Quantitative De Giorgi Methods in Kinetic Theory

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Kinetic Equations: from Modeling Computation to Analysis
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Summary

- We study quantitatively local regularity properties of solutions $f = f(t, x, v)$ to hypoelliptic divergence-form PDEs

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S \quad (t \in \mathbb{R},\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d)$$

- Assumptions on $A = A(t, x, v)$, $B = B(t, x, v)$ and $S = S(t, x, v)$:

$$\begin{cases} A \text{ measurable symmetric real matrix field with eigenvalues in } [\lambda, \Lambda] \\ B \text{ measurable vector field such that } |B| \leq \Lambda \\ S \text{ real scalar field in } L^\infty \end{cases}$$

- This equation naturally appears in kinetic theory where it is called the kinetic Fokker-Planck equation; it is related to the class considered by Kolmogorov and Hörmander (see later) and to Langevin dynamics
- The coefficients are called rough because A and B are merely measurable and no further regularity is assumed on them

Historical detour: The 19th problem of Hilbert

- Hilbert 1900: existence of (analytic) minimizers of functional

$$\min_u \int_\Omega L(\nabla u)\,\mathrm{d}x \quad \text{with Lagrangian} \quad L : \mathbb{R}^d \to \mathbb{R}$$

  satisfying conditions of growth, regularity and convexity
- Euler-Lagrange equations on the minimizer

$$\partial_i \Big[(\partial_i L)\,(\nabla u) \Big] = 0 \quad \text{i.e.} \quad \underbrace{\Big[(\partial_{ij} L)(\nabla u)\Big]}_{a_{ij}} \partial_{ij} u = 0$$

- Dirichlet energy $L(p) = |p|^2$, minimal surfaces $L(p) = \sqrt{1 + |p|^2}$
- Under technical hypothesis on L and domain Ω, *a priori* pointwise bound on ∇u already known at the time of Hilbert
- However to go to higher regularity (necessary for existence), one needs more regularity than $\nabla u \in L^\infty$ in Euler-Lagrange equations

The result of De Giorgi and Nash (I)

- Another piece of the puzzle: Schauder 1934 proved that if coefficients $a_{ij} \in C^\alpha$ ($\alpha > 0$) then $a_{ij}\partial_{ij} u = 0$ implies $u \in C^{2,\alpha}$
- Iterating Schauder's estimate then yield C^∞ regularity, and finally analytic regularity is obtained by studying the Taylor series
- Remaining missing piece of the puzzle: $a_{ij} = (\partial_{ij} L)(\nabla u) \in C^\alpha$
- The equation on a partial derivative $f := \partial_k u$ is divergence-form

$$\partial_i \left[(\partial_{ij} L)(\nabla u) \partial_j f \right] = \partial_i \left(a_{ij} \partial_j f \right) = \nabla \cdot (A \nabla f) = 0$$

- De Giorgi 1956 – Nash 1958: if $A = (a_{ij})$ is measurable and

$$\lambda \mathrm{Id} \leq A \leq \Lambda \mathrm{Id}$$

  then $\nabla \cdot (A \nabla f) = 0$ implies f Hölder continuous
- This implies finally $a_{ij} \in C^\alpha$ and solves the problem

The result of De Giorgi and Nash (II)

- Proof of De Giorgi 1956: (1) iterate gain of integrability by Sobolev embedding (2) isoperimetric-type argument to control oscillations
- Proof of Nash 1958: based on the fundamental solutions and several functional inequalities including what is now called ‘Nash inequality’
- Proof of Moser 1964: (1) iterate gain of integrability similarly to De Giorgi but presented differently (2) control “integral” oscillations by an argument using a Poincaré inequality on the logarithm of the solution
- The proof of Moser also obtained Harnack inequality for such elliptic/parabolic equations, i.e. a universal inequality between upper and lower bounds. This proof was later simplified by Krüzkhov 1963-4
- Later a non-divergent version of this result was obtained by Krylov-Safonov 1981 through different methods: open problem to extend Krylov-Safanov theory to the hypoelliptic case

The theory of hypoellipticity of Hörmander (I)

- Theory associated with Hörmander 1967 but partial results by other mathematicians as early as the 1950s
- Starting point of Hörmander: Kolmogorov'1934 and Lewy'1957
- Kolmogorov'1934 considers a kinetic transport equation with drift-diffusion in velocity (i.e. kinetic Fokker-Planck equation)

$$\partial_t f + v \cdot \partial_x f = \partial_v^2 f \quad \text{whose fundamental solution from } \delta_{0,0}$$

$$\text{is } G(t,x,v) = \left(\frac{3}{4\pi^2 t^4} \right)^{\frac{1}{2}} \exp\left[- \frac{3\left|x - \frac{t}{2} v^2\right|}{t^3} - \frac{|v|^2}{4t} \right] \quad (t > 0)$$

