Thick sprays

Laurent Desvillettes, Université de Paris, IMJ-PRG

March 24, 2021

In collaboration with

Laurent Boudin, Sorbonne Univ.

Christophe Buet, CEA

Bruno Desprès, Sorbonne Univ.

François Golse, Ecole Polytechnique

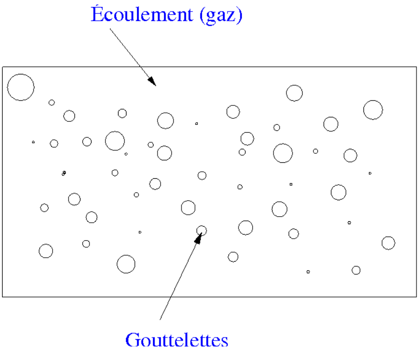
Julien Mathiaud, CEA and Univ. Bordeaux

Renaud Motte, CEA

Valeria Ricci, Univ. Palermo

Sprays

Dispersed Phase (i.-e. of small volume fraction, for example **droplets** or **dust**) in an Underlying Gas: their study is a subdomain of the study of multiphase flows



Examples : Clouds, Diesel Engines.

Modeling of sprays at the Mesoscopic Level (Eulerian-Lagrangian methods; fluid-kinetic coupling)

- Unknowns for the gas :

$$\alpha(t,x) \geq 0 \qquad \rho_g(t,x) \geq 0, \qquad u_g(t,x) \in \mathbb{R}^3, \qquad p(t,x) \geq 0;$$

- Unknown for the dispersed phase :

$$f(t,x,v) \geq 0;$$

Models explicitly written as PDEs: Williams 74; O'Rourke 81

Classification of sprays:Cf. P.J. O'Rourke

Critical quantity : Droplets volume fraction $1-\alpha(t,x)$.

Example of thin spray equations $1-\alpha(t,x) << 1$

Vlasov-compressible (barotropic) Euler equation with a two-way coupling by drag force, monodisperse (medical sprays in the mouth/throat)

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g\, u_g) = 0,$$

$$\partial_t(\rho_g\, u_g) + \nabla_x \cdot (\rho_g\, u_g \otimes u_g) + \nabla_x[p(\rho_g)] = \int -m\, F\, f\, dv,$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [F\, f] = 0,$$

$$m\, F = -D(\rho_g, |v - u_g|)\,(v - u_g).$$

Coherence of thin sprays equations

- **Well posedness**: There exist unique strong local-in time solutions to thin spray equations for smooth ($C_c^1 \cap H^s$) initial data and "reasonable" D, p, Baranger, LD.

- **Derivation from well-established models**: There exists a formal asymptotics leading from coupled Boltzmann equations towards thin spray equations (with evolution of the energy) Golse, LD, Ricci.

Example of thick spray equations: $1-\alpha(t,x) \sim 1 - 10\%$

Vlasov-Boltzmann-compressible (barotropic) Euler equations with two-way coupling through drag force and volume fraction (nuclear industry, pharmaceutical industry)

$$\partial_t(\alpha\,\rho_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g) = 0,$$

$$\partial_t(\alpha\,\rho_g\,u_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g \otimes u_g) + \alpha\,\nabla_x[p(\rho_g)] = \int D(\rho_g, |v-u_g|)\,(v-u_g)\,f\,dv,$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F\,f) = Q(f).$$

$$1 - \alpha = \int \frac{4}{3}\,\pi\,r^3\,f\,dv,$$

$$m\,F = -\frac{4}{3}\,\pi\,r^3\,\nabla_x[p(\rho_g)] - D(\rho_g, |v-u_g|)\,(v-u_g).$$

Collision kernel, with possibly inelastic collisions

$$Q(f)(t,x,v) = \int_{v_* \in \mathbb{R}^3} \int_{\sigma \in S^2} \Big\{ \beta^{-4}\, f(t,x,v')\, f(t,x,v'_*)$$
$$- f(t,x,v)\, f(t,x,v_*) \Big\}\, B\, dv_*\, d\sigma.$$

Cross section

$$B(|v - v_*|) = r^2 |v - v_*|\,.$$

Post collisional velocities ($\beta \in]0,1]$)

$$v' = \frac{v + v_*}{2} + \frac{1}{2\,\beta} |v - v_*|\, \sigma,$$

$$v'_* = \frac{v + v_*}{2} - \frac{1}{2\,\beta} |v - v_*|\, \sigma.$$

Where are thick sprays equations coming from: Macroscopic modeling (Eulerian-Eulerian methods/Multiphase Euler equations)

