

Large time asymptotics for evolution equations with mean field couplings

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Outline

- **Nonlinear diffusion equations**
 - ▷ the fast diffusion equation
 - ▷ The subcritical Keller-Segel model
 - ▷ A simple mean-field model
- **Vlasov-Fokker-Planck models**
 - ▷ Hypocoercivity methods
 - ▷ Hypocoercivity for the linear Vlasov-Fokker-Planck equation
 - ▷ The Vlasov-Poisson-Fokker-Planck system

Nonlinear diffusion equations

- ▷ Fast diffusion equation
- ▷ The subcritical Keller-Segel model
- ▷ A simple mean-field model

The fast diffusion equation

(Blanchet, Bonforte, JD, Gillo, Vázquez)
(Bonforte, JD, Nazaret, Simonov)

The fast diffusion equation

Consider the *fast diffusion* equation in \mathbb{R}^d , $d \geq 1$, $m < 1$

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad u|_{t=0} = u_0 \geq 0 \quad (\text{FDE})$$

With $p = \frac{1}{2m-1}$, $u = f^{2p}$

$$\frac{d}{dt} \underbrace{\int_{\mathbb{R}^d} u \, dx}_{=\|f\|_{2p}^{2p}} = 0, \quad \frac{d}{dt} \underbrace{\int_{\mathbb{R}^d} u^m \, dx}_{=\|f\|_{p+1}^{p+1}} = (p+1)^2 \int_{\mathbb{R}^d} |\nabla f|^2 \, dx$$

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\text{GNS}}(p) \|f\|_{2p} \quad (\text{GNS})$$

$t \rightarrow +\infty$ asymptotics: $u(t, x) \sim B(t, x) = t^{-d/\mu} \mathbf{g}(t^{-1/\mu} x)^{2p}$

B Barenblatt self-similar solutions, $\mu = 2 - d(1 - m) > 1$

$\mathbf{g}(x) = (1 + |x|^2)^{-\frac{1}{p-1}}$ Aubin-Talenti type function

Self-similar variables, entropy-entropy production inequality

In *self-similar variables* (FDE) becomes a *Fokker-Planck type equation*

$$\frac{\partial v}{\partial t} + \nabla \cdot \left(v (\nabla v^{m-1} - 2x) \right) = 0 \quad (1)$$

with (GNS) $\iff \mathcal{J}[v] \geq 4 \mathcal{F}[v]$ and $\frac{d}{dt} \mathcal{F}[v] = -\mathcal{J}[v]$

Generalized entropy (free energy) and Fisher information

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\mathcal{B}^{m-1} (v - \mathcal{B}) - \frac{v^m - \mathcal{B}^m}{m} \right) dx, \quad \mathcal{J}[v] := \int_{\mathbb{R}^d} v |\nabla v^{m-1} + 2x|^2 dx$$

where $\mathcal{B}(x) = \mathbf{g}^{2p}(x) = (1 + |x|^2)^{-\frac{1}{1-m}}$ (with appropriate normalizations)

Linearized entropy-entropy production inequality

(BBDGV)... Linearization: Let $v_\varepsilon = \mathcal{B} (1 + \varepsilon \mathcal{B}^{1-m} f)$

$$\underbrace{\mathcal{J}[v_\varepsilon]}_{\sim \varepsilon^2 \int_{\mathbb{R}^d} |\nabla f|^2 \mathcal{B} dx} \geq 4 \underbrace{\mathcal{F}[v_\varepsilon]}_{\sim \varepsilon^2 \int_{\mathbb{R}^d} |f|^2 \mathcal{B}^{2-m} dx}$$