- As suggested by the German title, the motivation comes from the study of the law of the Brownian motion (integrated in time)
- It shows that the solution is C^∞ even though the diffusion is degenerate in x, and Lewy's example shows that even with polynomial coefficients and smooth source term some PDEs have no solution

The theory of hypoellipticity of Hörmander (II)

- Hörmander 1967: Identifies necessary and sufficient commutator conditions between the vector fields in the equation for regularization
- Regularization Gevrey instead of analytic
- Two types of hypoelliptic equations to distinguish: "Type I" when no term of order 1 in the equation and "Type 2" when a skew-symmetric (conservative) operator is combined with a partial diffusion
- Simple commutator example for Kolmogorov's equation

$$\partial_t f + Bf + A^* A f = 0, \quad B = v \cdot \partial_x, \quad A = \partial_v$$

$$[A, B] = C = \partial_x, \quad \frac{\mathrm{d}}{\mathrm{d}t} \langle Af, Cf \rangle = -\| Cf \|^2 + \dots$$

Math-Physics motivation for extending De Giorgi theory

- In kinetic theory long-rang interactions means grazing collisions dominate and lead to singular Boltzmann collision operators
- Coulomb interactions ill-defined for the Boltzmann collision operator (cf. "fractional" Laplacian at order 2) but Landau 1936 derived

$$Q(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \mathrm{P}\Big(f(v_*) \nabla_v f(v) - f(v) \nabla_v f(v_*) \Big) |v - v_*|^{-1} \, \mathrm{d}v_* \right)$$

where P orthogonal projection on $(v - v_*)^\perp$

- Rewrites as a nonlinear non-local drift-diffusion operator

$$Q(f,f) = \nabla_v \cdot \big(A[f] \nabla_v f + B[f] f \big)$$

$$\begin{cases} A[f](v) = a \displaystyle\int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{-1} f(t,x,v-w) \, \mathrm{d}w \\ B[f](v) = b \displaystyle\int_{\mathbb{R}^3} |w|^{-3} \, w \, f(t,x,v-w) \, \mathrm{d}w \end{cases}$$

- Existence of global smooth solutions far from equilibrium opened

Extension of De Giorgi theory to the hypoelliptic setting

Consider f a weak L^2 solution to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S \quad (t \in \mathbb{R},\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d)$$

$$\begin{cases} A \text{ measurable symmetric real matrix field with eigenvalues in } [\lambda, \Lambda] \\ B \text{ measurable vector field such that } |B| \leq \Lambda \\ S \text{ real scalar field in } L^\infty \end{cases}$$

Pascucci-Polidoro'04: Boundedness by Moser iteration
Wang-Zhang'11: Hölder continuity by Moser-Krüzkhov approach
Golse-Imbert-Mouhot-Vasseur'19: Hölder regularity and Harnack inequality by De Giorgi approach (non-constructive)
Guerand-Imbert'21: Revisit approach of Wang-Zhang 2011

Theorems (Guerand-M'21)

Quantitative De Giorgi new argument for Hölder continuity and Harnack $\leq$

Detailed main results (I) - Invariances

- Our class of equations is invariant under translations in t, x and under Galilean translations, i.e. for $z_0 = (t_0, x_0, v_0)$ and $z = (t, x, v)$,

$$z \mapsto z_0 \circ z = (t_0 + t, x_0 + x + tv_0, v_0 + v)$$

- For any $r > 0$ it is invariant under the scaling

$$z = (t, x, v) \to rz := (r^2 t, r^3 x, rv)$$

- Using the invariances, we write for $z_0 \in \mathbb{R}^{1+2d}$ and $r > 0$:

$$Q_r(z_0) := z_0 \circ [rQ_1] = z_0 \circ Q_r$$
$$= \left\{ -r^2 < t - t_0 \leq 0,\ |x - x_0 - (t - t_0)v_0| < r^3,\ |v - v_0| < r \right\}$$

  and we simply write $Q_r(0) = Q_r$ when $z_0 = 0$

- We denote $\mathcal{T} = \partial_t + v \cdot \nabla_x$ the free transport operator