Coupling through drag force and volume fraction of equations of Euler type (here one compressible and barotropic, the other incompressible):

$$\partial_t(\alpha\,\rho_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g) = 0,$$

$$\partial_t(\alpha\,\rho_g\,u_g)+\nabla_x\cdot(\alpha\,\rho_g\,u_g\otimes u_g)+\alpha\,\nabla_x[p(\rho_g)] = D_*(\alpha,\rho_g,|u_p-u_g|)\,(u_p-u_g),$$

$$\rho_p := cst,$$

$$\partial_t((1-\alpha)\,\rho_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p) = 0,$$

$$\partial_t((1-\alpha)\,\rho_p\,u_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p \otimes u_p) + (1-\alpha)\nabla_x[p(\rho_g)]$$
$$= -D_*(\alpha,\rho_g,|u_p-u_g|)\,(u_p-u_g).$$

Where are thick sprays equations coming from: Passage from mesoscopic to macroscopic modeling

For **monodisperse** sprays, there is a formal passage from **thick spray** equations [with an equation for internal energy] towards **multiphase Euler** equations [with an equation for internal energy], based on the Hibert expansion

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot (F\, f^\varepsilon) = \frac{1}{\varepsilon}\, Q(f^\varepsilon).$$

When $\varepsilon \to 0$,

$$m\, f^\varepsilon(t,x,v) \to (1-\alpha(t,x))\rho_p\, \delta_{v=u_p(t,x)}$$

at the formal level, where ρ_g, u_g, α, u_p satisfy the multiphase Euler equations, provided that that the collision kernel is inelastic $\beta < 1$, LD, Mathiaud.

Known difficulties of the macroscopic modeling

$$\partial_t(\alpha\,\rho_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g) = 0,$$

$$\partial_t(\alpha\,\rho_g\,u_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g \otimes u_g) + \alpha\,\nabla_x[p(\rho_g)] = D_*(\alpha, \rho_g, |u_p - u_g|)\,(u_p - u_g),$$

$$\rho_p := cst; \qquad \partial_t((1-\alpha)\,\rho_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p) = 0,$$

$$\partial_t((1-\alpha)\,\rho_p\,u_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p \otimes u_p) + (1-\alpha)\nabla_x[p(\rho_g)]$$

$$= -D_*(\alpha, \rho_g, |u_p - u_g|)\,(u_p - u_g).$$

Traditional method of derivation: averaging of microscopic equations [uses empirical closures], Cf. Ishii.

Difficulties: α outside of gradients; hyperbolicity not always satisfied (sometimes cured with the use of turbulent diffusion).

Do thick sprays equations suffer from the same difficulties?

Question: Are the equations of thick sprays as badly behaved (from the mathematical point of view) as the multiphase Euler equations?

Answer 1: We expect that the problem of shocks in nonconservative systems disappears, thanks to the use of the formulation:

$$\partial_t(\alpha\,\rho_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g) = 0,$$

$$\partial_t(\alpha\,\rho_g\,u_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g \otimes u_g) + \nabla_x[p(\rho_g)] = \int\int -m\,F\,f\,dvdr.$$

Answer 2: We expect that, as in the Chapman-Enskog asymptotics, in the limit $\varepsilon \to 0$, the thick spray equations behave like

$$\rho_p := cst; \qquad \partial_t((1-\alpha)\,\rho_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p) = 0,$$

$$\partial_t((1-\alpha)\,\rho_p\,u_p) + \nabla_x \cdot ((1-\alpha)\,\rho_p\,u_p \otimes u_p) + (1-\alpha)\nabla_x[p(\rho_g)]$$
$$= -D_*(\alpha, \rho_g, |u_p - u_g|)\,(u_p - u_g) + cst\,\varepsilon\Delta_x u_p.$$

Rigorous answer 2: Linear stability of thick sprays equations

To simplify, we assume that the spray is monodisperse with $m = \frac{4}{3}\pi r^3$, $Q = 0$, p is a strictly increasing function of ρ_g, and $D := cst$. Then the thick spray system becomes

$$\partial_t(\alpha\,\rho_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g) = 0,$$

$$\partial_t(\alpha\,\rho_g\,u_g) + \nabla_x \cdot (\alpha\,\rho_g\,u_g \otimes u_g) + \nabla_x[p(\rho_g)] = \int -m\,F\,f\,dv,$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F\,f) = 0,$$

$$1 - \alpha = m \int f\,dv,$$

$$m\,F = -m\,\nabla_x[p(\rho_g)] - D\,(v - u_g).$$

Linear stability: explicit solution

For any $\rho_0 > 0$, $T_* > 0$, $n_0 > 0$, and a nonnnegative function G, an explicit (homogeneous, but time-dependent) solution to the thick spray equations is given by

