

# Partial Regularity in Time for the Landau Equation (with Coulomb Interaction)

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Kinetic Equations: from Modeling, Computation to Analysis

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# Landau Equation

**Landau equation** with unknown  $f \equiv f(t, v) \geq 0$ :

$$\partial_t f(t, v) = \operatorname{div}_v \int_{\mathbb{R}^3} a(v-w)(\nabla_v - \nabla_w)(f(t, v)f(t, w))dw, \quad v \in \mathbb{R}^3$$

with the notation:

$$a(z) := \frac{1}{8\pi} \nabla^2 |z| = \frac{1}{8\pi|z|} \Pi(z), \quad \Pi(z) := I - \left( \frac{z}{|z|} \right)^{\otimes 2}$$

**Nonconservative form**

$$\partial_t f(t, v) = (a_{ij} \star_v f(t, v)) \partial_{v_i} \partial_{v_j} f(t, v) + f(t, v)^2$$

**Open question** global existence of classical solutions or finite-time blow-up for the Cauchy problem with  $f|_{t=0} = f_{in}$ ?

**H-solution**  $0 \leq f \in C([0, T]; \mathcal{D}'(\mathbb{R}^3)) \cap L^1((0, T); L^1_{-1}(\mathbb{R}^3))$  s.t.

$$\int_{\mathbb{R}^3} (1, v, |v|^2) f(t, v) dv = \int_{\mathbb{R}^3} (1, v, |v|^2) f_{in}(t, v) dv$$

$$\int_{\mathbb{R}^3} f(t, v) \ln f(t, v) dv \leq \int_{\mathbb{R}^3} f_{in}(v) \ln f_{in}(v) dv$$

for a.e.  $t \geq 0$ , and

$$\int_{\mathbb{R}^3} f_{in}(v) \phi(0, v) dv + \int_0^T \int_{\mathbb{R}^3} f(t, v) \partial_t \phi(t, v) dv$$

$$= \int_0^T \int_{\mathbb{R}^6} (\Phi(t, v) - \Phi(t, w)) \cdot \Pi(v - w) (F(\nabla_v - \nabla_w) F)(t, v, w) dv dw$$

with  $\Phi(t, v) := \nabla_v \phi(t, v)$ ,  $F(t, v, w) := \sqrt{\frac{f(t, v) f(t, w)}{8\pi|v-w|}}$

**Notation**  $\|g\|_{L_k^p}^p := \int (1 + |v|^2)^{k/2} |g(v)|^p dv$  with  $p \geq 1$  and  $k \in \mathbb{R}$

## Definition

A  $(\mathcal{N}, q, C'_E)$ -suitable solution on  $[0, T) \times \mathbf{R}^3$  is an H-solution s.t.

$$\begin{aligned} H_+(f(t_2, \cdot) | \kappa) + C'_E \int_{t_1}^{t_2} \left\| \mathbf{1}_{f(t, v) > \kappa} \nabla_v f(t, v)^{1/q} \right\|_{L^q(\mathbf{R}^3)}^2 dt \\ \leq H_+(f(t_1, \cdot) | \kappa) + 2\kappa \int_{t_1}^{t_2} \int_{\mathbf{R}^3} (f(t, v) - \kappa)_+ dv dt \end{aligned}$$

for all  $t_1 < t_2 \in [0, T) \setminus \mathcal{N}$  and  $\kappa \geq 1$ , where

$$H_+(g | \kappa) := \int_{\mathbf{R}^3} \kappa h_+ \left( \frac{g(v)}{\kappa} \right) dv, \quad h_+(z) := z(\ln z)_+ - (z-1)_+$$

# Partial Regularity in Time

**Definition** A **regular time** of  $f$ , suitable solution on  $I \subset (0, +\infty)$ , is a time  $\tau \in I$  s.t.  $f \in L^\infty((\tau - \epsilon, \tau) \times \mathbf{R}^3)$  for some  $\epsilon \in (0, \tau)$ .

