Quasilinear approximation of Vlasov and Liouville equations.

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The Vlasov equation

$$\partial_t F + v\cdot\nabla_x F + E\cdot\nabla_v F = 0,\ (t,x,v)\in\mathbb{R}^+_t\times\mathbb{T}^d_x\times\mathbb{R}^d_v.$$

$$E = \nabla\Phi\,, -\Delta\Phi(t,x) = \rho - 1, \fint F_\varepsilon(x,v,t)dxdv = 1$$

$$\fint .dx = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} .dx$$

The quasilinear approximation

$$\partial_t\overline{F}(t,v) - \partial_v(\mathbb{D}(t,v)\partial_v\overline{F}(t,v)) = 0\,.$$

With $\mathbb{D}(t,v)$ being a symmetric nonnegative matrix.
This has been a very classical subject in Plasma Physic, cf. Krall and Trivelpiece 1973 and Plasma Physic in the 20th century as told by players EPJ vol 43 2018. Now related to very moderns ideas.

- $\mathbb{T}^d$ Because it is a bounded domain where the advection flow is ergodic. Explicit computations can be made. Extension to other domains would be useful.
- It is a natural approximation to grasp the averaged profile of velocity.
- It involves a natural Fick law.

$$\partial_t \overline{F}(t,v) + \nabla_v \overline{\fint E(t,x)F(t,x,v)dv} = 0\,,$$

$$J = \overline{E(t,x) \fint F(t,x,v)dv}\,.$$

- Comparison of the conservations law: $\mathbb{D}(t,v) > 0$ is a subtle issue. No strong convergence.

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d_v} dv \fint dx\,|f^\varepsilon|^2 = 0$$

Versus

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d_v} dv \fint dx\,|\overline{f^\varepsilon}|^2 + \int_{\mathbb{R}^d_v} dv \fint dx\left(\mathbb{D}(t,v)\nabla_v\overline{f^\varepsilon}\,,\,\nabla_v\overline{f^\varepsilon}\right) = 0\,.$$

- Weak convergence on the solution itself often by Duhamel formula (or iterated Duhamel formula)

$$S(t)f = f(t, x - vt, v)\,,$$
$$f(t,x,v) = S_t f(0) - \int_0^t d\sigma_1\, S_{t-\sigma_1} E(\sigma_1)\cdot\nabla_v f(\sigma_1)\ldots,$$

and almost compulsory introduction of some type of average.

1. The rescaled Vlasov and Liouville equations. Pathologies in convergence.
2. Convergence of the rescaled stochastic Liouville equation.
3. Return to Genuine Vlasov Equation.
4. Short time quasilinear approximation in the presence of instabilities.
5. Remarks, conclusion and open problems.

$$f^\varepsilon(0,x,v) \in \mathcal{S}(\mathbb{T}^d \times \mathbb{R}^d_v) \quad \partial_t f^\varepsilon + \frac{v}{\varepsilon^2}\cdot\nabla_x f^\varepsilon + \frac{E^\varepsilon}{\varepsilon}\cdot\nabla_v f^\epsilon = 0\,, \tag{1a}$$

$$v\cdot\nabla_x f^\varepsilon = -\varepsilon^2\partial_t f^\varepsilon - \varepsilon E^\varepsilon\cdot\nabla_v f^\varepsilon\,, \tag{1b}$$

$$\begin{aligned} &\partial_\tau F^\varepsilon + v\cdot\nabla_x F^\varepsilon + \varepsilon E^\varepsilon(\tau)\cdot\nabla_v F^\epsilon = 0 \\ &\tau = \frac{t}{\varepsilon^2}\,, \quad f^\varepsilon(t,x,v) = F^\varepsilon(\frac{t}{\varepsilon^2},x,v)\,. \end{aligned} \tag{1c}$$

Equation (1a) $\Rightarrow$ convergence in $L^\infty(\mathbb{R}^+_t \times \mathbb{T}^d \times \mathbb{R}^d_v)$weak$-\star$.
Equation (1b) and ergodicity of $S(t)$ in $\mathbb{T}^d \Rightarrow f^\varepsilon(t,x,v) \rightharpoonup \overline{f^\varepsilon}(t,v)$.
With Poisson relation $E^\varepsilon(x,t) \rightharpoonup 0$ in $L^2((0,T)\times\mathbb{T}^d)$ weak.
A "baby" version of the Landau damping.
From (1c) with initial data $F^\varepsilon(0,x,v) = G(v) + h(x,v)$ satisfying the standard Landau-damping hypothesis one would have $\|E^\varepsilon(\tau)\|_{L^2(\mathbb{T}^d)} \to 0$ exponentially fast that would imply exponential convergence to 0 in $L^2(\delta, T; L^2(\mathbb{T}^d))$ of $E^\varepsilon(t,x)$.