$$\rho_g(t,x) = \rho_0,$$

$$u_g(t,x) = 0,$$

$$\alpha(t,x) = 1 - m\, n_0,$$

$$f(t,x,v) = f_0(t,v) := e^{3(D/m)\, t}\, \frac{n_0}{(K\, T_\star)^{\frac{3}{2}}}\, G\left(\frac{e^{2(D/m)t}\, |v|^2}{2\, T_\star}\right),$$

as soon as $m\, n_0 < 1$ and

$$K^{3/2} := 4\pi \int_0^\infty G(u)\, \sqrt{2u}\, du.$$

Linear stability: linearization of the unknowns

$$\rho(t,x) = \rho_0 + \varepsilon\,\rho_1(t,x) + O(\varepsilon^2),$$

$$u(t,x) = \varepsilon\,u_1(t,x) + O(\varepsilon^2),$$

$$\alpha(t,x) = 1 - m\,n_0 + \varepsilon\,\alpha_1(t,x) + O(\varepsilon^2),$$

$$f(t,x,v) = f_0(t,v) + \varepsilon\,\sqrt{f_0(t,v)}\,e^{(D/m)\,t}\,g_1(t,x,v) + O(\varepsilon^2).$$

Linear stability: linearization of the equations

$$\alpha_0 \rho_0 \, \partial_t \tau_1 = \alpha_0 \, \nabla \cdot u_1 + m \nabla \cdot \int \sqrt{f_0} \, e^{(D/m)\, t} \, g_1 \, v \, dv,$$

$$\alpha_0 \rho_0 \, \partial_t u_1 = \alpha_0 \rho_0^2 c_0^2 \, \nabla \tau_1 + D \int v \sqrt{f_0} \, e^{(D/m)\, t} \, g_1 \, dv - \frac{D}{m} \, u_1 \int f_0 \, dv,$$

$$\partial_t g_1 + v \cdot \nabla_x g_1 - \frac{\rho_0^2 c_0^2}{T_\star} \sqrt{f_0} \, e^{(D/m)\, t} \, v \cdot \nabla \tau_1 \left(- \frac{G'}{G} \right) \left(\frac{|v|^2}{2\, T(t)} \right)$$

$$= \frac{D}{m\, T_\star} \sqrt{f_0} \, e^{(D/m)\, t} \, v \cdot u_1 \left(- \frac{G'}{G} \right) \left(\frac{|v|^2}{2\, T(t)} \right) - \frac{D}{m} g_1 + \frac{D}{m\, g_1} \nabla_v \cdot \left(\frac{1}{2} v g_1^2 \right),$$

where $\tau_1 = -\frac{\rho_1}{\rho_0^2}$ is the first order of the specific volume, and $c_0 = \sqrt{p'(\rho_0)}$ is the 0-th order of the speed of sound.

Proposition (Buet, Desprès, LD):

$$\partial_t \left(\alpha_0 \rho_0 \left(\frac{\rho_0^2 c_0^2}{2} \tau_1^2 + \frac{1}{2} |u_1|^2 \right) + m_\star T_\star \int \frac{1}{2} g_1^2 \, R \, dv \right)$$

$$= \nabla \cdot H - D_\star \, Q,$$

where

$$H := \alpha_0 \rho_0^2 c_0^2 \, \tau_1 \, u_1 + m_\star \, T_\star \int \frac{g_1^2}{2} \, v \, R \, dv + \rho_0^2 c_0^2 \, m_\star \tau_1 \int \sqrt{f_0} \, e^{d_\star t} \, g_1 \, v \, dv,$$

$$Q := T_\star \int \left| g_1 - \frac{1}{T_\star} u_1 \cdot v \sqrt{f_0} \, e^{d_\star t} \, R^{-1} \right|^2 R \, dv \geq 0,$$

and

$$R(t,v) = -\frac{G}{G'} \left(\frac{|v|^2}{2 \, T(t)} \right) \geq 0$$

provided that G is decreasing as a function of $|v|$ [Penrose-style condition].

Perspectives for the study of thick sprays

- Extend the linear stability computation in cases when collisions between droplets are taken into account, and when an energy equation is present.
- Obtain a theorem of existence of local in time solutions, or of global in time solutions when the initial data are close to homogeneous
- Investigate modeling issues for spray equations as well as multiphase Euler systems: Find conditions under which the equations can be formally obtained from a kinetic system, typically involving Enskog equations (LD, Golse, Ricci).