\Gamma(-s)|}$$

$$(-\Delta_v)^s f(v) := \mathcal{F}\left[|k|^{2s} \mathcal{F}[f](k)\right](v)$$

- $s = 1$ reduces to the classical Fokker-Planck operator
- application: nonlocal effect in plasma turbulence

Properties of $\mathcal{L}^s$

- conservation of mass
- equilibrium

$$\mathcal{L}^s(\mathcal{M}) = 0, \quad \int_{\mathbb{R}^d} \mathcal{M}(v)\mathrm{d}v = 1, \quad \mathcal{M}(v) \sim \frac{C_0}{|v|^{d+2s}} \text{ as } |v| \to \infty\,.$$

- entropy dissipation: $\partial_t f = \mathcal{L}^s f$ [1]

$$G^{\Phi}_{\mathcal{M}}(f) := \int_{\mathbb{R}^d} \Phi(f)\mathcal{M}\mathrm{d}v - \Phi\left(\int_{\mathbb{R}^d} f\mathcal{M}\mathrm{d}v\right)$$

$$E_\Phi(t) = G^{\Phi}_{\mathcal{M}}\left(\frac{f}{\mathcal{M}}\right), \qquad E_\Phi(t) \le e^{-\frac{t}{C}} E_\Phi(0), \quad t \ge 0$$

[1] Gentil, Imbert, Asymptotic Analysis 2008.

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v)\,. \end{cases}$$

Q: why anomalous scaling?

$$\varepsilon^2\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) + \Delta_v f^\varepsilon$$

$$\varepsilon \to 0 \Longrightarrow f \to \rho e^{-\frac{|v|^2}{2}}$$

$$\int_{\mathbb{R}^N} \mathrm{d}v \Longrightarrow \partial_t \rho + \frac{1}{\varepsilon}\nabla_x \cdot j = 0$$

$$\int_{\mathbb{R}^N} v \cdot \;\mathrm{d}v \Longrightarrow \varepsilon\partial_t j + \nabla_x \cdot \int_{\mathbb{R}^N} v \otimes v f \mathrm{d}v = -\frac{1}{\varepsilon} j$$

$$\Longrightarrow \quad \partial_t \rho + \nabla_x \cdot (D \nabla_x \rho) = 0, \quad D = \int v \otimes v e^{-\frac{|v|^2}{2}} \mathrm{d}v$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v) \,. \end{cases}$$

Q: why anomalous scaling?

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) + \Delta_v f^\varepsilon$$

$$\varepsilon \to 0 \Longrightarrow f \to \rho e^{-\frac{|v|^2}{2}}$$

$$\int_{\mathbb{R}^N} \mathrm{d}v \Longrightarrow \partial_t \rho + \frac{1}{\varepsilon} \nabla_x \cdot j = 0$$

$$\int_{\mathbb{R}^N} v \cdot \; \mathrm{d}v \Longrightarrow \varepsilon \partial_t j + \nabla_x \cdot \int_{\mathbb{R}^N} v \otimes v f \mathrm{d}v = -\frac{1}{\varepsilon} j$$

$$\Longrightarrow \quad \partial_t \rho + \nabla_x \cdot (D \nabla_x \rho) = 0, \quad D = \int v \otimes v e^{-\frac{|v|^2}{2}} \mathrm{d}v$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v)\,. \end{cases}$$

Q: why anomalous scaling?

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) + \Delta_v f^\varepsilon$$

$$\varepsilon \to 0 \Longrightarrow f \to \rho e^{-\frac{|v|^2}{2}}$$

$$\int_{\mathbb{R}^N} \,\mathrm{d}v \Longrightarrow \partial_t \rho + \frac{1}{\varepsilon}\nabla_x \cdot j = 0$$

$$\int_{\mathbb{R}^N} v \cdot \,\mathrm{d}v \Longrightarrow \varepsilon \partial_t j + \nabla_x \cdot \int_{\mathbb{R}^N} v \otimes v f \mathrm{d}v = -\frac{1}{\varepsilon} j$$

$$\Longrightarrow \quad \partial_t \rho + \nabla_x \cdot (D \nabla_x \rho) = 0, \quad D = \int v \otimes v e^{-\frac{|v|^2}{2}} \mathrm{d}v$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v)\,. \end{cases}$$

Theorem (Cesbron, Mellet, Trivisa 2012)

Assume that $f_0 \in L^2(\mathbb{R}^N, \mathcal{M}(v)^{-1}\mathrm{d}v\mathrm{d}x)$*, where* $\mathcal{M}(v)$ *is the unique normalized equilibrium distribution. Then, up to a subsequence, the solution* f^ε *converges weakly in* $L^\infty(0,T; L^2(\mathbb{R}^{2d}, \mathcal{M}(v)^{-1}\mathrm{d}v\mathrm{d}x))$ *to* $\rho(t,x)\mathcal{M}(v)$ *as* $\varepsilon \to 0$*, where* $\rho(t,x)$ *solves*

$$\begin{cases} \partial_t \rho + (-\Delta_x)^s \rho = 0\,, \\ \rho(0,x) = \rho_{in}(x) := \int_{\mathbb{R}^d} f_{in}(x,v)\mathrm{d}v\,. \end{cases}$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v)\,. \end{cases}$$