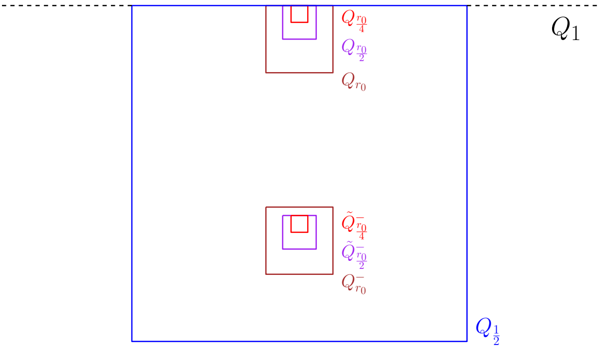
Detailed main results (II) - Notion of solutions

- Given $\mathcal{U} = (a,b) \times \Omega_x \times \Omega_v$ with Ω_x and Ω_v open sets of $\mathbb{R}^d$, $f : \mathcal{U} \to \mathbb{R}$ is a weak solution on $\mathcal{U}$ if it belongs to the energy space $\mathcal{E} = L^\infty((a,b); L^2(\Omega_x \times \Omega_v)) \cap L^2((a,b) \times \Omega_x; H^1(\Omega_v))$ and the equation is satisfied in the sense of distributions in $\mathcal{U}$
- f is a weak sub-solution if $f \in \mathcal{E}$ and for all $G \in C^2$ with $G' \geq 0$ bounded and $G'' \geq 0$, and any non-negative $\varphi \in C_c^\infty(\mathcal{U})$

$$-\int_{\mathcal{U}} G(f)\mathcal{T}\varphi \,\mathrm{d}z \leq -\int_{\mathcal{U}} A\nabla_v G(f) \cdot \nabla_v \varphi \,\mathrm{d}z + \int_{\mathcal{U}} \left[B \cdot \nabla_v G(f) + SG'(f)\right]$$

- It is a weak super-solution if $-f$ is a weak sub-solution
- Equivalent to previous definitions in the case of solutions, but slightly weaker in the case of sub- and super-solutions: extra assumptions $\mathcal{T}f \in L^2((a,b) \times \Omega_x \times \Omega_v)$ or $\mathcal{T}f \in L^2((a,b) \times \Omega_x; H^{-1}(\Omega_v))$ were made before for energy estimates
- It allows to include important sub-solutions such as $f = f(t) = 1_{t \leq 0}$
- Our definition is equivalent to that of De Giorgi in the elliptic case

Detailed main results (III) - Figure



- Given invariances, we only state results in a unit centred cylinder
- f (sub/super)-solution in Q_1
- $r_0 \in (0, \frac{1}{20})$ explicit from the proof
- Intermediate Value Lemma relates $Q^-_{r_0}$ and Q_{r_0}
- Weak Harnack inequality relates $\tilde{Q}^-_{\frac{r_0}{2}}$ and $Q_{\frac{r_0}{2}}$
- Harnack inequality relates $\tilde{Q}^-_{\frac{r_0}{4}}$ and $Q_{\frac{r_0}{4}}$

Detailed main results (IV) - Statements (with $S = 0$)

Intermediate Value Lemma. Given $\delta_1, \delta_2 \in (0,1)$, there are $r_0 = \frac{1}{20}$, $\nu \gtrsim (\delta_1\delta_2)^{5d+8}$ and $\theta \gtrsim (\delta_1\delta_2)^{6d+15}$, such that any sub-solution f in Q_1 so that $f \leq 1$ in $Q_{\frac{1}{2}}$ and $|\{f \leq 0\} \cap Q^-_{r_0}| \geq \delta_1 |Q^-_{r_0}|$ and $|\{f \geq 1-\theta\} \cap Q_{r_0}| \geq \delta_2 |Q_{r_0}|$ satisfies

$$\left|\{0 < f < 1-\theta\} \cap Q_{\frac{1}{2}}\right| \geq \nu |Q_{\frac{1}{2}}|$$

Weak Harnack Inequality. There is $\zeta > 0$ depending only λ, Λ such that any non-negative weak super-solution f in Q_1 satisfies, for $r_0 = \frac{1}{20}$,

$$\left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta(z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}v \right)^{\frac{1}{\zeta}} \lesssim_{\lambda,\Lambda} \inf_{Q_{\frac{r_0}{2}}} f + \|S\|_{L^\infty(Q_1)}$$

Harnack inequality. Any non-negative weak solution f in Q_1 satisfies

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}} f \lesssim_{\lambda,\Lambda} \inf_{Q_{\frac{r_0}{4}}} f + \|S\|_{L^\infty(Q_1)}$$

(Both IVL & Harnack $\leq$ imply Hölder continuity quantitatively)