The set of singular (i.e. nonregular) times of  $f$  on  $I$  is denoted  $\mathbf{S}[f, I]$ .

**Main Thm** Let  $f$  be a suitable solution to the Landau equation on  $[0, T) \times \mathbf{R}^3$  for all  $T > 0$ , with initial data  $f_{in}$  satisfying

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \quad \text{for all } k > 3$$

Then

$$\text{Hausdorff dim } \mathbf{S}[f, (0, +\infty)] \leq \frac{1}{2}$$

**Prop 1** For all  $0 \leq f_{in} \in L^1(\mathbf{R}^3)$  s.t.

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \quad \text{for some } k > 3$$

there exists an  $(\mathcal{N}, q, C'_E)$ -suitable solution  $f$  on  $[0, T]$  with initial data  $f_{in}$  and

$$C'_E \equiv C'_E[T, q, f_{in}] > 0, \quad q := \frac{2k}{k+3}$$

**Notation**  $\|g\|_{L_k^p(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} (1 + |v|^2)^{k/2} |g(v)|^p \right)^{1/p}$

**Thm** For each  $0 \leq f \in L^1_2(\mathbb{R}^3)$  s.t.  $f \ln f \in L^1(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \frac{|\nabla \sqrt{f(v)}|^2 dv}{(1+|v|^2)^{3/2}} \leq C_D + C_D \int_{\mathbb{R}^6} \frac{|\Pi(v-w)(\nabla_v - \nabla_w) \sqrt{f(v)f(w)}|^2}{|v-w|} dv dw$$

with

$$C_D \equiv C_D \left[ \int_{\mathbb{R}^3} (1, v, |v|^2, |\ln f(v)|) f(v) dv \right] > 0$$

**Corollary** Let  $0 \leq f_{in} \in L^1_k(\mathbb{R}^3)$  with  $k > 2$  s.t.  $f_{in} |\ln f_{in}| \in L^1(\mathbb{R}^3)$ .

$$f \text{ H-solution s.t. } f|_{t=0} = f_{in} \implies f \in L^\infty(0, T; L^1_k(\mathbb{R}^3))$$

# (Formal) Truncated H Theorem

One has

$$\begin{aligned}
 & \frac{d}{dt} H_+(f(t, \cdot) | \kappa) \\
 & + \underbrace{\int \frac{f(t, v) f(t, w)}{16\pi |v-w|} \left| \Pi(v-w) \left( \frac{\mathbf{1}_{f(t, v) > \kappa} \nabla_v f(t, v)}{f(t, v)} - \frac{\mathbf{1}_{f(t, w) > \kappa} \nabla_w f(t, w)}{f(t, w)} \right) \right|^2}_{D_1} dv dw \\
 & = - \int f(t, v) f(t, w) a(v-w) : \nabla_v \left( \ln \frac{f(t, v)}{\kappa} \right)_+ \otimes \nabla_w \left( \ln \frac{f(t, w)}{\kappa} \right)_- dv dw \\
 & = - \int a(v-w) : \nabla_v f(t, v) \mathbf{1}_{f(t, v) \geq \kappa} \otimes \nabla_w f(t, w) \mathbf{1}_{f(t, w) < \kappa} dv dw \\
 & = \int \underbrace{- \operatorname{div}_v (\operatorname{div}_w a(v-w))}_{\geq 0 \text{ (in fact } = \delta(v-w))} (f(t, v) - \kappa)_+ (\kappa - (f(t, w) - \kappa)_-) dv dw \\
 & \leq \underbrace{\kappa \int (f(t, v) - \kappa)_+}_{\text{depleted NL}} dv
 \end{aligned}$$



# Sketch of the Proof of Prop 1

- Replace  $a$  with its truncated variant

$$a_n(z) = \frac{1}{8\pi} \left( \frac{1}{|z|} \wedge n \right) \Pi(z), \quad \text{satisfying } \operatorname{div}(\operatorname{div} a_n) \geq 0$$