- $f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)\mathcal{M}(v)$ weakly in $L^2_{\mathcal{M}^{-1}}(\mathbb{R}^N \times (0,\infty))$, $\rho = \int f \mathrm{d}v$
- consider a test function $\phi(t,x,v) = \varphi(t, x + \varepsilon v)$

$$\int f^\varepsilon \left[\varepsilon^{2s}\partial_t \phi + \varepsilon v \cdot \nabla_x \phi - v \cdot \nabla_v \phi + (-\Delta_v)^s \phi\right] \mathrm{d}x\mathrm{d}v\mathrm{d}t$$

$$+ \varepsilon^{2s} \int f_0(x,v)\phi(0,x,v)\mathrm{d}x\mathrm{d}v = 0$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s} \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0, x, v) = f_{in}(x, v) \, . \end{cases}$$

- $f^\varepsilon(t, x, v) \rightharpoonup \rho(t, x)\mathcal{M}(v)$ weakly in $L^2_{\mathcal{M}^{-1}}(\mathbb{R}^N \times (0, \infty))$, $\rho = \int f \mathrm{d}v$
- consider a test function $\phi(t, x, v) = \varphi(t, x + \varepsilon v)$

$$\int f^\varepsilon \left[\varepsilon^{2s} \partial_t \phi + \varepsilon v \cdot \nabla_x \phi - v \cdot \nabla_v \phi + (-\Delta_v)^s \phi \right] \mathrm{d}x \mathrm{d}v \mathrm{d}t + \varepsilon^{2s} \int f_0(x, v) \phi(0, x, v) \mathrm{d}x \mathrm{d}v = 0$$

Fractional diffusion limit

$x \mapsto \varepsilon x, \quad t \mapsto \varepsilon^{2s} t$

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (v f^\varepsilon) - (-\Delta_v)^s f^\varepsilon, \\ f^\varepsilon(0,x,v) = f_{in}(x,v)\,. \end{cases}$$

- $f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)\mathcal{M}(v)$ weakly in $L^2_{\mathcal{M}^{-1}}(\mathbb{R}^N \times (0,\infty))$, $\rho = \int f \mathrm{d}v$
- consider a test function $\phi(t,x,v) = \varphi(t, x+\varepsilon v)$

$$\int f^\varepsilon \Big[\varepsilon^{2s}\partial_t\varphi + \underbrace{\varepsilon v \cdot \nabla_x \varphi - v \cdot \nabla_v \varphi}_{=0} + \underbrace{(-\Delta_v)^s \phi}_{\varepsilon^{2s}(-\Delta_x)^s\varphi} \Big] \mathrm{d}x\mathrm{d}v\mathrm{d}t$$

$$+ \varepsilon^{2s} \int f_0(x,v)\phi(0,x,v)\mathrm{d}x\mathrm{d}v = 0$$

$$\Longrightarrow \int f^\varepsilon \left[\partial_t\varphi + (-\Delta_x)^s\varphi\right] \mathrm{d}x\mathrm{d}v\mathrm{d}t + \int f_0(x,v)\varphi(0,x+\varepsilon v)\mathrm{d}x\mathrm{d}v = 0$$

$$\stackrel{\varepsilon\to 0}{\Longrightarrow} \int \rho(t,x)[\partial_t\varphi + (-\Delta_x)^s\varphi](t,x)\mathrm{d}x\mathrm{d}t + \int \rho_0(x)\varphi(0,x)\mathrm{d}x = 0$$

Outline

Numerical challenges

$$\begin{cases} \varepsilon^{2s}\partial_t f + \varepsilon v \cdot \nabla_x f = \nabla_v \cdot (vf) - (-\Delta_v)^s f \\ f(0,x,v) = f_{in}(x,v) \end{cases}$$

$$\begin{cases} \partial_t \rho + (-\Delta_x)^s \rho = 0 \\ \rho(0,x) = \int f_{in}(x,v)\mathrm{d}v \end{cases}$$

- Nonlocality $(-\Delta_v)^s \frac{1}{|v|^{d+2s}}$
  - finite difference/finite element + domain truncation
    [Huang, Oberman, 2014] [Zhang, Deng, Karniadakis, 2018]
  - spectral method
    [Mao, Shen, 2017] *Hermite polynomial*
    [Cayama, Cuesta, Hoz, 2019] *mapped Chebyshev polynomial*
    [Sheng, Shen, Tang, Wang, Yuan, 2020] *Donford-Taylor formula*
  - review
    [Bonito et al., 2018] [Lishke et al., 2020]
- Stiffness

  [W., Yan, 2016], [Crouseilles, Hivert, Lemou, 2016], [W., Yan, 2019]

Numerical challenges

$$\begin{cases} \varepsilon^{2s}\partial_t f + \varepsilon v \cdot \nabla_x f = \nabla_v \cdot (vf) - (-\Delta_v)^s f \\ f(0,x,v) = f_{in}(x,v) \end{cases}$$

$$\begin{cases} \partial_t \rho + (-\Delta_x)^s \rho = 0 \\ \rho(0,x) = \int f_{in}(x,v)\mathrm{d}v \end{cases}$$

- Nonlocality $(-\Delta_v)^s \frac{1}{|v|^{d+2s}}$
  - finite difference/finite element + domain truncation
    [Huang, Oberman, 2014] [Zhang, Deng, Karniadakis, 2018]
  - spectral method
    [Mao, Shen, 2017] *Hermite polynomial*
    [Cayama, Cuesta, Hoz, 2019] *mapped Chebyshev polynomial*
    [Sheng, Shen, Tang, Wang, Yuan, 2020] *Donford-Taylor formula*
  - review
    [Bonito et al., 2018] [Lishke et al., 2020]
- Stiffness
  [W., Yan, 2016], [Crouseilles, Hivert, Lemou, 2016], [W., Yan, 2019]