Structure of the method (for f sub/super sol. and $S = 0$)

$$f \in L^{\zeta} \xrightarrow{(1)} f \in L^{\infty} \cap L^1_{t,v} W_x^{\frac{1}{3}-0,1} \xrightarrow{(2)} \text{Weak } L^1\text{-Poincaré inequality}$$

$$\xrightarrow{(3)} \text{Intermediate Value Lemma} \xrightarrow{(4)} \text{Measure-to-pointwise estimate}$$

$$\xrightarrow{(5)} \text{Weak log-Harnack estimate} \xrightarrow{(6)} \text{Weak Harnack estimate}$$

[Once these steps are proved, Harnack inequality follows (6)+(1)]

Step (1) inspired by [PP'04] and uses Kolmogorov fundamental solutions
Step (2) is the most novel step and introduces an argument based on trajectories and the previous Sobolev regularity to "noise" the x-dependency
Step (3) is novel and based on simple energy estimates
Step (4) is standard and only sketched for obtaining quantitative constants
Step (5) is semi-novel but immediate when constants are quantified
Step (6) is novel in the context of hypoelliptic equations but inspired from a conceptually similar idea in elliptic equations; it uses an induction, Vitali's covering lemma and Step (5) at every scale

Step 1: The L^2 energy estimate

- Starting point of all methods
- Consider f non-negative sub-solution in an open set $\mathcal{U} \in \mathbb{R}^{1+2d}$ and $Q_r(z_0) \subset Q_R(z_0) \subset \mathcal{U}$ with $0 < r < R$
- Integrate then the equation against $f\varphi^2$ with an appropriate smooth localisation function φ to get

$$\sup_{\tau \in (-r^2+t_0, t_0)} \int_{Q_r^\tau(z_0)} f^2 + \int_{Q_r(z_0)} |\nabla_v f|^2 \lesssim_{\lambda,\Lambda,r,R} \int_{Q_R(z_0)} f^2 + \|S\|^2_{L^2(Q_R(z_0))}$$

  where $z_0 = (t_0, x_0, v_0)$, $Q_r^\tau(z_0) = \{(x, v) \in \mathbb{R}^{2d} : (\tau, x, v) \in Q_r(z_0)\}$
- Unlike the elliptic or parabolic case, the energy estimate does not yield Sobolev regularity in all variables
- Addressed before by averaging lemma, here simpler systematic optimal calculation based on Kolmogorov solutions inspired from [PP'04]

Step 1: The L^1 mass estimate

- Less well-known but simple and useful
- Consider again f non-negative sub-solution in an open set $\mathcal{U} \in \mathbb{R}^{1+2d}$ and $Q_r(z_0) \subset Q_R(z_0) \subset \mathcal{U}$ with $0 < r < R$
- Write $m \geq 0$ the defect measure:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S - m$$

- Integrate then the equation against φ^2 with an appropriate smooth localisation function φ to get

$$\|m\|_{L^1(Q_r(z_0))} \lesssim_{\lambda,\Lambda,r,R} \int_{Q_R(z_0)} f + \int_{Q_R(z_0)} |\nabla_v f| + \int_{Q_R(z_0)} |S|$$

- Hence the mass of the defect measure is controlled, i.e. intuitively the total amount of jump in discontinuities is constrained

Step 1: Kolmogorov fundamental solutions (I)

- Consider $f \geq 0$ locally integrable so that

$$\mathcal{K}f := \partial_t + v \cdot \nabla_x f - \Delta_v f = \nabla_v \cdot F_1 + F_2 - m$$

with $F_1, F_2 \in L^1 \cap L^2(\mathbb{R}_- \times \mathbb{R}^{2d})$, $m \geq 0$ measure with finite mass on $\mathbb{R}_- \times \mathbb{R}^{2d}$, and F_1, F_2, m have compact support in time $[-T, 0]$

- Then for $p \in [2, 2 + \frac{1}{d})$ and $\sigma \in [0, \frac{1}{3})$

$$\|f\|_{L^p(\mathbb{R}_- \times \mathbb{R}^{2d})} \lesssim_{\lambda,\Lambda,T,p} \|F_1\|_{L^2(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|F_2\|_{L^2(\mathbb{R}_- \times \mathbb{R}^{2d})}$$

$$\|f\|_{L^1_{t,v} W^{\sigma,1}_x(\mathbb{R}_- \times \mathbb{R}^{2d})} \lesssim_{\lambda,\Lambda,T,\sigma} \|F_1\|_{L^1(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|F_2\|_{L^1(\mathbb{R}_- \times \mathbb{R}^{2d})} + \|m\|_{M^1(\mathbb{R}_- \times \mathbb{R}^{2d})}$$