- Use the Desvillettes theorem to bound

$$\frac{1}{C_D''} \int_{\mathbb{R}^3} \frac{|\nabla_v \sqrt{f(t,v)}|^2}{(1+|v|)^3} \mathbf{1}_{f(t,v) > \kappa} dv \leq D_1 + \int_{\mathbb{R}^3} (f(t,w) - \kappa)_+ dw$$

- Using the Desvillettes corollary with  $p' = 2/q$  (recall  $q \in (1, 2)$ )

$$\begin{aligned} & \left\| \mathbf{1}_{f(t,v) > \kappa} \nabla_v f(t,v)^{1/q} \right\|_{L^q(\mathbb{R}^3)}^q \\ & \leq \left( \frac{2}{q} \right)^q \|f(t, \cdot)\|_{L_{3p/2p'}^p(\mathbb{R}^3)} \left( \int_{\mathbb{R}^3} \frac{|\nabla_v \sqrt{f(t,v)}|^2 \mathbf{1}_{f(t,v) \geq \kappa}}{(1+|v|^2)^{3/2}} dv \right)^{1/p'} \end{aligned}$$

# The 1st De Giorgi Type Lemma

**Prop 2** Let  $f$  be a  $(\mathcal{N}, q, C'_E)$ -suitable solution to the Landau equation for  $t \in [0, 1]$  with  $C'_E > 0$  and  $q \in (\frac{6}{5}, 2)$

Then there exists  $\eta_0 \equiv \eta_0[q, C'_E] > 0$  s.t.

$$\int_{1/8}^1 H_+(f(t, \cdot) | \frac{1}{2}) dt < \eta_0 \implies f(t, v) \leq 2 \quad \text{a.e. on } [\frac{1}{2}, 1] \times \mathbf{R}^3$$

# Proof of Prop 2

Set

$$\begin{cases} t^k := \frac{1}{2} - \frac{1}{4} \cdot 2^{-k}, & \kappa_k := (1 + (2^{1/q} - 1)(1 - 2^{-k}))^q \\ f_k^+(t, v) := \mu((f(t, v)^{1/q} - \kappa_k^{1/q})_+) & \text{with } \mu(r) := \min(r, r^2) \end{cases}$$

The quantity

$$A_k := \operatorname{ess\,sup}_{t^k \leq t \leq 1} \frac{C_h}{2} \int_{\mathbb{R}^3} f_k^+(t, v)^q dv + \frac{1}{4} C'_E \int_{t^k}^1 \|\nabla_v f_k^+(t, v)\|_{L^q(\mathbb{R}^3)}^2 dt$$

satisfies

$$A_{k+1} \leq M[q, C'_E] \Lambda^k A_k^\beta, \quad \beta := \frac{8}{3} - \frac{2}{q} > 1 \text{ and } \Lambda := 2 \cdot 4^{\frac{5q}{3}}$$

so that  $A_0 < M^{-\frac{1}{\beta-1}} \Lambda^{-\frac{1}{(\beta-1)^2}} \implies A_k \rightarrow 0$  as  $k \rightarrow +\infty$

• Control  $A_0$  by truncated entropy + conclude by Fatou's lemma

# The Improved De Giorgi Type Lemma

**Prop 3** Let  $f$  be a  $(\mathcal{N}, q, C'_E)$ -suitable solution to the Landau equation on  $[0, 1]$  with  $q \in (\frac{4}{3}, 2)$ . Let  $\gamma := \frac{5q-6}{2q-2}$ . Then there exists  $\eta_1 \equiv \eta_1[q, C'_E] > 0$  and  $\delta_1 \in (0, 1)$  such that

$$\begin{aligned} \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{\gamma-3} \int_{1-\epsilon^\gamma}^1 \left\| \mathbf{1}_{f(T,V) > \epsilon^{-\gamma}} \nabla_V f(T, V)^{\frac{1}{q}} \right\|_{L^q(\mathbb{R}^3)}^2 dT < \eta_1 \\ \implies f \in L^\infty((1-\delta_1, 1) \times \mathbb{R}^3) \end{aligned}$$