Numerical challenges

$$\begin{cases} \varepsilon^{2s}\partial_t f + \varepsilon v \cdot \nabla_x f = \nabla_v \cdot (vf) - (-\Delta_v)^s f \\ f(0,x,v) = f_{in}(x,v) \end{cases}$$

$$\begin{cases} \partial_t \rho + (-\Delta_x)^s \rho = 0 \\ \rho(0,x) = \int f_{in}(x,v)\mathrm{d}v \end{cases}$$

- Nonlocality $(-\Delta_v)^s \; \frac{1}{|v|^{d+2s}}$
  - finite difference/finite element + domain truncation

    [Huang, Oberman, 2014] [Zhang, Deng, Karniadakis, 2018]
  - spectral method

    [Mao, Shen, 2017] *Hermite polynomial*

    [Cayama, Cuesta, Hoz, 2019] *mapped Chebyshev polynomial*

    [Sheng, Shen, Tang, Wang, Yuan, 2020] *Donford-Taylor formula*
  - review

    [Bonito et al., 2018] [Lishke et al., 2020]
- Stiffness

  [W., Yan, 2016], [Crouseilles, Hivert, Lemou, 2016], [W., Yan, 2019]

Computation of $(-\Delta_v)^s$

[Cayama, Cuesta, Hoz, 2019]

- mapping $v \mapsto \xi$

$$\xi = \frac{v}{\sqrt{L_v^2 + v^2}} \in (-1,1) \Longleftrightarrow v = \frac{L_v \xi}{\sqrt{1-\xi^2}} \in (-\infty, \infty)\,,$$

- Chebyshev polynomials

$$T_k(\xi) = \cos(k \arccos(\xi))\,, \qquad \xi \in [-1,1]$$

- further change of variable $q = \arccos(\xi) \in [0,\pi]$

$$T_k(\xi) = \cos(kq)$$

- In the new variable q ($v = \frac{L_v \xi}{\sqrt{1-\xi^2}} = L_v \cot(q)$)

$$(-\Delta_q)^s f(q) = \begin{cases} -\frac{1}{L_v \pi} \int_0^\pi \frac{f'(p)}{\cot(q)-\cot(p)} \mathrm{d}p, & s = \frac{1}{2}\,, \\ \frac{C_{s,d}}{2L_v^{2s} s(1-2s)} \int_0^\pi \frac{\sin^2(p) f''(p) + 2\sin(p)\cos(p) f'(p)}{|\cot(q)-\cot(p)|^{2s-1}} \mathrm{d}p, & s \neq \frac{1}{2}\,. \end{cases}$$

Computation of $(-\Delta_v)^s$

[Cayama, Cuesta, Hoz, 2019]

- mapping $v \mapsto \xi$

$$\xi = \frac{v}{\sqrt{L_v^2 + v^2}} \in (-1,1) \Longleftrightarrow v = \frac{L_v \xi}{\sqrt{1-\xi^2}} \in (-\infty, \infty)\,,$$

- Chebyshev polynomials

$$T_k(\xi) = \cos(k \arccos(\xi))\,, \qquad \xi \in [-1,1]$$

- further change of variable $q = \arccos(\xi) \in [0, \pi]$

$$T_k(\xi) = \cos(kq)$$

- In the new variable q ($v = \frac{L_v \xi}{\sqrt{1-\xi^2}} = L_v \cot(q)$)

$$(-\Delta_q)^s f(q) = \begin{cases} -\frac{1}{L_v \pi} \int_0^\pi \frac{f'(p)}{\cot(q)-\cot(p)} \mathrm{d}p, & s = \frac{1}{2}\,, \\ \frac{C_{s,d}}{2L_v^{2s} s(1-2s)} \int_0^\pi \frac{\sin^2(p) f''(p) + 2\sin(p)\cos(p) f'(p)}{|\cot(q)-\cot(p)|^{2s-1}} \mathrm{d}p, & s \neq \frac{1}{2}\,. \end{cases}$$

Computation of $(-\Delta_v)^s$

[Cayama, Cuesta, Hoz, 2019]

- mapping $v \mapsto \xi$

$$\xi = \frac{v}{\sqrt{L_v^2 + v^2}} \in (-1,1) \Longleftrightarrow v = \frac{L_v \xi}{\sqrt{1-\xi^2}} \in (-\infty, \infty)\,,$$

- Chebyshev polynomials

$$T_k(\xi) = \cos(k \arccos(\xi))\,, \qquad \xi \in [-1,1]$$

- further change of variable $q = \mathrm{arc}\cos(\xi) \in [0,\pi]$

$$T_k(\xi) = \cos(kq)$$

- In the new variable q ($v = \frac{L_v \xi}{\sqrt{1-\xi^2}} = L_v \cot(q)$)

$$(-\Delta_q)^s f(q) = \begin{cases} -\frac{1}{L_v \pi} \int_0^\pi \frac{f'(p)}{\cot(q)-\cot(p)} \mathrm{d}p, & s = \frac{1}{2}\,, \\ \frac{C_{s,d}}{2L_v^{2s} s(1-2s)} \int_0^\pi \frac{\sin^2(p) f''(p) + 2\sin(p)\cos(p) f'(p)}{|\cot(q)-\cot(p)|^{2s-1}} \mathrm{d}p, & s \neq \frac{1}{2}\,. \end{cases}$$

Computation of $(-\Delta_v)^s$

[Cayama, Cuesta, Hoz, 2019]

- mapping $v \mapsto \xi$

$$\xi = \frac{v}{\sqrt{L_v^2 + v^2}} \in (-1,1) \Longleftrightarrow v = \frac{L_v \xi}{\sqrt{1-\xi^2}} \in (-\infty, \infty)\,,$$