- Bounds on p and σ seem optimal & constants like inverse distance
- Note that defect measure appears in the second (regularity) estimate but not in the first (integrability) estimate

Step 1: Kolmogorov fundamental solutions (II)

- Localize the sub-solution f and write

$$\mathcal{K} f = \nabla_v \cdot ((A - \mathrm{Id}) \nabla_v f) + B \cdot \nabla_v f + S - m = \nabla_v \cdot F_1 + F_2 - m$$

- Use the L^2 energy estimate and the L^1 mass estimate to get L^2 bounds on F_1 and F_2 and L^1 bounds on F_1, F_2 and m
- Express solution f with the fundamental solution

$$f(t,x,v) = \int_{t' \in \mathbb{R}} \int_{x',v' \in \mathbb{R}^d} G(t-t', x-x'-(t-t')v', v-v')(\mathcal{K} f)(t',x',v')$$

$$G(t,x,v) := \begin{cases} \left(\frac{3}{4\pi^2 t^4}\right)^{\frac{d}{2}} \exp\left[-\frac{3\left|x-\frac{t}{2}v\right|^2}{t^3} - \frac{|v|^2}{4t}\right] & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

- Since $G \geq 0$, drop the defect measure for the gain of integrability
- Use $G \in L^q$ and $\nabla_v G \in L^q$ and $t \nabla_x G \in L^q$ to gain $L^2 \to L^p$
- Since $f^{\frac{p}{2}}$ sub-solution, iteration gives $L^2 \to L^\infty$
- Additional iteration easily yields $L^\zeta \to L^\infty$ for any $\zeta > 0$
- Decompose G in t and use higher-order estimates to gain regularity

Step 2: Weak Poincaré Inequality (with $S = 0$) (I)

- The key step to the Intermediate Value Lemma is to measure variations above the mean in terms of $\|\nabla_v f\|_{L^1}$
- Given $\varepsilon \in (0,1)$, $\sigma \in (0, \frac{1}{3})$, and f non-negative sub-solution on Q_5

$$\left\| \left(f - \langle f \rangle_{Q_1^-} \right)_+ \right\|_{L^1(Q_1)} \lesssim_{\lambda,\Lambda} \frac{1}{\varepsilon^{d+2}} \|\nabla_v f\|_{L^1(Q_5)} + \varepsilon^\sigma \|f\|_{L^2(Q_5)}$$

where $Q_1^- := Q_1(-1,0,0) = (-3,-2] \times B_1 \times B_1$ and

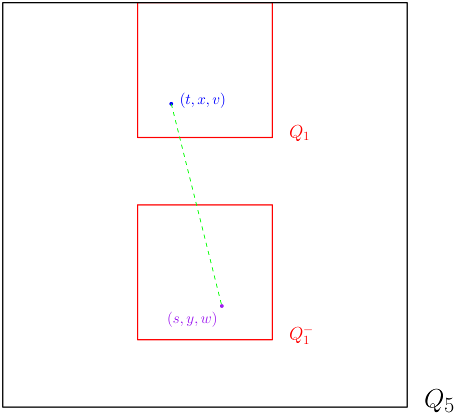
$$\langle f \rangle_{Q_1^-} := \fint_{Q_1^-} f := \frac{1}{|Q_1^-|} \int_{Q_1^-} f$$

- Such inequality is reminiscent of the Moser approach, however our proof is a new simpler argument based on trajectories

Step 2: Weak Poincaré Inequality (with $S = 0$) (II)

$$\left\| \left(f - \langle f \rangle_{Q_1^-} \right)_+ \right\|_{L^1(Q_1^+)} \lesssim \left\| \left(f - \langle f \varphi_\varepsilon \rangle_{Q_1^-} \right)_+ \right\|_{L^1(Q_1^+)}$$

$$\lesssim \int_{(t,x,v)\in Q_1^+} \left[\fint_{(s,y,w)\in Q_1^-} (f(t,x,v) - f(s,y,w))\, \varphi_\varepsilon(y,w) \right]_+ + \varepsilon^{2d} \|f\|_{L^2(Q_1^+)}$$



Step 2: Weak Poincaré Inequality (with $S = 0$) (III)

- We decompose the trajectory $(t, x, v) \to (s, y, w)$ into four sub-trajectories in Q_5:
  - a trajectory of length $O(\varepsilon)$ along ∇_x in the direction w
  - two trajectories of length $O(1)$ along ∇_v
  - one trajectory of length $O(1)$ along $\mathcal{T} := \partial_t + v \cdot \nabla_x$
- This yields the diagram

$$(t, x, v) \xrightarrow[\nabla_x]{} (t, x + \varepsilon w, v) \xrightarrow[\nabla_v]{} \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right)$$
$$\xrightarrow[\mathcal{T}]{} \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) \xrightarrow[\nabla_v]{} (s, y, w)$$