Start from:

$$\partial_t \fint f^\varepsilon(t,x,v)dx + \nabla_v \fint \frac{E^\varepsilon(t,x)f^\varepsilon(t,x,v)}{\varepsilon}dx = 0$$
$$\partial_t \overline{f^\varepsilon}(t,v)dx + \nabla_v \overline{\fint \frac{E^\varepsilon(t,x)f^\varepsilon(t,x,v)}{\varepsilon}dx} = 0 \qquad (2)$$
$$f^\varepsilon(t) = S^\varepsilon_t f^\varepsilon_0 - \frac{1}{\varepsilon}\int_0^t S^\varepsilon_{t-\sigma}E^\varepsilon(\sigma)\cdot\nabla_v f^\varepsilon(\sigma)d\sigma\,.$$

After introduction of test function, integration by part and change of variables:

$$\int_{\mathbb{R}^d_v} dv\phi(v)\nabla_v \fint \frac{E^\varepsilon(t,x)f^\varepsilon(t,x,v)}{\varepsilon}dx =$$
$$\overline{-\int_{\mathbb{R}^d_v} dv \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma f^\varepsilon(t-\sigma\varepsilon^2,x,v)} \qquad (3)$$
$$\nabla_v\cdot\Big(E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+\sigma v)\Big)\nabla_v\phi(v)\Big)$$

$$\mathbb{D}^\varepsilon(t,v) = \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma \big(E^\varepsilon(t-\varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x+\sigma v) \big)$$

a tensor which in the present field ("plasma turbulence") plays a role very similar to the Reynolds stress tensor

$$u(t,x) \otimes u(s,y)$$

in fluid turbulence. Hence 2 natural issues

1. Does
$$\overline{\int_{\mathbb{R}^d_v} \nabla\phi(v)\mathbb{D}^\varepsilon(t,v)\nabla\psi(v)dv} \to \int_{\mathbb{R}^d_v} \nabla\phi(v)\overline{\mathbb{D}^\varepsilon(t,v)}\nabla\psi(v)dv$$
with $\overline{\mathbb{D}^\varepsilon(t,v)}$ being a non degenerate, non singular, diffusion ??
2. What about decorelation between two "weakly converging terms"

$$\overline{\int_{\mathbb{R}^d_v} dv \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma f^\varepsilon(t-\sigma\varepsilon^2, x, v)\nabla_v \mathbb{D}^\varepsilon(t,v))\nabla_v\phi(v))}$$

$$= ??? \int_{\mathbb{R}^d_v} dv \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma \overline{f^\varepsilon(t-\sigma\varepsilon^2, x, v)}\nabla_v \overline{\mathbb{D}^\varepsilon(t,v))}\nabla_v\phi(v))\,.$$

Assuming that $E(t,v) = \nabla_x \Phi(t,x)$ is given , smooth with respect to the x variable but oscillating with respect to the time.

Proposition

$$Assume\,\Phi^\varepsilon(t,k) = \underline{\Phi}^\varepsilon(t,k) e^{-i\frac{\omega(k)}{\varepsilon^2}t} and \sum_{k\in\mathbb{Z}^d} |k|^4 |\underline{\Phi}^\varepsilon(t,k)|^2 \leq C$$

$$Then\,\mathbb{D}^\varepsilon(t,v) = \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma\, E^\varepsilon(t, x+\sigma v) \otimes E^\varepsilon(t-\varepsilon^2\sigma, x) =$$

$$\sum_{k\in\mathbb{Z}^d} k\otimes k\, |\underline{\Phi}^\varepsilon(t,k)|^2 \,\frac{\sin\left((\omega(k)-k\cdot v)\frac{t}{\varepsilon^2}\right)}{\omega(k)-k\cdot v}\,. \,And:$$

$$\int_{\mathbb{R}^d_v} (\nabla_v \phi(v))^T\, \overline{\mathbb{D}^\varepsilon(t,v)} \nabla_v \psi(v)\, dv =$$

$$\pi \sum_{k\in\mathbb{Z}^d} \overline{|\underline{\Phi}^\varepsilon_k|^2} \int_{k\cdot v=\omega(k)} k\cdot\nabla_v\phi(v)\, k\cdot\nabla_v\psi(v)\, dv\,.$$

Singular and\ or degenerate.