Hardy-Poincaré inequality: with $\mathcal{B}^{2-m} = \frac{\mathcal{B}}{1+|x|^2}$

$$\Lambda_{m,d} \int_{\mathbb{R}^d} f^2 \mathcal{B}^{2-m} dx \leq \int_{\mathbb{R}^d} |\nabla f|^2 \mathcal{B} dx \quad \forall f \in H^1(\mathcal{B} dx), \quad \int_{\mathbb{R}^d} f \mathcal{B}^{2-m} dx = 0$$

asymptotic decay rates = rates of the linearized FDE equation

$$0 = \partial_t v + \nabla \cdot \left(v \nabla (v^{m-1} - \mathcal{B}^{m-1}) \right) \\ \sim \varepsilon \mathcal{B}^{2-m} \left(\partial_t f - (1-m) \mathcal{B}^{m-2} \nabla \cdot (\mathcal{B} \nabla f) \right)$$

same rate in the nonlinear regime (Bakry-Emery)

much more (stability results)... but the difficulty lies in the justification of the Taylor expansion

The subcritical Keller-Segel model

(Campos, JD)
(Dávila, JD, del Pino, Musso, Wei)

The subcritical Keller-Segel model

$M = \int_{\mathbb{R}^2} n_0 dx \leq 8\pi$: global existence (W. Jäger, S. Luckhaus 1992),
(JD, B. Perthame 2004)

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u (\nabla (\log u) - \nabla v)]$$

the *free energy*

$$F[u] := \int_{\mathbb{R}^2} u \log u dx - \frac{1}{2} \int_{\mathbb{R}^2} u v dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u |\nabla (\log u) - \nabla v|^2 dx$$

The *logarithmic HLS inequality* (E. Carlen, M. Loss 1992)

F is bounded from below if and only if $M \leq 8\pi$

Time-dependent rescaling

$$u(x, t) = \frac{1}{R^2(t)} n \left(\frac{x}{R(t)}, \tau(t) \right) \quad \text{and} \quad v(x, t) = c \left(\frac{x}{R(t)}, \tau(t) \right)$$

with $R(t) = \sqrt{1 + 2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n(\nabla c - x)) & x \in \mathbb{R}^2, t > 0 \\ c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2, t > 0 \\ n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

(A. Blanchet, JD, B. Perthame 2006)

The convergence in self-similar variables

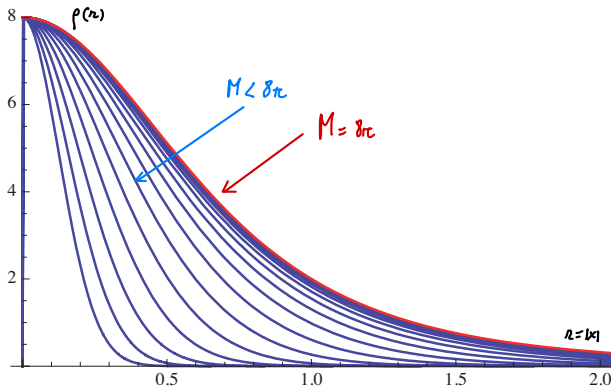
$$\lim_{t \rightarrow \infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)} = 0$$

means *intermediate asymptotics* in original variables:

$$\left\| u(x, t) - \frac{1}{R^2(t)} n_\infty \left(\frac{x}{R(t)}, \tau(t) \right) \right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} dx} = -\Delta c_\infty, \quad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$



Linearization

We can introduce two functions f and g such that

$$n = n_\infty (1 + f) \quad \text{and} \quad c = c_\infty (1 + g) = (-\Delta)^{-1} n$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + \frac{1}{n_\infty} \nabla \cdot (f n_\infty \nabla (c_\infty g))$$

where the linearized operator is

$$\mathcal{L} f = \frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f - c_\infty g))$$

and

$$-\Delta (c_\infty g) = n_\infty f$$

Spectrum of \mathcal{L} (lowest eigenvalues only)

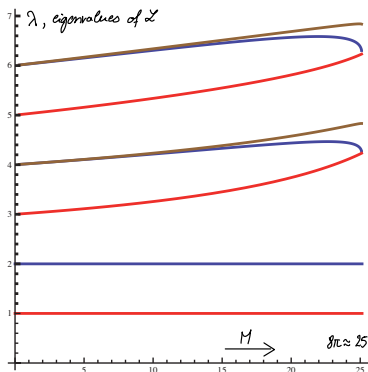


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n \cdot f)$

Functional setting...



Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log \left(\frac{n}{n_\infty} \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty) (c - c_\infty) dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$Q_1[f] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f n_\infty dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality. For any $f \in H^1(\mathbb{R}^2, n_\infty dx)$ such that $\int_{\mathbb{R}^2} f n_\infty dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f n_\infty)|^2 n_\infty dx = \int_{\mathbb{R}^2} |\nabla(g c_\infty)|^2 n_\infty dx \leq \int_{\mathbb{R}^2} |f|^2 n_\infty dx$$



... and eigenvalues

With g such that $-\Delta(g c_\infty) = f n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1 f_2 n_\infty dx - \int_{\mathbb{R}^2} f_1 n_\infty (g_2 c_\infty) dx$$

on the orthogonal space to f_0 in $L^2(n_\infty dx)$

$$Q_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g c_\infty)|^2 n_\infty dx \quad \text{with} \quad g = -\frac{1}{c_\infty} \frac{1}{2\pi} \log |\cdot| * (f n_\infty)$$

is a positive quadratic form, whose polar operator is the **self-adjoint** operator \mathcal{L}

$$\langle f, \mathcal{L} f \rangle = Q_2[f] \quad \forall f \in \mathcal{D}(L_2)$$

Lemma (J. Campos, JD)

\mathcal{L} has pure discrete spectrum and its lowest eigenvalue is 1

A simple Cucker-Smale mean-field model

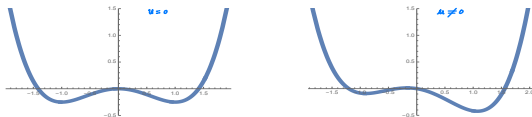
(Xingyu Li)

A simple version of the Cucker-Smale model

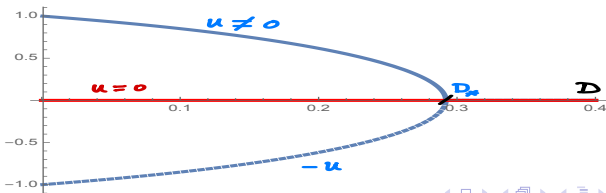
(J. Tugaut, 2014), (A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016), (X. Li)

A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D \Delta_v f + \nabla_v \cdot (\nabla_v \phi(v) f - \mathbf{u}_f f)$$



where $\mathbf{u}_f = \int v f dv$ is the average velocity and $\phi(v) = \frac{1}{4} |v|^4 - \frac{1}{2} |v|^2$



Relative entropy and related quantities

$$\frac{d}{dt} \mathcal{F}_{\mathbf{u}}[f(t, \cdot)] = -\mathcal{J}[f]$$

- Relative entropy with respect to a stationary solution $f_{\mathbf{u}}$

$$\mathcal{F}_{\mathbf{u}}[f] = D \int_{\mathbb{R}^d} f \log \left(\frac{f}{f_{\mathbf{u}}} \right) dv - \frac{1}{2} |\mathbf{u}_f - \mathbf{u}|^2$$

- Relative Fisher information

$$\mathcal{J}[f] := \int_{\mathbb{R}^d} \left| D \frac{\nabla f}{f} + \alpha v |v|^2 + (1 - \alpha) v - \mathbf{u}_f \right|^2 f dv$$

- Non-equilibrium Gibbs state

$$G_f(v) := \frac{e^{-\frac{1}{D} \left(\frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left(\frac{1}{2} |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} |v|^4 - \frac{\alpha}{2} |v|^2 \right)} dv}$$

Stability and coercivity

(X. Li)

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[f_{\mathbf{u}}(1 + \varepsilon g)] = D \int_{\mathbb{R}^d} g^2 f_{\mathbf{u}} dv - D^2 |\mathbf{v}_g|^2$$

$$\text{where } \mathbf{v}_g := \frac{1}{D} \int_{\mathbb{R}^d} v g f_{\mathbf{u}} dv$$

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathcal{J}[f_{\mathbf{u}}(1 + \varepsilon g)] = D^2 \int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2 f_{\mathbf{u}} dv$$

Stability: $Q_{1,\mathbf{u}} \geq 0$?