- Chebyshev polynomials

$$T_k(\xi) = \cos(k \arccos(\xi))\,, \qquad \xi \in [-1,1]$$

- further change of variable $\quad q = \text{arc}\cos(\xi) \in [0,\pi]$

$$T_k(\xi) = \cos(kq)$$

- In the new variable q ($v = \frac{L_v \xi}{\sqrt{1-\xi^2}} = L_v \cot(q)$)

$$(-\Delta_q)^s f(q) = \begin{cases} -\frac{1}{L_v \pi} \int_0^\pi \frac{f'(p)}{\cot(q)-\cot(p)} \mathrm{d}p, & s = \frac{1}{2}\,, \\ \frac{C_{s,d}}{2L_v^{2s} s(1-2s)} \int_0^\pi \frac{\sin^2(p) f''(p) + 2\sin(p)\cos(p) f'(p)}{|\cot(q)-\cot(p)|^{2s-1}} \mathrm{d}p, & s \neq \frac{1}{2}\,. \end{cases}$$

Computation of $(-\Delta_v)^s$

- even extension

$$\tilde{f}(q) = \begin{cases} f(q), & q \in [0, \pi] \\ f(2\pi - q), & q \in [\pi, 2\pi] \end{cases}$$

- discrete Fourier transform

$$f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k e^{ikq_j}, \quad q_j = \frac{\pi(2j+1)}{2N_v}, \qquad 0 \le j \le N_v - 1$$

- compute $(-\Delta_v)^s f$

$$(-\Delta_q)^s f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k (-\Delta_q)^s e^{ikq_j}, \qquad j = 0, \cdots, 2N_v - 1\,.$$

- compute $(-\Delta_v)^s \left(e^{imq}\right)$

Computation of $(-\Delta_v)^s$

- even extension

$$\tilde{f}(q) = \begin{cases} f(q), & q \in [0, \pi] \\ f(2\pi - q), & q \in [\pi, 2\pi] \end{cases}$$

- discrete Fourier transform

$$f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k e^{ikq_j}, \quad q_j = \frac{\pi(2j+1)}{2N_v}, \qquad 0 \leq j \leq N_v - 1$$

- compute $(-\Delta_v)^s f$

$$(-\Delta_q)^s f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k (-\Delta_q)^s e^{ikq_j}, \qquad j = 0, \cdots, 2N_v - 1.$$

- compute $(-\Delta_v)^s \left(e^{imq}\right)$

Computation of $(-\Delta_v)^s$

- even extension

$$\tilde{f}(q) = \begin{cases} f(q), & q \in [0, \pi] \\ f(2\pi - q), & q \in [\pi, 2\pi] \end{cases}$$

- discrete Fourier transform

$$f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k e^{ikq_j}\,, \quad q_j = \frac{\pi(2j+1)}{2N_v}, \qquad 0 \le j \le N_v - 1$$

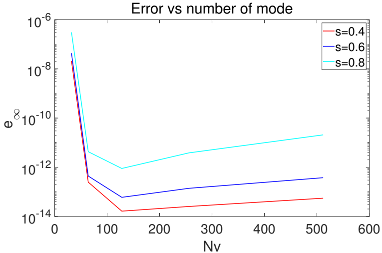
- compute $(-\Delta_v)^s f$

$$(-\Delta_q)^s f(q_j) = \sum_{k=-N_v}^{N_v-1} \hat{\tilde{f}}_k (-\Delta_q)^s e^{ikq_j}\,, \qquad j = 0, \cdots, 2N_v - 1\,.$$

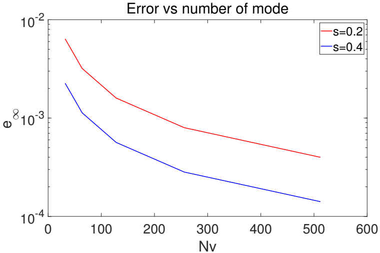
- compute $(-\Delta_v)^s \left(e^{imq}\right)$

Computation of $\mathcal{L}^s$

- Examples: $(-\Delta_v)^s f$



$$f(v) = e^{-v^2}$$



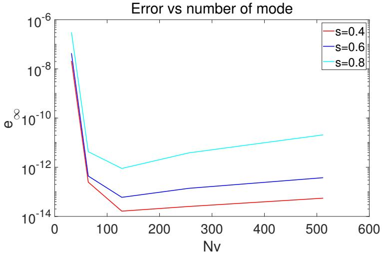
$$f(v) = (1+v^2)^{-\frac{1-2s}{2}}$$

- $\mathcal{L}$ in new variable q

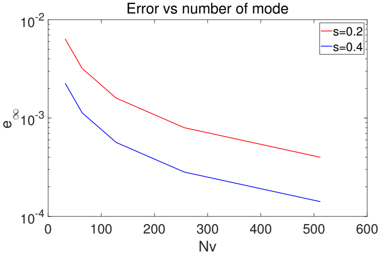
$$\mathcal{L}_q^s(f) := f - \cos(q)\sin(q)\partial_q f - (-\Delta_q)^s f$$