The stochastic electric field

$$\mathbb{E}[f] = \int_\Omega f(\omega) d\mathbb{P}(\omega)\,. \tag{4}$$

- **H1**. Stochastic average of E^ε set equal to 0, i.e.

$$\forall\,(t,x), \quad \mathbb{E}[E^\varepsilon(t,x)] = 0\,. \tag{5}$$

- **H2**. Finite time decorelation:
  $\overline{\tau}$ such that

$$\forall (t,s) \quad |t-s| \geq \overline{\tau}\varepsilon^2 \implies \mathbb{E}[E^\varepsilon(t,x) \otimes E^\varepsilon(s,y)] = 0, \quad \forall (x,y)\,. \tag{6}$$

- **H3**. Time and space homogeneity.

$$E^\varepsilon(t,k) = \fint E^\varepsilon(t,x) e^{-ik\cdot x} dx = \underline{E}^\varepsilon(t,k) e^{-i\frac{\omega_k t}{\varepsilon^2}} = ik\underline{\Phi}^\varepsilon(t,k) e^{-i\frac{\omega_k t}{\varepsilon^2}}\,. \tag{7}$$

$\underline{\Phi}^\varepsilon(t,k)$ are complex random variables, while frequencies ω_k are real and (t,ε)-independent. We have also the following parity properties,

$$\forall k \in \mathbb{Z}^d, \quad \underline{\Phi}^\varepsilon(t,-k) = (\underline{\Phi}^\varepsilon(t,k))^\star \quad \text{and} \quad \omega_{-k} = -\omega_k\,.$$

$\forall k \in \mathbb{Z}^d \backslash \{0\}$, $\sigma \mapsto A_k(\sigma)$ with:

$$\mathbb{E}[\underline{\Phi}^\varepsilon(t,k)\underline{\Phi}^\varepsilon(s,k')] = A_k\Big(\frac{|t-s|}{\varepsilon^2}\Big)\delta(k+k')\,, \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}}|k|^3|A_k(\sigma)|d\sigma < C,$$
$$\hat{A}_k(s) = \int_{\mathbb{R}} A_k(\sigma)e^{-is\sigma}d\sigma\,. \tag{8}$$

$\hat{A}_k(s)$ is analytic (Paley-Wiener and non negative Bochner, Theorems) diffusion matrix which symmetric, non singular and non degenerate.

Proposition

The stochastic diffusion matrix is given by the formula:

$$\mathbb{D}(v) = \overline{\mathbb{E}[\mathbb{D}^\varepsilon(t,v)]} = \frac{1}{2}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} k\otimes k\ \hat{A}_k(\omega_k - k\cdot v)$$

Proof.

With **H2**. Finite time decorelation:

$$\mathbb{E}[\mathbb{D}^\varepsilon(t,v)] = \fint dx \int_0^{\frac{t}{\varepsilon^2}} d\sigma \mathbb{E}[E^\varepsilon(t-\varepsilon^2\sigma, x)\otimes E^\varepsilon(t, x+\sigma v)] =$$

$$\sum_{k\in\mathbb{Z}^d} k\otimes k \int_0^\infty \mathbb{E}[\underline{\Phi}^\varepsilon(t-\varepsilon^2\sigma,k)(\underline{\Phi}^\varepsilon(t,k))^\star]e^{i(\omega_k - k\cdot v)\sigma}d\sigma$$

With **H3** Space time homogeneity.

$$= \sum_{k\in\mathbb{Z}^d} k\otimes k\int_0^\infty A_k(\sigma)e^{i(\omega_k-k\cdot v)\sigma}d\sigma = \frac{1}{2}\sum_{k\in\mathbb{Z}^d} k\otimes k\int_{-\infty}^\infty A_k(\sigma)e^{i(\omega_k-k\cdot v)\sigma}d\sigma$$