• Prop 3 and a Vitali type covering argument imply that

$$\mathcal{H}^{\frac{3-\gamma}{\gamma} = \frac{q}{5q-6}}(\mathbf{S}[f, [\frac{1}{2}, 1]]) < \infty$$

The Main Thm follows by translation+letting  $q \rightarrow 2^-$

# Proof of Prop 3: (a) Scaling

- Set

$$f_n(t, v) := \epsilon_n^\gamma f(1 + \epsilon^\gamma(t - 1), \epsilon v) \quad \text{with } \epsilon_n := 2^{-n}$$

$$F_n(t, v) := \mu((f_n(t, v))^{1/q} - 1)_+, \quad \int F_n(t, v) dv \leq \epsilon_n^{\gamma-3}$$

- Observe that  $f_n$  is a suitable solution of the Landau eq. on  $[0, 1]$  with the same constant  $C'_E$  as  $f$
- There exists  $N$  large enough so that

$$\begin{aligned} n \geq N &\implies \int_0^1 \left( \int |\nabla_v F_n(t, v)|^q dv \right)^{2/q} dt \\ &\leq 4\epsilon_n^{\gamma-3} \int_{1-\epsilon_n^\gamma}^1 \left( \int |\mathbf{1}_{f \geq \epsilon_n^{-\gamma}} \nabla_v f(T, V)^{1/q}|^q dV \right)^{2/q} dT < 8\eta_1 \end{aligned}$$

## Proof of Prop 3: (b) Iteration

- Use the Hölder inequality + Sobolev inequality as in the proof of Prop 2, isolating the term  $\|\nabla_v F_{n+1}\|_{L_t^2 L_v^q} = O(\eta_1)$  shows that

$$X_m := \operatorname{ess\,sup}_{\frac{1}{2} < t < 1} \int F_{N+m}(t, v)^q dv$$

satisfies

$$X_{m+1} < \rho(\max(1, X_m)^\alpha + \max(1, X_{m-1})^\alpha), \quad X_0, X_1 \leq M$$

with  $\alpha := q/3$ ,  $\rho := D(q)\eta_1^{q/2}$ ,  $M := 2^{(N+3)(3-\gamma)}$

- With  $\eta_1$  small so that  $\rho < \frac{1}{2}$ , an easy induction shows that

$$X_{2m}, X_{2m+1} \leq \max\left(2\rho, (2\rho)^{\frac{1-\alpha^m}{1-\alpha}} M^{\alpha^m}\right) \implies X_{m_0} < 2D(q)\eta_1^{\frac{q}{2}} \ll 1$$

- Hence  $f_{N+m_0+3}$  satisfies the assumption in Prop 2, q.e.d.

# Final Remarks/Perspectives

- The Desvillettes theorem puts the Landau equation in the same class as 3d Navier-Stokes in terms of Lebesgue exponents — except for the  $(1 + |v|)^{-3}$  weight

$$\begin{array}{ll} \text{Navier-Stokes} & u \in L_t^\infty L_x^2, \quad \nabla_x u \in L_t^2 L_x^2 \\ \text{Landau} & \sqrt{f} \in L_t^\infty L_v^2, \quad \nabla_v \sqrt{f} \in L_t^2 L_{-3}^2 \end{array}$$

- Partial regularity in  $(t, v)$  à la Caffarelli-Kohn-Nirenberg (1982)?
- Conditional regularity à la Serrin (1962)? for instance

$$\sqrt{f} \in L_t^p L_k^q \text{ with } \frac{2}{p} + \frac{3}{q} < 1 \implies f \in L_{loc}^\infty?$$

Yes for  $p = \infty$  and  $k > 5$  [Silvestre 2017, Gualdani-Guillen 2016 for radial nonincreasing solns, Ben Porath 2021 (local in  $(t, v)$ )]