Computation of $\mathcal{L}^s$

- Examples: $(-\Delta_v)^s f$



$$f(v) = e^{-v^2}$$



$$f(v) = (1+v^2)^{-\frac{1-2s}{2}}$$

- $\mathcal{L}$ in new variable q

$$\mathcal{L}_q^s(f) := f - \cos(q)\sin(q)\partial_q f - (-\Delta_q)^s f$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t,x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) = \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M})$$
$$= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \mathcal{M}(-\Delta_v)^s\eta - I(\eta,\mathcal{M}) + \eta\mathcal{L}^s(\mathcal{M})$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t, x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\begin{aligned}\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) &= \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M}) \\ &= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \mathcal{M}(-\Delta_v)^s\eta - I(\eta,\mathcal{M}) + \eta\mathcal{L}^s(\mathcal{M})\end{aligned}$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t, x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) = \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M})$$
$$= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \underbrace{\mathcal{M}(-\Delta_v)^s\eta}_{\varepsilon^{2s}\mathcal{M}(-\Delta_x)^s\eta} - I(\eta,\mathcal{M}) + \underbrace{\eta\mathcal{L}^s(\mathcal{M})}_{=0}$$

$$I(f,g) = (-\Delta_v)^s(fg) - g(-\Delta_v)^s f - f(-\Delta_v)^s g$$
$$= C_{1,s}\int_{\mathbb{R}^d} \frac{(f(v)-f(w))(g(w)-g(v))}{|v-w|^{d+2s}}\mathrm{d}w\,.$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t, x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\begin{aligned} \varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) &= \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M}) \\ &= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \underbrace{\mathcal{M}(-\Delta_v)^s\eta}_{\varepsilon^{2s}\mathcal{M}(-\Delta_x)^s\eta} - I(\eta,\mathcal{M}) + \underbrace{\eta\mathcal{L}^s(\mathcal{M})}_{=0} \end{aligned}$$

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x g = \mathcal{L}^s(g) - \varepsilon^{2s}(-\Delta_x)^s\eta\mathcal{M} - I(\eta,\mathcal{M})$$

- Split system

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v\partial_x g = \mathcal{L}^s(g) - I(\eta,\mathcal{M}), \\ \partial_t\eta = -(-\Delta_x)^s\eta, \end{cases}$$

- Initial data

$$\eta_{in}(x,v) = \rho_{in}(x+\varepsilon v) \qquad g_{in}(x,v) = f_{in}(x,v) - \eta_{in}(x,v)\mathcal{M}(v).$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t, x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\begin{aligned}\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) &= \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M}) \\ &= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \underbrace{\mathcal{M}(-\Delta_v)^s\eta}_{\varepsilon^{2s}\mathcal{M}(-\Delta_x)^s\eta} - I(\eta,\mathcal{M}) + \underbrace{\eta\mathcal{L}^s(\mathcal{M})}_{=0}\end{aligned}$$

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x g = \mathcal{L}^s(g) - \varepsilon^{2s}(-\Delta_x)^s\eta\mathcal{M} - I(\eta,\mathcal{M})$$

- Split system

$$\begin{cases}\varepsilon^{2s}\partial_t g + \varepsilon v\partial_x g = \mathcal{L}^s(g) - I(\eta,\mathcal{M}), \\ \partial_t\eta = -(-\Delta_x)^s\eta,\end{cases}$$

- Initial data

$$\eta_{in}(x,v) = \rho_{in}(x+\varepsilon v) \qquad g_{in}(x,v) = f_{in}(x,v) - \eta_{in}(x,v)\mathcal{M}(v).$$

System decomposition

- Micro-macro decomposition

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v)$$
$$\eta(t,x,v) = h(t, x+\varepsilon v)$$

- Plug the decomposition to the equation

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M} + g) + \varepsilon v\partial_x(\eta\mathcal{M} + g) = \mathcal{L}^s(g) + \mathcal{L}^s(\eta\mathcal{M})$$
$$= \mathcal{L}^s(g) + v\mathcal{M}\cdot\partial_v\eta - \underbrace{\mathcal{M}(-\Delta_v)^s\eta}_{\varepsilon^{2s}\mathcal{M}(-\Delta_x)^s\eta} - I(\eta,\mathcal{M}) + \underbrace{\eta\mathcal{L}^s(\mathcal{M})}_{=0}$$
$$\varepsilon^{2s}\partial_t(\eta\mathcal{M} + g) + \varepsilon v\partial_x g = \mathcal{L}^s(g) - \varepsilon^{2s}(-\Delta_x)^s\eta\mathcal{M} - I(\eta,\mathcal{M})$$

- Split system

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v\partial_x g = \mathcal{L}^s(g) - I(\eta,\mathcal{M}) \\ \partial_t\eta = -(-\Delta_x)^s\eta \end{cases}$$

- Initial data

$$\eta_{in}(x,v) = \rho_{in}(x+\varepsilon v) \qquad g_{in}(x,v) = f_{in}(x,v) - \eta_{in}(x,v)\mathcal{M}(v)$$

Energy stability

Proposition

If (η, g) solves split system, then $f = \eta\mathcal{M} + g$ solves the original equation. Both system has the energy dissipation property. That is, define the total energy

$$E_f^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f^2}{\mathcal{M}} \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\eta\mathcal{M} + g)^2}{\mathcal{M}} \mathrm{d}v\mathrm{d}x\,,$$

then $\frac{dE_f}{dt} \leq 0$.

Proposition

$$E_\eta^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^2\mathcal{M}\mathrm{d}x\mathrm{d}v, \qquad E_g^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g^2}{\mathcal{M}} \mathrm{d}x\mathrm{d}v$$

are both uniformly bounded in time.