- The first sub-trajectory is estimated by the integral regularity $L^1_{t,v} W^{\sigma,1}_x$
- The other trajectories are estimated by the vector fields in the equation
- Note that we are implicitly using the Hörmander commutator condition: $\nabla_v, \mathcal{T}, [\nabla_v, \mathcal{T}]$ span all the vector fields on $\mathbb{R}^{2d+1}$

Step 2: Weak Poincaré Inequality (with $S = 0$) (IV)

First sub-trajectory

$$I_1 \lesssim \int_{(t,x,v)\in Q_1} |f(t,x,v) - f(t,x+\varepsilon w,v)| \lesssim \varepsilon^\sigma \|f\|_{L^1_{t,v} W^{\sigma,1}_x}$$

Second and fourth trajectories

$$I_2 + I_4 \lesssim \int |\nabla_v f|$$

Third and hardest trajectory

$$I_3 \lesssim \int_{(t,x,v)\in Q_1} \left[\int_{(s,y,w)} \int_{\tau\in[0,1]} \nabla_v \cdot (A \nabla_v f)\,(s^*, y^*, w^*) \varphi_\varepsilon(y,w) \right]_+$$

and the change of variable $(s,y,w) \to (s^*, y^*, w^*)$ has bounded Jacobian thanks to the "noise" εw of the first trajectory, which allows to integrate by parts the divergence on φ_ε inside Q_1^-

Step 3: Proof of the intermediate value lemma (I)

Take $S = 0$ and f sub-solution on Q_1 so that for $\delta_1, \delta_2 > 0$ and $r_0 = \frac{1}{20}$

$$|\{f \leq 0\} \cap Q_{r_0}^-| \geq \delta_1 |Q_{r_0}^-| \quad \text{and} \quad |\{f \geq 1-\theta\} \cap Q_{r_0}| \geq \delta_2 |Q_{r_0}|$$

Then the Poincaré inequality implies

$$⨍_{Q_{r_0}} \left(f_+ - \langle f_+ \rangle_{Q_{r_0}^-} \right)_+ \lesssim \frac{1}{\varepsilon^{d+2}} \int_{Q_{5r_0}} |\nabla_v f_+| + \varepsilon^\sigma$$

Since

$$\langle f_+ \rangle_{Q_{r_0}^-} \leq \frac{\left|\{f > 0\} \cap Q_{r_0}^-\right|}{|Q_{r_0}^-|} \leq 1 - \delta_1$$

the left hand side is bounded below:

$$\begin{aligned} ⨍_{Q_{r_0}} \left(f_+ - \langle f_+ \rangle_{Q_{r_0}^-} \right)_+ &\geq \frac{1}{|Q_{r_0}|} \int_{(t,x,v) \in Q_{r_0}} \left[f(t,x,v) - (1-\delta_1) \right]_+ \\ &\geq \frac{1}{|Q_{r_0}|} \int_{\{f \geq 1-\theta\} \cap Q_{r_0}} (\delta_1 - \theta)_+ \geq \delta_2 (\delta_1 - \theta) \end{aligned}$$

Step 3: Proof of the intermediate value lemma (II)

We now bound from above the right hand side

$$\int_{Q_{5r_0}} |\nabla_v f_+| \leq \underbrace{\int_{\{f=0\}\cap Q_{5r_0}} \cdots}_{=0} + \underbrace{\int_{\{0<f<1-\theta\}\cap Q_{5r_0}} \cdots}_{I_1} + \underbrace{\int_{\{f\geq 1-\theta\}\cap Q_{5r_0}} \cdots}_{I_2}$$

The first term takes advantage of the fact that Poincaré $\leq$ was in L^1:

$$I_1 \leq |\{0 < f < 1-\theta\}\cap Q_{5r_0}|^{\frac{1}{2}} \left(\fint_{Q_{5r_0}} |\nabla_v f_+|^2 \right)^{\frac{1}{2}} \lesssim |\{0 < f < 1-\theta\}\cap Q_{\frac{1}{2}}|^{\frac{1}{2}}$$

The second term is small when θ is small:

$$I_2 = \int_{Q_{5r_0}} \left|\nabla_v \left[(f-(1-\theta))_+ + (1-\theta)\right]\right| = \int_{Q_{5r_0}} \left|\nabla_v \left[f-(1-\theta)\right]_+\right|$$

$$\lesssim \left(\int_{Q_{5r_0}} \left|\nabla_v \left[f-(1-\theta)\right]_+\right|^2 \right)^{\frac{1}{2}} \lesssim \int_{Q_{\frac{1}{2}}} \left[f-(1-\theta)\right]_+^2 \lesssim \theta$$