Coercivity: $Q_{2,\mathbf{u}} \geq \lambda Q_{1,\mathbf{u}}$ for some $\lambda > 0$?

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2 |\mathbf{u}|^2} Q_{1,\mathbf{u}}[g]$$

$\kappa(D) < 1$ and as a special case, if $\mathbf{u} = \mathbf{u}[f]$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D (1 - \kappa(D)) Q_{1,\mathbf{u}}[g]$$

An exponential rate of convergence for partially symmetric solutions in the polarized case

Proposition (X. Li)

Let $\alpha > 0$, $D > 0$ and consider a solution $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$ with initial datum $f_{\text{in}} \in L^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[f_{\text{in}}] < \mathcal{F}[f_0]$ and $\mathbf{u}_{f_{\text{in}}} = (u, 0 \dots 0)$ for some $u \neq 0$. We further assume that $f_{\text{in}}(v_1, v_2, \dots, v_{i-1}, v_i, \dots) = f_{\text{in}}(v_1, v_2, \dots, v_{i-1}, -v_i, \dots)$ for any $i = 2, 3, \dots, d$. Then

$$0 \leq \mathcal{F}[f(t, \cdot)] - \mathcal{F}[f_{\mathbf{u}}] \leq C e^{-\lambda t} \quad \forall t \geq 0$$

holds with $\lambda = \mathcal{C}_D (1 - \kappa(D)) > 0$

The Vlasov-Poisson-Fokker-Planck system

- ▷ Hypocoercivity methods
- ▷ Linearized Vlasov-Poisson-Fokker-Planck system
- ▷ A result in the non-linear case, $d = 1$

The Vlasov-Poisson-Fokker-Planck system: linearization and hypoocoercivity

- (JD, Mouhot, Schmeiser, 2015)
- (Bouin, JD, Mischler, Mouhot, Schmeiser, 2020) Hypoocoercivity without confinement
- (Arnold, JD, Schmeiser, Wöhrer) Sharpening of decay rates in Fourier based hypoocoercivity methods
- (Addala, JD, Li, Tayeb) L^2 -Hypoocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. Preprint hal-02299535 and arxiv: 1909.12762

Hypoocoercivity methods

H^1 -hypocoercivity: an example

$$\frac{\partial f}{\partial t} + \mathbb{T}f = \Delta_v f + \nabla_v \cdot (v f), \quad \mathbb{T}f := v \cdot \nabla_x f - x \cdot \nabla_v f$$

(JD, X. Li) take $h = (f/f_*)^{2/p}$, $p \in (1, 2)$

$$\frac{\partial h}{\partial t} + \mathbb{T}h = \mathbb{L}h + \frac{2-p}{p} \frac{|\nabla_v h|^2}{h}, \quad \mathbb{L}h := \Delta_v h - v \cdot \nabla_v h$$

Twisted Fisher information

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu$$

Theorem (JD, Li)

For an appropriate choice of $t \mapsto \lambda(t)$, there is a function $t \mapsto \rho(t) > 1$ a.e.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -\rho(t) \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \quad \forall t \geq 0$$

and $\mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{J}_{1/2}[h_0] \exp\left(-\int_0^t \rho(s) ds\right)$