Semi-discrete scheme

To solve

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v \partial_x g = \mathcal{L}^s(g) - I(\eta, \mathcal{M}) \\ \partial_t \eta = -(-\Delta_x)^s \eta \end{cases}$$

we propose

$$\begin{cases} \frac{\varepsilon^{2s}}{\Delta t}(g^* - g^n) = \mathcal{L}^s(g^*) - \gamma g^* - I(\eta^n, \mathcal{M}) \\ \frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1} \\ \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^n \end{cases}$$

Compare to $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+1}) - I(\eta^n, \mathcal{M})$

- alleviate ill-conditioning

	$Cond(\mathsf{A}_0)$	$Cond(\mathsf{A}_{\frac{1}{2}})$	$Cond(\mathsf{A}_1)$
$s = 0.4,\ \varepsilon = 1$	4.73	4.54	4.36
$s = 0.6,\ \varepsilon = 1$	4.93	4.73	4.55
$s = 0.8,\ \varepsilon = 1$	13.28	12.66	12.11
$s = 0.4,\ \varepsilon = 1e-3$	7.74e3	158	52
$s = 0.6,\ \varepsilon = 1e-3$	1.24e5	187.42	63.44
$s = 0.8,\ \varepsilon = 1e-3$	5.97e6	564.32	190.75
$s = 0.4,\ \varepsilon = 1e-5$	3.57e5	181.66	55.84
$s = 0.6,\ \varepsilon = 1e-5$	3.15e7	188.98	63.67
$s = 0.8,\ \varepsilon = 1e-5$	9.46e9	564.62	190.79

Semi-discrete scheme

To solve

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v \partial_x g = \mathcal{L}^s(g) - I(\eta, \mathcal{M}) \\ \partial_t \eta = -(-\Delta_x)^s \eta \end{cases}$$

we propose

$$\begin{cases} \frac{\varepsilon^{2s}}{\Delta t}(g^* - g^n) = \mathcal{L}^s(g^*) - \gamma g^* - I(\eta^n, \mathcal{M}) \\ \frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1} \\ \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^n \end{cases}$$

Compare to $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+1}) - I(\eta^n, \mathcal{M})$

- alleviate ill-conditioning

	$Cond(\mathsf{A}_0)$	$Cond(\mathsf{A}_{\frac{1}{2}})$	$Cond(\mathsf{A}_1)$
$s = 0.4,\ \varepsilon = 1$	4.73	4.54	4.36
$s = 0.6,\ \varepsilon = 1$	4.93	4.73	4.55
$s = 0.8,\ \varepsilon = 1$	13.28	12.66	12.11
$s = 0.4,\ \varepsilon = 1e-3$	7.74e3	158	52
$s = 0.6,\ \varepsilon = 1e-3$	1.24e5	187.42	63.44
$s = 0.8,\ \varepsilon = 1e-3$	5.97e6	564.32	190.75
$s = 0.4,\ \varepsilon = 1e-5$	3.57e5	181.66	55.84
$s = 0.6,\ \varepsilon = 1e-5$	3.15e7	188.98	63.67
$s = 0.8,\ \varepsilon = 1e-5$	9.46e9	564.62	190.79

Semi-discrete scheme

To solve

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v \partial_x g = \mathcal{L}^s(g) - I(\eta, \mathcal{M}) \\ \partial_t \eta = -(-\Delta_x)^s \eta \end{cases}$$

we propose

$$\begin{cases} \frac{\varepsilon^{2s}}{\Delta t}(g^* - g^n) = \mathcal{L}^s(g^*) - \gamma g^* - I(\eta^n, \mathcal{M}) \\ \frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1} \\ \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^n \end{cases}$$

Compare to $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+1}) - I(\eta^n, \mathcal{M})$

- alleviate ill-conditioning

	$Cond(\mathsf{A}_0)$	$Cond(\mathsf{A}_{\frac{1}{2}})$	$Cond(\mathsf{A}_1)$
$s = 0.4,\ \varepsilon = 1$	4.73	4.54	4.36
$s = 0.6,\ \varepsilon = 1$	4.93	4.73	4.55
$s = 0.8,\ \varepsilon = 1$	13.28	12.66	12.11
$s = 0.4,\ \varepsilon = 1e-3$	7.74e3	158	52
$s = 0.6,\ \varepsilon = 1e-3$	1.24e5	187.42	63.44
$s = 0.8,\ \varepsilon = 1e-3$	5.97e6	564.32	190.75
$s = 0.4,\ \varepsilon = 1e-5$	3.57e5	181.66	55.84
$s = 0.6,\ \varepsilon = 1e-5$	3.15e7	188.98	63.67
$s = 0.8,\ \varepsilon = 1e-5$	9.46e9	564.62	190.79

Semi-discrete scheme

To solve

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v \partial_x g = \mathcal{L}^s(g) - I(\eta, \mathcal{M}) + \eta \mathcal{L}^s(\mathcal{M}) \,, \\ \partial_t \eta = -(-\Delta_x)^s \eta \,, \end{cases}$$

we propose

$$\begin{cases} \frac{\varepsilon^{2s}}{\Delta t}(g^* - g^n) = \mathcal{L}^s(g^*) - \gamma g^* - I(\eta^n, \mathcal{M}) \\ \frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1} \\ \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^n \end{cases}$$

Compare to $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+1}) - I(\eta^n, \mathcal{M})$

- alleviate ill-conditioning
- reduce computational cost
- asymptotic property

Semi-discrete scheme

To solve

$$\begin{cases} \varepsilon^{2s}\partial_t g + \varepsilon v \partial_x g = \mathcal{L}^s(g) - I(\eta, \mathcal{M}) + \eta \mathcal{L}^s(\mathcal{M}) \,, \\ \partial_t \eta = -(-\Delta_x)^s \eta \,, \end{cases}$$

we propose

$$\begin{cases} \frac{\varepsilon^{2s}}{\Delta t}(g^* - g^n) = \mathcal{L}^s(g^*) - \gamma g^* - I(\eta^n, \mathcal{M}) \\ \frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1} \\ \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^n \end{cases}$$