Proposition

Decorelation

$$-\nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] =$$
$$\int_0^{\overline{\tau}} d\sigma \fint dx\, \mathbb{E}\left[E^\varepsilon(t)\nabla_v(S^\varepsilon_{\varepsilon^2\sigma}(E^\varepsilon(t-\varepsilon^2\sigma)\nabla_v(S^\varepsilon_{-\varepsilon^2\sigma})))\right]\mathbb{E}\left[f^\varepsilon(t-2\varepsilon^2\overline{\tau})\right]$$
$$+\mu^\varepsilon_t\,. \tag{9}$$

$$\mu^\varepsilon_t = \varepsilon \int_0^{\overline{\tau}}\int_0^{2\overline{\tau}-\sigma} ds d\sigma \fint dx\, \mathbb{E}[E^\varepsilon(t)\nabla_v(E^\varepsilon(t-\varepsilon^2\sigma)\cdot \nabla_v(S^\varepsilon_{\varepsilon^2(s+\sigma)}(E^\varepsilon(t-\varepsilon^2(\sigma+s))\nabla_v(f^\varepsilon(t-\varepsilon^2(\sigma+s))))))]. \tag{10}$$

Follows after a second order Duhamel formula from the hypothesis

$$|t-s| \geq \overline{\tau}\varepsilon^2 \implies \mathbb{E}[E^\varepsilon(t,x)\otimes E^\varepsilon(s,y)] = 0,$$

After multiplication by $\phi(t,v)$ observe, using space regularity, that with

$$|\langle\phi(t,v)\,,\,\mu_t^\varepsilon\rangle| \le \varepsilon C(\phi)(1+\tau^3)\|E\|_{L^3(0,\tau;W^{2,\infty}(\mathbb{T}^d))} \tag{11}$$

the reminder can be ignored. Then $\left[f^\varepsilon(t-2\varepsilon^2\overline{\tau})\right]$ converges in L^∞ weak-$\star$. $\forall \quad \psi(t,v)$ integration by part in (x,v) and use of the space time homogeneity hypothesis shows strong L^1_{loc} convergence for

$$\mathbb{E}\left[E^\varepsilon(t)\nabla_v(S^\varepsilon_{\varepsilon^2\sigma}(E^\varepsilon(t-\varepsilon^2\sigma)\nabla_v(S^\varepsilon_{-\varepsilon^2\sigma})))\right]^t\psi(v,t)$$

leading to

Proposition

For any $\phi(t,v)\in\mathcal{D}(\mathbb{R}^+_t\times\mathbb{R}^d_v)$ one has:

$$\begin{aligned}&\overline{\int_{\mathbb{R}^+_t\times\mathbb{R}^d_v}\phi(t,v)\nabla_v\cdot\mathbb{E}\left[\fint dx\,\frac{E^\varepsilon(t)f^\varepsilon(t)}{\varepsilon}\right]}\\&=-\int_{\mathbb{R}^+_t\times\mathbb{R}^d_v}\overline{f^\varepsilon}(t,v)\nabla_v\mathbb{D}(t,v)\nabla\phi(t,v)\,dtdv\end{aligned} \tag{12}$$

and to the :

Theorem

A family of x regular stochastic, ε, t dependent vector fields, The inverse Fourier transform of the function $\tilde{A}_k(|\sigma|)$ is non negative. The diffusion matrix

$$\mathbb{D}(v) = \sum_{k\in\mathbb{Z}^d\backslash 0} (k\otimes k)\tilde{A}_k((k\cdot v+\omega_k)) \tag{13}$$

is symmetric and non negative. The family of $\mathbb{E}[f^\varepsilon]$ of expectation of f^ε solution of the stochastic Liouville equation

$$\varepsilon^2\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + E^\varepsilon\nabla_x f^\varepsilon = 0\,, f^\varepsilon(0,x,v) = f_0(x,v)\,. \tag{14}$$

converges in the weakstar $L^\infty_{loc}(\mathbb{R}^+_t; L^\infty(\mathbb{R}^d_v)) \cap C^1\mathbb{R}^+_t; H^{-1}(\mathbb{R}^d_v))$ topology to a function $\overline{f^\varepsilon}(t,v)$ independent of x and solution of the diffusion equation:

$$\partial_t\overline{f^\varepsilon} - \nabla_v(\mathbb{D}(v)(\nabla_v\overline{f^\varepsilon}) = 0 \quad \overline{f^\varepsilon}(0,v) = \fint f_0(x,v)dx \tag{15}$$

