Stable and unstable steady states for the HMF model

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The HMF model

The Hamiltonian mean field model (HMF) is a caricature of the Vlasov-Poisson system. Particles in interaction, moving on the circle, are described by their distribution function $f(t, \theta, v)$ which is solution of

$$\partial_t f + v\partial_\theta f - \partial_\theta \phi_f \partial_v f = 0, \qquad (t, \theta, v) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}$$
$$f(0, \theta, v) = f_{init}(\theta, v)$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and the potential $\phi_f(t, \theta)$ is given by

$$\phi_f(t, \theta) = -\int_0^{2\pi} \rho_f(t, \theta') \cos(\theta - \theta')d\theta', \qquad \rho_f(t, \theta) = \int_{\mathbb{R}} f(t, \theta, v)dv$$

The magnetization M_f is the vector

$$M_f = \left(\int_0^{2\pi} \rho_f \cos\theta d\theta, \int_0^{2\pi} \rho_f \sin\theta d\theta\right)^T = (M_f^1, M_f^2)^T$$

and we have

$$\phi_f(t, \theta) = -M_f^1(t) \cos\theta - M_f^2(t) \sin\theta$$

Basic properties of the HMF model

- The following quantities do not depend on time
  - The Casimir functions

$$\iint G(f(t,\theta,v))d\theta dv$$

  - The nonlinear energy

$$\mathcal{H} = \frac{1}{2}\iint v^2 f(t,\theta,v)d\theta dv - \frac{1}{2}|M_f(t)|^2$$

- Galilean invariance: if $f(t,\theta,v)$ is solution, then so is $f(t,\theta+v_0t, v+v_0)$
- Any function of the form

$$f(\theta,v) = F\left(\frac{v^2}{2} + \phi_f(\theta)\right)$$

(where ϕ_f depends on f) is a steady state of the system.

Main results (more detailed statements later)

Let $f_0 = F\left(\frac{v^2}{2} + \phi_{f_0}(\theta)\right)$ be a steady state and consider the quantity

$$\kappa_0 = -\int_0^{2\pi}\int_{-\infty}^{+\infty} F'(e_0(\theta,v)) \left(\frac{\int_{\mathcal{D}} (\cos\theta - \cos\theta')(e_0(\theta,v) - \phi_{f_0}(\theta'))^{-1/2} d\theta'}{\int_{\mathcal{D}} (e_0(\theta,v) - \phi_{f_0}(\theta'))^{-1/2} d\theta'} \right)^2 d\theta dv$$

where

$$e_0(\theta,v) = \frac{v^2}{2} + \phi_{f_0}(\theta) \quad \text{and} \quad \mathcal{D} = \left\{\theta' \in \mathbb{T} \,:\, \phi_{f_0}(\theta') < e_0(\theta,v)\right\}$$

Then, we have the following alternative

- if $\kappa_0 < 1$ and F is decreasing, f_0 is nonlinearly orbitally stable[1]
- if $\kappa_0 > 1$, f_0 is linearly unstable and, under additional assumptions on its support, f_0 is nonlinearly unstable

[1] The same critierion, with a different expression, was found by Ogawa (PRE 2013) who proved formally the linear stability

Outline

Outline

Rearrangements

For all function $f \in L^1(\mathbb{T} \times \mathbb{R})$, let

$$\mu_f(s) = \text{meas}\,\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : f(\theta, v) > s\}, \quad \text{for all } s \geq 0$$

and its pseudo-inverse

$$f^\sharp(s) = \inf\{t \geq 0,\ \mu_f(t) \leq s\}, \qquad \text{for all } s \geq 0$$

The Schwarz rearrangement of f is the function

$$f^*(\theta, v) = f^\sharp\left(\text{meas}\left\{B(0, \sqrt{\theta^2 + v^2}) \cap \mathbb{T} \times \mathbb{R}\right\}\right)$$

and is equimeasurable with f i.e., for all $s \geq 0$,

$$\text{meas}\,\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : f(\theta, v) > s\} = \text{meas}\,\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : f^*(\theta, v) > s\}$$

Important fact. As a consequence of the conservation of the Casimir functions, all these quantities are conserved by the HMF flow

$$\mu_{f(t)} = \mu_{f_{init}}, \qquad f(t)^\sharp = f^\sharp_{init}, \qquad f(t)^* = f^*_{init}$$