The conclusion follows from taking ε and θ small enough

Step 4: The measure-to-pointwise estimate (I)

Given $S = 0$, $\delta \in (0,1)$ and $r_0 = \frac{1}{20}$ there is $\mu := \mu(\delta) \sim \delta^{2(1+\delta^{-10d-16})} > 0$ such that any sub-solution f in Q_1 so that $f \leq 1$ in $Q_{\frac{1}{2}}$ and

$$\left|\{f \leq 0\} \cap Q_{r_0}^-\right| \geq \delta \left|Q_{r_0}^-\right|$$

satisfies $f \leq 1 - \mu$ in $Q_{\frac{r_0}{2}}$

Proof follows the standard De Giorgi argument, only more quantitative:

- There is $\delta' > 0$ universal such that for any $r > 0$, any sub-solution f on Q_{2r} so that $\int_{Q_r} f_+^2 \leq \delta' |Q_r|$ satisfies $f \leq \frac{1}{2}$ in $Q_{\frac{r}{2}}$
- Define $\nu, \theta > 0$ as in the IVL with $\delta_1 = \delta$ and $\delta_2 = \delta'$ and define the sub-solutions $f_k := \theta^{-k}[f - (1 - \theta^k)]$ for $k \geq 0$
- The sets $\{0 < f_k < 1 - \theta\} = \{1 - \theta^k < f < 1 - \theta^{k+1}\}$ are disjoints and each f_k satisfies the assumptions of the IVL
- If $\int_{Q_{r_0}} (f_k)_+^2 \leq \delta' |Q_{r_0}|$ then $f_k \leq \frac{1}{2}$ in $Q_{\frac{r_0}{2}}$ so $f \leq 1 - \mu$ with $\mu = \frac{\theta^k}{2}$ which concludes the proof

Step 4: The measure-to-pointwise estimate (II)

- Consider $1 \leq k_0 \leq 1 + \nu^{-1}$ such that $\int_{Q_{r_0}} (f_k)_+^2 > \delta' |Q_{r_0}|$ for any $0 \leq k \leq k_0$. Then for $0 \leq k \leq k_0 - 1$

  $$|\{f_k \geq 1 - \theta\} \cap Q_{r_0}| = |\{f_{k+1} \geq 0\} \cap Q_{r_0}| \geq \int_{Q_{r_0}^+} (f_{k+1})_+^2 > \delta' |Q_{r_0}|$$

  $$\left|\{f_k \leq 0\} \cap Q_{r_0}^-\right| \geq \left|\{f \leq 0\} \cap Q_{r_0}^-\right| \geq \delta |Q_{r_0}^-|$$

  IVL then implies

  $$\left|\{0 < f_k < 1 - \theta\} \cap Q_{\frac{1}{2}}\right| \geq \nu |Q_{\frac{1}{2}}|$$

- Summing these estimates we have

  $$|Q_{\frac{1}{2}}| \geq \sum_{k=0}^{k_0 - 1} \left|\{0 < f_k < 1 - \theta\} \cap Q_{\frac{1}{2}}\right| \geq k_0 \nu |Q_{\frac{1}{2}}|.$$

  So $k_0 \leq \nu^{-1}$, and we deduce in $Q_{\frac{1}{2}}$

  $$f \leq 1 - \frac{\theta^{k_0+1}}{2} \leq 1 - \frac{\theta^{\frac{1+\nu}{\nu}}}{2} \implies \mu(\delta) := \frac{\theta^{1+\frac{1}{\nu}}}{2} \sim \delta^{2(1+\delta^{-10d-16})}$$

Steps 5: Weak log-Harnack inequality (with $S = 0$)

- Given h non-negative super-solution, the contraposition on the sub-solution $g := 1 - \frac{h}{M}$ of the measure-to-pointwise estimate implies for any $\delta \in (0,1)$, there is $M \sim \delta^{-2(1+\delta^{-10d-16})}$ so that

$$\frac{|\{h > M\} \cap Q_r(z)|}{|Q_r(z)|} > \delta \quad \Longrightarrow \quad \inf_{Q^+_{\frac{r}{2}}(z)} h \geq 1$$

  where $Q_r(z) \mapsto Q^+_{\frac{r}{2}}(z)$ is the inverse of the operation $Q_{\frac{r}{2}}(z) \mapsto Q^-_r(z)$ in the previous statement
- Assuming $\inf_{Q_{\frac{r_0}{2}}} h \leq 1$ and inverting the function $\delta \mapsto M(\delta)$, this gives upper bounds on the upper level sets, and the layer-cake representation finally yields

$$\int_{Q^-_{r_0}} \left[\ln\left(1 + h\right)\right]^{\frac{1}{10d+18}} \lesssim 1$$