Compare to $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+1}) - I(\eta^n, \mathcal{M})$

- alleviate ill-conditioning
- reduce computational cost
- asymptotic property

AP property

Proposition

The numerical solution $\rho^n = \langle f^n \rangle = \langle \eta^n \mathcal{M} + g^n \rangle$ *satisfies*

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = (-\Delta_x)^s \rho^n, \qquad \text{as} \quad \varepsilon \to 0 .$$

- From the reconstruction formula

$$\begin{aligned} \frac{\rho^{n+1} - \rho^n}{\Delta t} &= \left\langle \frac{\eta^{n+1} - \eta^n}{\Delta t} \mathcal{M} \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\ &= -\left\langle (-\Delta_x)^s \eta^n \mathcal{M} \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\ &= (-\Delta_x)^s \left\langle f^n - g^n \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\ &= (-\Delta_x)^s \rho^n - (-\Delta_x)^s \left\langle g^n \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \end{aligned}$$

AP property

Proposition

The numerical solution $\rho^n = \langle f^n \rangle = \langle \eta^n \mathcal{M} + g^n \rangle$ *satisfies*

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = (-\Delta_x)^s \rho^n, \qquad as \quad \varepsilon \to 0\,.$$

- From the reconstruction formula

$$\begin{aligned}
\frac{\rho^{n+1} - \rho^n}{\Delta t} &= \left\langle \frac{\eta^{n+1} - \eta^n}{\Delta t} \mathcal{M} \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\
&= - \left\langle (-\Delta_x)^s \eta^n \mathcal{M} \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\
&= (-\Delta_x)^s \left\langle f^n - g^n \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle \\
&= (-\Delta_x)^s \rho^n - (-\Delta_x)^s \left\langle g^n \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle
\end{aligned}$$

AP property

Proposition

The numerical solution $\rho^n = \langle f^n \rangle = \langle \eta^n \mathcal{M} + g^n \rangle$ *satisfies*

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = (-\Delta_x)^s \rho^n, \qquad as \quad \varepsilon \to 0\,.$$

- From the reconstruction formula

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = \left\langle \frac{\eta^{n+1} - \eta^n}{\Delta t} \mathcal{M} \right\rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle$$
$$= (-\Delta_x)^s \rho^n - (-\Delta_x)^s \langle g^n \rangle + \left\langle \frac{g^{n+1} - g^n}{\Delta t} \right\rangle$$

- $\left(\frac{\varepsilon^{2s}}{\Delta t} + \gamma + \mathcal{L}^s \right) g^* = \frac{\varepsilon^{2s}}{\Delta t} g^n - I(\eta^n, \mathcal{M})$

  $\|g^*(x,\cdot)\|_{L^\infty_v} \lesssim \|I(\eta^n, \mathcal{M})\|_{L^\infty_v} + \frac{\varepsilon^{2s}}{\Delta t} \|g^n\|_{L^\infty_v}$

- $\frac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^*) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1}$

  $|g^{n+1}(x,v)| \leq \frac{2\varepsilon^{2s}}{\Delta t} \|g^*(\cdot, v)\|_{L^\infty_x}$

Outline

Long time behavior

$$\partial_t f = \partial_v(vf) - (-\Delta_v)^s f\,, \qquad f(0,v) = e^{-v^2}$$

$s = 0.5, \quad \mathcal{M}(v) = \pi^{-0.5}(1+v^2)^{-1}$:

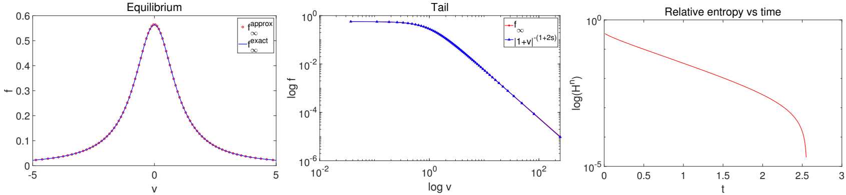


Figure: Here $\Delta t = 0.01$ and $N_v = 128$

Long time behavior

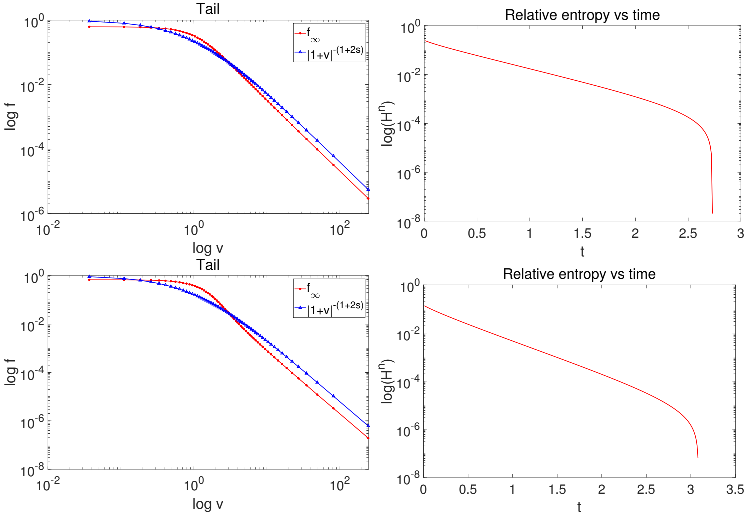


Figure: Here use $N_v = 128$, $\Delta t = 0.01$. top: $s = 0.6$. bottom: $s = 0.8$.