The stability result

Theorem

Let (for simplicity) $f_0(\theta, v) = F(\frac{v^2}{2} - m_0 \cos\theta)$ be a compactly supported steady state with $F \in \mathcal{C}^1(\mathbb{R})$ decreasing. Assume that $\kappa_0 < 1$. Then there exists $\delta > 0$ such that, for all f satisfying $|M_f - M_{f_0(\cdot - \theta_f)}| < \delta$, we have

$$\|f(\theta, v) - f_0(\theta - \theta_f, v)\|_{L^1}^2 \leq C\left(\mathcal{H}(f) - \mathcal{H}(f_0)\right) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1}$$

where θ_f is the angle such that $M_f = |M_f|(\cos\theta_f, \sin\theta_f)^T$

Note that all the quantities in the right-hand side are conserved by the HMF flow. Therefore, its solution $f(t, \theta, v)$ satisfies

$$\|f(t, \theta, v) - f_0(\theta - \theta_f(t), v)\|_{L^1}^2 \leq C\left(\mathcal{H}(f_{init}) - \mathcal{H}(f_0)\right) + C(1 + \|f_{init}\|_{L^1})\|f_{init}^* - f_0^*\|_{L^1}$$

Corollary (orbital stability)

Under the same assumptions, for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\|(1+v^2)(f_{init} - f_0)\|_{L^1} \leq \eta \Longrightarrow \forall t \geq 0, \quad \|(1+v^2)(f(t) - f_0(\cdot - \theta_f))\|_{L^1} \leq \varepsilon$$

The main ingredients of the proof[2]

(i) Symmetric rearrangements with respect to the microscopic energy

➠ Generalization of the classical Schwarz symmetric rearrangement

(ii) Monotonicity of the Hamiltonian with respect to this rearrangement

➠ Reduced energy functional depending on the magnetization only

(iii) Coercivity of this new functional near m_0 (the magnetization of f_0)

(iv) Control of the whole distribution function

[2] Adaptation of techniques developed in Lemou-M.-Raphaël (Inventiones 2012)

Symmetric rearrangement with respect to the microscopic energy

- The standard Schwarz symmetization.

  Let $f \in L^1(\mathbb{T} \times \mathbb{R})$, then $f^* \in L^1(\mathbb{T} \times \mathbb{R})$ is the unique nonincreasing spherically symmetric function such that f^* is equimeasurable with f.

- Generalization: rearrangement with respect to the microscopic energy.

  Let $\phi(\theta)$ be a potential.
  Let $f \in L^1 \cap L^\infty(\mathbb{T} \times \mathbb{R})$, then we may define its rearrangement $f^{*\phi}$ which is:

  - a nonincreasing function of $\frac{v^2}{2} + \phi(\theta)$
  - such that $f^{*\phi}$ is equimeasurable with f

- Key fact: the "fundamental identity of the steady state"

  Our assumptions on f_0 are enough to show that

$$f_0^{*\phi_{f_0}} = f_0$$

Explicit construction of $f^{*\phi}$

We first define the Jacobian function

$$\begin{aligned} a_\phi(e) &= \text{meas}\left\{(x,v)\in\mathbb{T}\times\mathbb{R} : \frac{|v|^2}{2}+\phi(x)<e\right\} \\ &= 2\sqrt{2}\int_0^{2\pi}\sqrt{(e-\phi(\theta))_+}\,d\theta \end{aligned}$$

Then we have

$$f^{*\phi}(\theta,v) := f^\sharp\left(a_\phi\left(\frac{v^2}{2}+\phi(\theta)\right)\right)$$

The key monotonicity property

Introduce the following reduced energy functional

$$\mathcal{J}(m) = \frac{m^2}{2} + \iint \left(\frac{v^2}{2} + \phi\right) f_0^{*\phi} d\theta dv \qquad \text{with } \phi = -m\cos\theta$$

Proposition

$$\mathcal{H}(f) - \mathcal{H}(f_0) = \mathcal{J}(|M_f|) - \mathcal{J}(m_0) + \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f^{*\phi_f} - f_0^{*\phi_f}\right) d\theta dv$$
$$+ \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f - f^{*\phi_f}\right) d\theta dv$$

- The red term is bounded by $C\,(1 + \|f\|_{L^1})\,\|f^* - f_0^*\|_{L^1}$
- The green term is nonnegative

The positivity of the green term is reminiscent from the following property of the standard Schwarz symmetrization

$$\int_{\mathbb{R}^3} |x| f(x) dx \geq \int_{\mathbb{R}^3} |x| f^*(x) dx$$

- The next step is a Taylor expansion of $\mathcal{J}$ near m_0, using the following identities