- This logarithmic integrability is weaker than the usual weak Harnack inequality but it can strenghtened by a simple iterative argument

Step 6: The weak Harnack inequality (I)

- Consider as before $S = 0$ (the source term can be re-introduced in the end of the proof anyway easily)
- We prove by induction on a sequence of cylinders $\mathcal{Q}^k$ that satisfy $\tilde{Q}^-_{\frac{r_0}{2}} \subset \mathcal{Q}^k \subset \bar{\mathcal{Q}}^k \subset \mathring{\mathcal{Q}}^{k-1} \subset Q^-_{r_0}$ for all $k \geq 1$, that for $\delta_0 > 0$ small enough, any non-negative super-solution h with $\inf_{Q_{\frac{r_0}{2}}} h < 1$ satisfies

$$\forall\, k \geq 1, \quad \frac{\left|\{h \geq M^k\} \cap \mathcal{Q}^k\right|}{|\mathcal{Q}^k|} \leq \frac{\delta_0}{210^{(4d+2)k}}$$

  where $M \sim \delta^{-2(1+\delta^{-10d-16})}$ with $\delta := \frac{\delta_0}{210^{4d+2}}$
- If the latter is true it implies $\int_{\tilde{Q}^-_{\frac{r_0}{2}}} h^\zeta \lesssim 1$ for some $\zeta \gtrsim \delta_0^{10d+17} > 0$ which concludes the proof

Step 6: Weak Harnack inequality (II)

- To propagate the induction we cover $A_{k+1} := \{h > M^{k+1}\} \cap \mathcal{Q}^{k+1}$ with translations of centered cylinders of the form

$$\mathfrak{C}_r[z] := z \circ Q_{2r}((2r^2, 0, 0)) = z \circ \left(-2r^2, 2r^2\right] \times B_{(2r)^3} \times B_{2r}$$

- We construct a sequence $\mathfrak{C}_{r_\ell}[z_\ell]$, $\ell \geq 1$, so that:
  1. $r_\ell \in (0, \frac{r_0}{30 \cdot 7^{k-1}})$
  2. $|A_{k+1} \cap \mathfrak{C}_{15 r_\ell}[z_\ell]| \leq \delta_0 |\mathfrak{C}_{15 r_\ell}[z_\ell]|$
  3. $|A_{k+1} \cap \mathfrak{C}_{r_\ell}[z_\ell]| > \delta_0 |\mathfrak{C}_{r_\ell}[z_\ell]|$
  4. the cylinders $\mathfrak{C}_{3 r_\ell}[z_\ell]$ are disjoint
  5. A_{k+1} is covered by the family $\mathfrak{C}_{15 r_\ell}[z_\ell]$
- This construction is based on Vitali's covering lemma and the geometry of the cylinders
- The measure-to-pointwise estimate at every scale then implies that $\mathfrak{C}_{r_\ell}[z_\ell]^+ \subset A_k$ and since $\mathfrak{C}_{r_\ell}[z_\ell]^+ \subset \mathfrak{C}_{3 r_\ell}[z_\ell]$, these "+" cylinders are disjoint

Step 6: Weak Harnack inequality (III)

- We deduce

$$
\begin{aligned}
|A_{k+1}| &\leq \sum_{\ell \geq 1} |A_{k+1} \cap \mathfrak{C}_{15 r_\ell}[z_\ell]| \leq \delta_0 \sum_{\ell \geq 1} |\mathfrak{C}_{15 r_\ell}[z_\ell]| \\
&\leq 15^{4d+2} \delta_0 \sum_{\ell \geq 1} |\mathfrak{C}_{r_\ell}[z_\ell]| \leq 30^{4d+2} \delta_0 \sum_{\ell \geq 1} |\mathfrak{C}_{r_\ell}[z_\ell]^+| \\
&\leq 30^{4d+2} \delta_0 |A_k| \leq \frac{30^{4d+2} \delta_0^2}{210^{(4d+2)k}} \leq \frac{\delta_0}{210^{(4d+2)(k+1)}} \left| \mathcal{Q}^{k+1} \right|
\end{aligned}
$$

for δ_0 small enough which proves the induction claim and concludes the whole proof