Uniform accuracy in time

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (vf) - (-\Delta_v)^s f := \mathcal{L}^s(f)\,, \quad s \in (0,1) \\ f(0,x,v) = \pi^{-0.5} e^{-15x^2} e^{-v^2}\,, \quad x \in [-5,5] \end{cases}$$

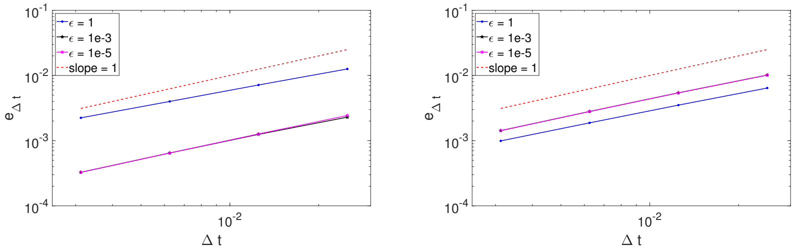


Figure: Convergence test for $s = 0.4$ (left) and $s = 0.8$ (right). $L_x = 5$, $N_x = 200$, $L_v = 3$ and $N_v = 128$, $\Delta t = 0.025, 0.0125, 0.00625, 0.003125, 0.0015625$.

Stability

$$E_f = \sum_{i=1}^{N_x} \sum_{j=1}^{N_v} \frac{f_{i,j}^2}{\mathcal{M}_j} w_j \Delta q \Delta x\,, \quad w_j = \frac{L_v}{(\sin(q_j))^2}$$

$$E_g = \sum_{i=1}^{N_x} \sum_{j=1}^{N_v} \frac{g_{i,j}^2}{\mathcal{M}_j} w_j \Delta q \Delta x\,, \quad E_\eta = \sum_{i=1}^{N_x} \sum_{j=1}^{N_v} \eta_{i,j}^2 \mathcal{M}_j w_j \Delta q \Delta x\,.$$

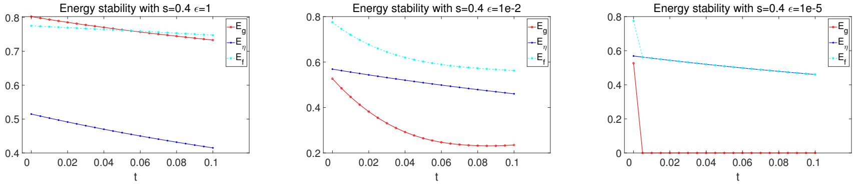


Figure: $L_x = 5,\ L_v = 3,\ N_x = 100,\ N_v = 128,\ \Delta t = 0.01.$

Kinetic regime

Reference solution:

$$\begin{cases} \frac{f^* - f^n}{\Delta t} + v\partial_x f^n = 0, \\ \frac{f^{n+1} - f^*}{\Delta t} = \partial_v(v f^{n+1}) - (-\Delta_v)^s f^{n+1}. \end{cases}$$

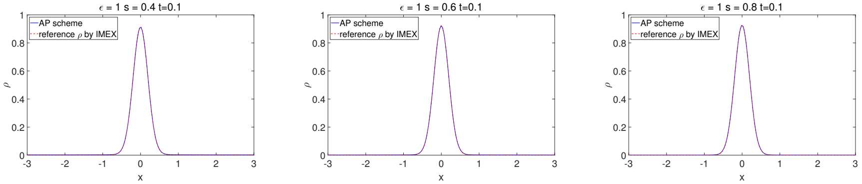


Figure: For our AP scheme, $N_x = 200$, $N_v = 128$, $\Delta t = 0.01$. For reference solution, $N_x = 800$, $N_v = 256$, $\Delta t = 1e-4$.

Diffusive regime

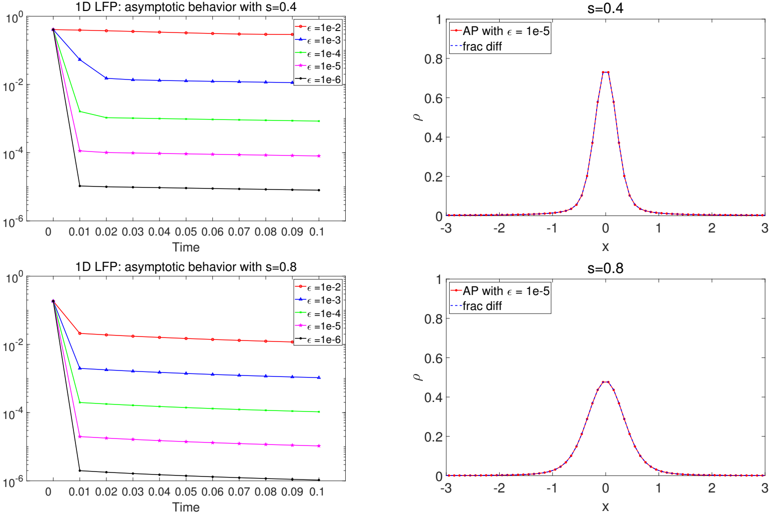


Figure: $T = 0.1$, $N_x = 100$, $\Delta t = 0.01$, $N_v = 128$.

Outline

Conclusion

We have developed an *asymptotic preserving* scheme for Lévy Fokker Planck equation with *fractional* diffusion limit.

Key ideas:

- spectral method with nonlocal basis
- a new macro-micro decomposition
- an operator splitting

Thank you!

Conclusion

We have developed an *asymptotic preserving* scheme for Lévy Fokker Planck equation with *fractional* diffusion limit.

Key ideas:

- spectral method with nonlocal basis
- a new macro-micro decomposition
- an operator splitting

Thank you!