Proposition

$$\mathcal{J}'(m_0) = 0 \quad \text{and} \quad \mathcal{J}''(m_0) = 1 - \kappa_0$$

Hence, if $1 - \kappa_0 > 0$, we can control locally

$$||M_f| - m_0| \leq C\left(\mathcal{H}(f) - \mathcal{H}(f_0)\right) + C\left(1 + \|f\|_{L^1}\right)\|f^* - f_0^*\|_{L^1}$$

- The last step is a functional inequality [3]

Proposition

$$\left(\|f - f_0\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1}\right)^2 \leq C\left|M_f - M_{f_0}\right| + C\left(\mathcal{H}(f) - \mathcal{H}(f_0)\right) + C\|f^* - f_0^*\|_{L^1}$$

Hence we can control $f - f_0$ (up to a translation shift) and conclude

[3] obtained in Lemou (CMP 2016)

Outline

The linear instability result

Consider the linearized HMF operator near f_0

$$Lf = -v\partial_\theta f + \partial_\theta \phi_{f_0} \partial_v f + \partial_\theta \phi_f \partial_v f_0$$

Theorem

Let $f_0 = F(\frac{v^2}{2} - m_0 \cos\theta)$ be a steady state of the HMF system such that $F'(\frac{v^2}{2} - m_0 \cos\theta) \in L^1$. Assume that $\kappa_0 > 1$. Then there exists an eigenvalue $\lambda > 0$ and an eigenfunction $f \in L^1$ such that $Lf = \lambda f$.

In particular, the function $g(t, \theta, v) = e^{\lambda t} f(\theta, v)$ is a growing mode of the equation $\partial_t g = Lg$. We have exhibited an instability for the linearized HMF equation

The method used to prove this theorem is inspired from the works of Strauss, Guo, Lin[4] for the Vlasov-Poisson system in the plasma case or in the gravitational case

[4] Guo-Strauss (Ann. IHP 1995), Guo-Lin (CMP 2008)

Sketch of the proof

- We introduce the characteristics associated to the equation

$$\partial_t f + v\partial_\theta f - \partial_\theta \phi_{f_0} \partial_v f = 0$$

$$\frac{d\Theta}{ds} = V, \qquad \frac{dV}{ds} = \partial_\theta \phi_{f_0}(\Theta(s)) = m_0 \sin\Theta(s)$$

- Assume that $Lf = \lambda f$ and that $\phi_f(\theta) = -m\cos\theta$, then

$$\begin{aligned}\frac{d}{ds}\left(e^{\lambda s} f\left(\Theta(s), V(s)\right)\right) &= e^{\lambda s}\left(\partial_\theta \phi_f \partial_v f_0\right)\left(\Theta(s), V(s)\right)\\ &= e^{\lambda s} m \sin\Theta(s) V(s) F'(e(\Theta(s), V(s)))\end{aligned}$$

where $e(\Theta(s), V(s)) = \frac{V(s)^2}{2} - m\cos\Theta(s)$ is independent of s. Hence after an integration between $s = -\infty$ and $s = 0$,

$$\begin{aligned}f(\theta, v) &= mF'(e)\int_{-\infty}^{0} e^{\lambda s}\sin\Theta(s)V(s)ds\\ &= -mF'(e)\cos\theta - mF'(e)\int_{-\infty}^{0}\lambda e^{\lambda s}\cos\Theta(s)ds\end{aligned}$$

- Hence the equation of the magnetization $m = \iint f(\theta, v) \cos\theta d\theta dv$ yields $G(\lambda) = 0$, with

$$G(\lambda) := 1 + \iint F'(e)\cos^2\theta d\theta dv + \iint F'(e)\cos\theta \int_{-\infty}^{0} \lambda e^{\lambda s} \cos\Theta(s) ds d\theta dv$$

  Conversely, if $G(\lambda) = 0$, one can see that λ is an eigenfunction of L

- It is not difficult to see that $\lim_{\lambda\to+\infty} G(\lambda) = 1$ by dominated convergence

- The crucial point is now to show that $\lim_{\lambda\to 0} G(\lambda) = 1 - \kappa_0$. This calculation uses the solution of the characteristics equations, which are nothing but the equations of the pendulum

$$\dot{\Theta}(s) = V(s) = \pm\sqrt{2(e + m\cos\Theta(s))}$$

- Then, if $\kappa_0 > 1$, by a continuity argument, G admits a zero on $\mathbb{R}_+^*$, i.e. there exists an eigenvalue $\lambda > 0$ corresponding to a growing mode