L^2 -hypoocoercivity: the strategy

(JD, Mouhot, Schmeiser) Π is the orthogonal projection on $\text{Ker}(\mathbf{L})$

$$\varepsilon \frac{dF}{dt} + \mathbf{T}F = \frac{1}{\varepsilon} \mathbf{L}F$$

$F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \rightarrow 0_+$, $u = F_0 = \Pi F_0$

$$\partial_t u + (\mathbf{T}\Pi)^* (\mathbf{T}\Pi) u = 0$$

▷ Main assumption: *macroscopic coercivity* (Poincaré inequality)

$$\|\mathbf{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2$$

$\varepsilon = 1$: the estimate $\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle \mathbf{L}F, F \rangle \leq -\lambda_m \|(1 - \Pi)F\|^2$ is not enough to conclude that $\|F(t, \cdot)\|^2$ decays exponentially

The operator $\mathbf{A} := (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} (\mathbf{T}\Pi)^*$ is such that

$$\langle \mathbf{A}\mathbf{T}\Pi F, F \rangle \geq \frac{\lambda_M}{1 + \lambda_M} \|\Pi F\|^2$$

and we can use the L^2 entropy / Lyapunov functional

$$\mathbf{H}[F] := \frac{1}{2} \|F\|^2 + \delta \text{Re} \langle \mathbf{A}F, F \rangle$$

Linearized Vlasov-Poisson-Fokker-Planck system

In collaboration with Lanoir Addala, Xingyu Li and Lazhar M. Tayeb

- L^2 -Hypoocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. Preprint hal-02299535 and arxiv: 1909.12762
(Hérau, Thomann, 2016), (Herda, Rodrigues, 2018)

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f) \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f \, dv \end{aligned} \quad (\text{VPFP})$$

Linearized problem around f_\star : $f = f_\star (1 + \eta h)$, $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0$

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= \eta \nabla_x \psi_h \cdot \nabla_v h \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv \end{aligned}$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck system*

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0 \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0 \end{aligned} \quad (\text{VPFPlin})$$

Hypoocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 f_\star dx dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 dx$$

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants \mathcal{C} and λ such that any solution h of (VPFPlin) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

Diffusion limit

Linearized problem in the parabolic scaling

$$\begin{aligned} \varepsilon \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi h - \frac{1}{\varepsilon} (\Delta_v h - v \cdot \nabla_v h) &= 0 \\ -\Delta_x \psi h &= \int_{\mathbb{R}^d} h f_\star dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv = 0 \end{aligned}$$

(VPFPscal)

Expand $h_\varepsilon = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \rightarrow 0_+$. With $W_\star = V + \phi_\star$

$$\varepsilon^{-1} : \quad \Delta_v h_0 - v \cdot \nabla_v h_0 = 0$$

$$\varepsilon^0 : \quad v \cdot \nabla_x h_0 - \nabla_x W_\star \cdot \nabla_v h_0 + v \cdot \nabla_x \psi h_0 = \Delta_v h_1 - v \cdot \nabla_v h_1$$

$$\varepsilon^1 : \quad \partial_t h_0 + v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1 = \Delta_v h_2 - v \cdot \nabla_v h_2$$

With $u = \Pi h_0$, $-\Delta \psi = u \rho_\star$, $w = u + \psi$,

$$u = h_0, \quad v \cdot \nabla_x w = \Delta_v h_1 - v \cdot \nabla_v h_1$$

from which we deduce that $h_1 = -v \cdot \nabla_x w$ and

$$\partial_t u - \Delta w + \nabla_x W_\star \cdot \nabla u = 0$$

Rates of convergence

Results in the diffusion limit / in the non-linear case

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. For any $\varepsilon > 0$ small enough, there exist two positive constants \mathcal{C} and λ , which do not depend on ε , such that any solution h of (VPFPscal) with an initial datum h_0 of zero average satisfies

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

Corollary

Assume that $d = 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. If f solves (VPFP) with initial datum $f_0 = (1 + h_0) f_\star$ such that h_0 has zero average, $\|h_0\|^2 < \infty$ and $(1 + h_0) \geq 0$, then

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

holds with $h = f/f_\star - 1$ for some positive constants \mathcal{C} and λ

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Thank you for your attention !