- Deriving a diffusion equation from an Hamiltonian dynamic is the most classical problem to statistical goes back to Einstein and to the brownian motion.
- Only weak convergence and introduction of randomness does work.
- Analysis should be made on the solution rather than on the equation.
- As such the reiteration of the Duhamel formula leading to the Duhamel series is an essential tool, used by for instance by Lanford for Boltzmann in O. Lanford in 1975, L. Erdos– Yau in (2000) and many others.
- However a breakthrough was made in the contributions of Poupaud–Vasseur and Loeper–Vasseur in 2004 where it was shown that using the second term of the Duhamel formula is enough to obtain the fondamental decorelation property .
- And finally in some sense the introduction of the second order Duhamel to obtain a closure can be viewed is in many case equivalent to the notion of propagation of chaos.

Vlasov equations before any rescaling

$$\partial_t f + v\cdot\nabla_x f + E\nabla_v f = 0\,,$$
$$-\Delta\Phi = \int_{\mathbb{R}^d_v} f(x,v,t)dv - 1\,, E(x,t) = -\nabla_x\Phi$$
$$,.\ \forall t\in R^+_t \quad f(x,v,t)\geq 0 \quad \forall 1\leq p\leq\infty \quad \frac{d}{dt}\int_{\mathbb{R}^d_v\times\mathbb{T}^d} f(x,v,t)^p dxdv = 0\,,$$
$$\forall t\in R^+_t \frac{d}{dt}\ (\int_{\mathbb{R}^d_v}\fint \frac{|v|^2}{2} f(x,v,t)dxdv + \frac{1}{2}\fint |E(x,t)|^2 dx = 0\,,$$

$$\Rightarrow \text{ for any } d \quad \|E(x,t)\|_{L^\infty(\mathbb{R}^+;W^{1,1+2/d}(\mathbb{T}^d))}\leq c_0<\infty\,. \tag{16}$$

Prove the validity of a short time approximation in connection with the local in time dynamic, ie in the presence of unstable (in the Penrose sense) eigenvalues for the linearized problem. Hence for a special class of solutions in particular with analytic initial data.

$$\begin{aligned} &f^\varepsilon(t,x,v) = G(t,v) + \varepsilon h(t,x,v)\,, \\ &G(t,v) \geq 0 \quad \int_{\mathbb{R}^d_v} G(t,v)dv = 1 \quad \fint h(t,x,v)dx = 0\,, \end{aligned} \tag{17}$$

Then the Vlasov equation is equivalent to the system:

$$\begin{aligned} &\partial_t G(t,v) + \varepsilon^2 \nabla_v \cdot \Big(\fint E[h]hdx \Big) = 0\,, \quad E[h] = \nabla \Delta^{-1} \int_{\mathbb{R}^d_v} h(t,x,v)dv\,, \\ &\partial_t h + v \cdot \nabla_x h + E[h] \cdot \nabla_v G = -\varepsilon \nabla_v \cdot \Big(E[h]h - \fint E[h]hdx \Big)\,. \end{aligned} \tag{18}$$

With (18):

$$\partial_t G(t,v) = O(\varepsilon^2) \quad \text{and} \quad G(t,v) = G(0,v) + tO(\varepsilon^2) \tag{19}$$

The goal is the construction of an approximation that will improve the order of accuracy from $tO(\varepsilon^2)$ to $tO(\varepsilon^3)$. Compare two objects.

1. The solution $f^\varepsilon(t,x,v)$ of the Vlasov equation with initial data:

$$f^\varepsilon(0,x,v) = G(0,v) + \varepsilon h(0,x,v) = G_0(v) + \varepsilon \frac{E(0,k)\cdot\nabla_v G(0,v)}{\lambda(0) + ik\cdot v} e^{ik\cdot x}\,. \tag{20}$$

2. The ansatz $\tilde{f}^\varepsilon(t,x,v)$

$$\tilde{f}^\varepsilon(t,x,v) = G(t,v) + \varepsilon\tilde{h}(t,x,v) = G(t,v) + \varepsilon \frac{E(0,k)e^{\int_0^t ds\lambda(s)}\cdot\nabla_v G(t,v)}{\lambda(t) + ik\cdot v} e^{ik\cdot x}\,. \tag{21}$$