Outline

The nonlinear instability result

The nonlinear instability results relies on Grenier's technique[5], adapted to kinetic equations by Han-Kwan and Hauray[6]

Theorem

Let $f_0 = F(\frac{v^2}{2} - m_0 \cos\theta)$ be a steady state of the HMF system such that $F \in \mathcal{C}^\infty$ and such that $F(e) = 0$ for all $e \geq e_*$, with $e_* < m_0$. Assume that $\kappa_0 > 1$. Then there exists $\delta_0 > 0$ such that for all $\delta > 0$, there exists a nonnegative solution of the HMF system satisfying

$$\|f(0) - f_0\|_{L^1} \leq \delta$$

and

$$\|f(t_\delta) - f_0\|_{L^1} \geq \delta_0$$

for some $t_\delta = \mathcal{O}(|\log \delta|)$.

Remark: one can prove that the set of such steady states is not empty...

[5] Grenier CPAM 2000

[6] Han-Kwan, Hauray (CMP 2015)

Sketch of the proof

- Let f_1 be an eigenfunction of L associated to the eigenvalue $\lambda > 0$. We construct an 'almost' solution of the nonlinear HMF equation of the form

$$f_{app} = f_0 + \delta e^{\lambda t} f_1(\theta, v) + \sum_{k=2}^{N} \delta^k f_k(t, \theta, v)$$

where δ is small. The f_k are recursively defined by

$$(\partial_t - L) f_k + \sum_{j=1}^{k-1} \partial_\theta \phi_{f_j} \partial_v f_{k-j} = 0, \qquad f_k(0, \theta, v) = 0$$

- To estimate the f_k's, we first use an estimate on the linearized semi-group

$$\|e^{tL} f\|_{W^{k,1}} \leq C e^{t\beta} \|f\|_{W^{k,1}}$$

for all $\beta > \max\{\mathrm{Re}\,\mu : \mu \text{ is an eigenvalue of } L\}$

For this estimate, we use that the period of the characteristics is uniformly bounded on the support of the considered functions, which is true with our assumption on the support of F

- We prove that

$$\|f_k\|_{W^{N-k+1,1}} \leq Ce^{kt\lambda}$$

- Then, if N is large enough,

$$\|f(t) - f_{app}(t)\|_{L^1} \leq C\left(\delta e^{t\lambda}\right)^{N+1}$$

- And finally

$$\begin{aligned}\|f(t) - f_0\|_{L^1} &\geq \|f_{app}(t) - f_0\|_{L^1} - \|f(t) - f_{app}(t)\|_{L^1} \\ &\geq \delta e^{t\lambda}\|f_1\|_{L^1} - \sum_{k=2}^{N} \delta^k \|f_k\|_{L^1} - \|f(t) - f_{app}(t)\|_{L^1} \\ &\geq \delta e^{t\lambda}\|f_1\|_{L^1}\left(1 - \frac{C}{\|f_1\|_{L^1}} \sum_{k=1}^{N} \delta^k e^{kt\lambda}\right)\end{aligned}$$

- To conclude, it suffices to choose

$$\delta_0 = \min\left(\frac{\|f_1\|^2}{8C}, \frac{\|f_1\|_{L^1}}{4}\right) \quad \text{and} \quad \delta e^{t_\delta \lambda}\|f_1\|_{L^1} = 2\delta_0$$

Technicalities...

In practice, one has to face several technical difficulties, in particular

- Since f_0 is compactly supported, one has to truncate the function f_1 to guarantee that $f(0)$ is nonnegative
- In fact, it is not possible to use directly the function f_1 constructed during the linear instability proof. Instead, one has to use the eigenfunction $\widetilde{f}_1$ associated to an eigenvalue of maximal real part. Hence one has to deal with complex-valued functions

Conclusion

- We have exhibited a sharp stability criterion for the HMF model
- For the stability proof, our technique relies on nonincreasing rearrangements and is specially adapted to deal with nonincreasing functions F. An extension to more general profiles would probably require another technique
- The linear instability proof was done under quite general hypotheses
- The nonlinear instability proof is the first one for non homogeneous steady states in the kinetic context. It requires an assumption on the support of the steady state

THANK YOU FOR YOUR ATTENTION