$(\lambda(0), k)$ is a simple root of the dispersion equation

$$1 - \frac{1}{|k|^2} \int_{\mathbb{R}^d_v} \frac{ik \cdot \nabla_v G(0,v)}{\lambda(0) + ik \cdot v} dv = 0\,, \tag{22}$$

while $\lambda(t)$ is the analytic continuation on a finite time interval as solutions of

$$1 - \frac{1}{|k|^2} \int_{\mathbb{R}^d_v} \frac{ik \cdot \nabla_v G(t,v)}{\lambda(t) + ik \cdot v} dv = 0\,, \tag{23}$$

Then $\quad \partial_t G(t) = O(\varepsilon^2) \Rightarrow \partial_t \lambda(t) = O(\varepsilon^2)$

$$\partial_t \tilde{h} + v \cdot \nabla_x \tilde{h} + E[\tilde{h}] \cdot \nabla_v G(t) = O(\varepsilon^2)\,.$$

$$\partial_t (h - \tilde{h}) + v \cdot \nabla_x (h - \tilde{h}) + E[h - \tilde{h}] \cdot \nabla_v G = -\varepsilon \nabla_v \cdot \Big(E[h]h - \fint E[h]h dx \Big) + \mathcal{O}(\varepsilon^{2}$$

$$(h - \tilde{h})(0,x,v) = 0 \Rightarrow h(t) - \tilde{h}(t) = \mathcal{O}(\varepsilon)$$

$$\partial_t G + \varepsilon^2 \nabla_v \cdot \Big(\fint E[\tilde{h}]\tilde{h} dx \Big) = \varepsilon^2 \nabla_v \cdot \fint \Big(E[\tilde{h}] - E[h]h \Big) dx = \mathcal{O}(\varepsilon^3)\,.$$

(λ, k) solutions of (22) implies that the same is true for $(\lambda^*, -k)$ and one can extend comparison between genuine solutions with initial data

$$f^\varepsilon(0,x,v) = G(v) + \varepsilon R(h(0,x,v))\,, \tag{24}$$

and the approximate solutions given by

$$\tilde{f}^\varepsilon(t,x,v) = G(t,v) + \varepsilon R\tilde{h}(t,x,v)\,.$$

Since the functions $\tilde{h}$ satisfy the relation

$$(\lambda + ik\cdot v)\tilde{h}(t,k,v) + E[\tilde{h}(t,k,v)]\cdot\nabla_v G(t,v) = 0\,, \tag{25}$$

one compute for such solutions explicitly $\fint E[\tilde{h}]\tilde{h}dx$ and obtain:

$$\partial_t G(t,v) - \varepsilon^2\nabla_v\cdot\left(\frac{E(0,k)\otimes(E(0,k))^\star e^{2R\int_0^t ds\lambda(s)}}{(k\cdot v - I\lambda)^2 + (R\lambda)^2}\nabla_v G(t,v)\right) = \mathcal{O}(\varepsilon^3)\,.$$

Remark

First observe that the starting point of the above construction is the consideration of the unstable modes (eigenvalues) of the operator

$$\mathcal{T} = -v \cdot \nabla_x + E(h) \nabla_v G(0, v)$$

In particular the formula (21) can be extended to all the (λ_k, k_λ) *solutions of the dispersion equation*

$$1 - \frac{1}{|k_m|^2} \int_{\mathbb{R}^d_v} \frac{ik_m \cdot \nabla_v G(0, v)}{\lambda_m + ik_m \cdot v} dv = 0\,,$$

leading to the formula:

$$\partial_t G(t, v) - \varepsilon^2 \nabla_v \cdot \sum_m \left(\frac{E(0, k_m) \otimes (E(0, k_m))^\star e^{2R \int_0^t ds \lambda(s)}}{(k_m \cdot v - I\lambda_m)^2 + (R\lambda_m)^2} \nabla_v G(t, v) \right) = \mathcal{O}(\varepsilon^3)$$

Remark

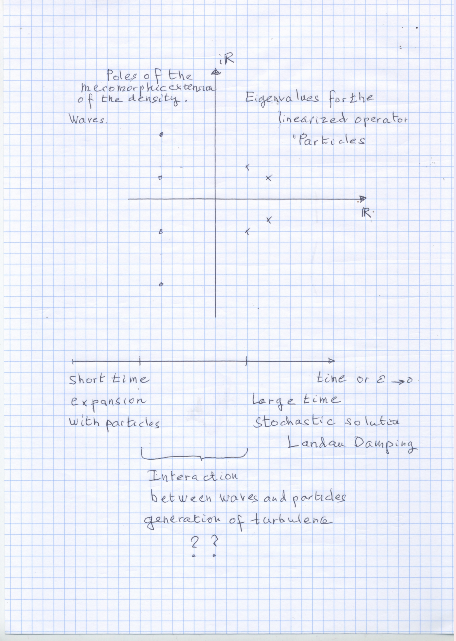
Then the introduction of the $\tilde{h}(t, x, v)$ *can be obtained by considering the linearized operator with* $s \mapsto G(s, v)$ *being frozen near* $t = 0$. *In term of spectral analysis the* λ_m *correspond to pure point spectra. Then for the resolvent obtained after meromorphic extension in the half plane* $R\lambda < 0$ *the pole would be resonancies . With the physic interpretation the point spectra are called particles and the resonancies waves. The Villani Mouhot and others proofs of the Landau damping correspond to the case where there is no point spectrum!*
More refined analysis would consider the interaction between resonnacies (waves) and particles point spectrum.

Remark

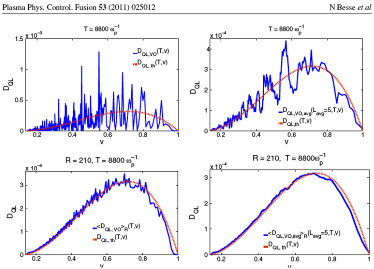
For short time in the quasi linear approximation the term $e^{2R\int_0^t ds\lambda(s)}$ *enhances the dissipation and as long as it is active forces the profile* $G(t,v)$ *to become smooth.*

$$\partial_t G(t,v) - \varepsilon^2 \nabla_v \cdot \left(\frac{E(0,k)\otimes (E(0,k))^\star e^{2R\int_0^t ds\lambda(s)}}{(k\cdot v - I\lambda)^2 + (R\lambda)^2} \nabla_v G(t,v) \right) = \mathcal{O}(\varepsilon^3)\,.$$

When the profile is too smooth the effects of the point spectrum becomes less dominant with respect to the effect of the interaction wave particles. In such region of time the interaction this wave particles interaction makes the system stochastic and then the description given by the stochastic approach becomes valid.
Details can be found for instance in the paper : Validity of quasilinear theory: refutations and new numerical confirmation Nicolas Besse , Yves Elskens , D F Escande and Pierre Bertrand1 Plasma Phys. Control. Fusion 53 (2011)


iR
Poles of the
meromorphic extension
of the density.
Wares.
Eigenvalues for the
linearized operator
'Particles
R.
Short time
expansion
with particles
time or ε → 0
Large time
Stochastic solution
Landau Damping
Interaction
between wares and particles
generation of turbulence
? ?

Bump of the "bump in tail profile" : Top Single realizations. Bottom ensemble averaged over 210 simulations.



FINAL REMARKS

- 1 The goal of the QL linear approximation is to produce a global in x v dependent equation for purpose of the the solution.
- 2 This has to be compared with the introduction of x independent equations for instance homogenous Boltzmann or Fokker Planck equation which may be view as an intermediate step in a numerical simulation.
- 3 But they can also fulfill the first purpose for instance with also with Fokker Planck. and a good exemple is the linearized Balescu-Lenhard for fluctuations near and absolute Maxwellian:

$$\partial_t f = \nabla_v (D(v)(\nabla_v f + vf)) \tag{26}$$

derived directly from classical N tagged particles dynamic cf. Duerinckx-Saint Raymond .

FINAL REMAKS-Continuing

What about weak turbulence:
Should appear for larger time than the short time approximation and before the Landau Damping regime.. with many real eigenvalues converging to the imaginary axis.

In Duerinck-Saint Raymond, the introduction of a god given probability as Vasseur ... is replaced by a detailed analysis of cumulants in the Duhamel serie and quoting: The following main result provides a fully rigorous derivation of (26) starting from particle dynamics, although only on an intermediate timescale $t \simeq N^r$ with $r < 1$ small enough. The reason for this limitation is that we do not manage to rule out possible resonant effects .

THANK YOU FOR YOUR ATTENTION!