Collective Dynamics of Quantized Vortex in Superfluidity and Superconductivity

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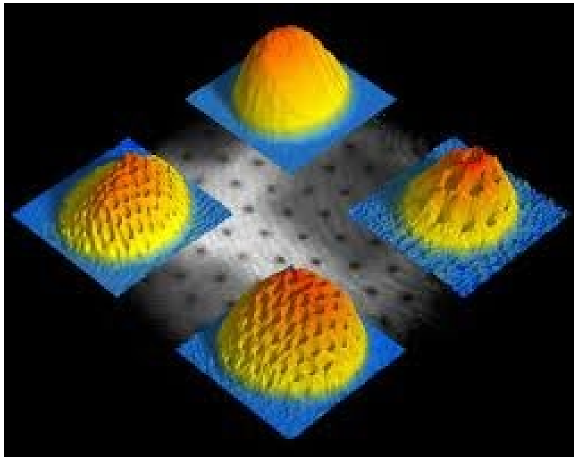

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Outline

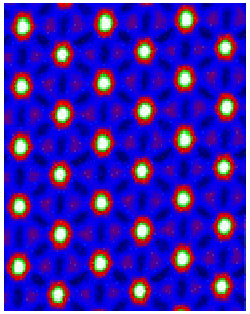
- Motivation
- Mathematical models
- Central vortex states and stability
- Reduced dynamic laws (RDL)
- Collective dynamics based on RDL for GLE
- Extension to NLSE, NLWE & bounded domains
- Conclusions

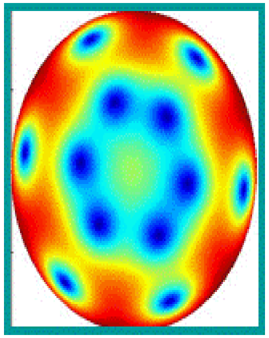
Motivation

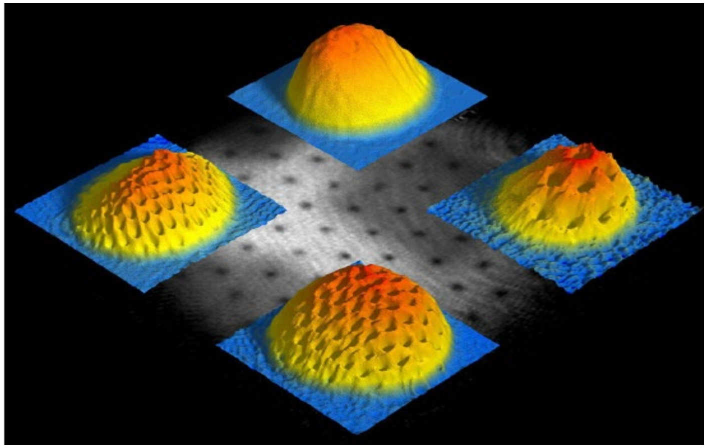
- Quantized Vortex: Particle-like (topological) defect
  - Zero of the complex scalar field $\psi := \psi(\vec{x},t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$
  - localized phase singularities with integer topological charge:

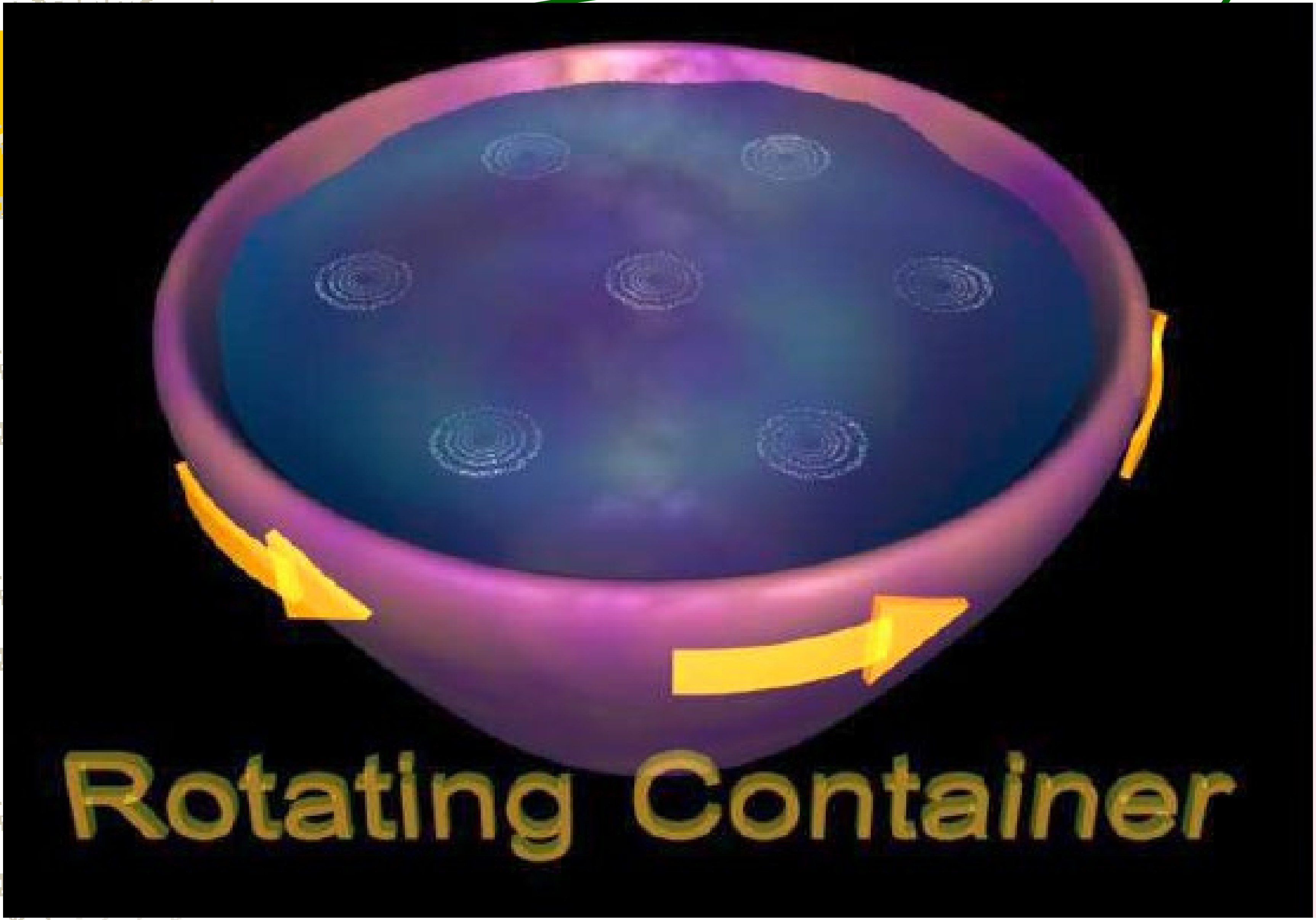
$$\psi(\vec{x}_0) = 0, \quad \psi(\vec{x}) = \sqrt{\rho}\, e^{i\phi}, \quad \int_\Gamma d(\arg\psi) = \int_\Gamma d\phi = 2\pi n \neq 0$$

  - Key of superfluidity: ability to support dissipationless flow

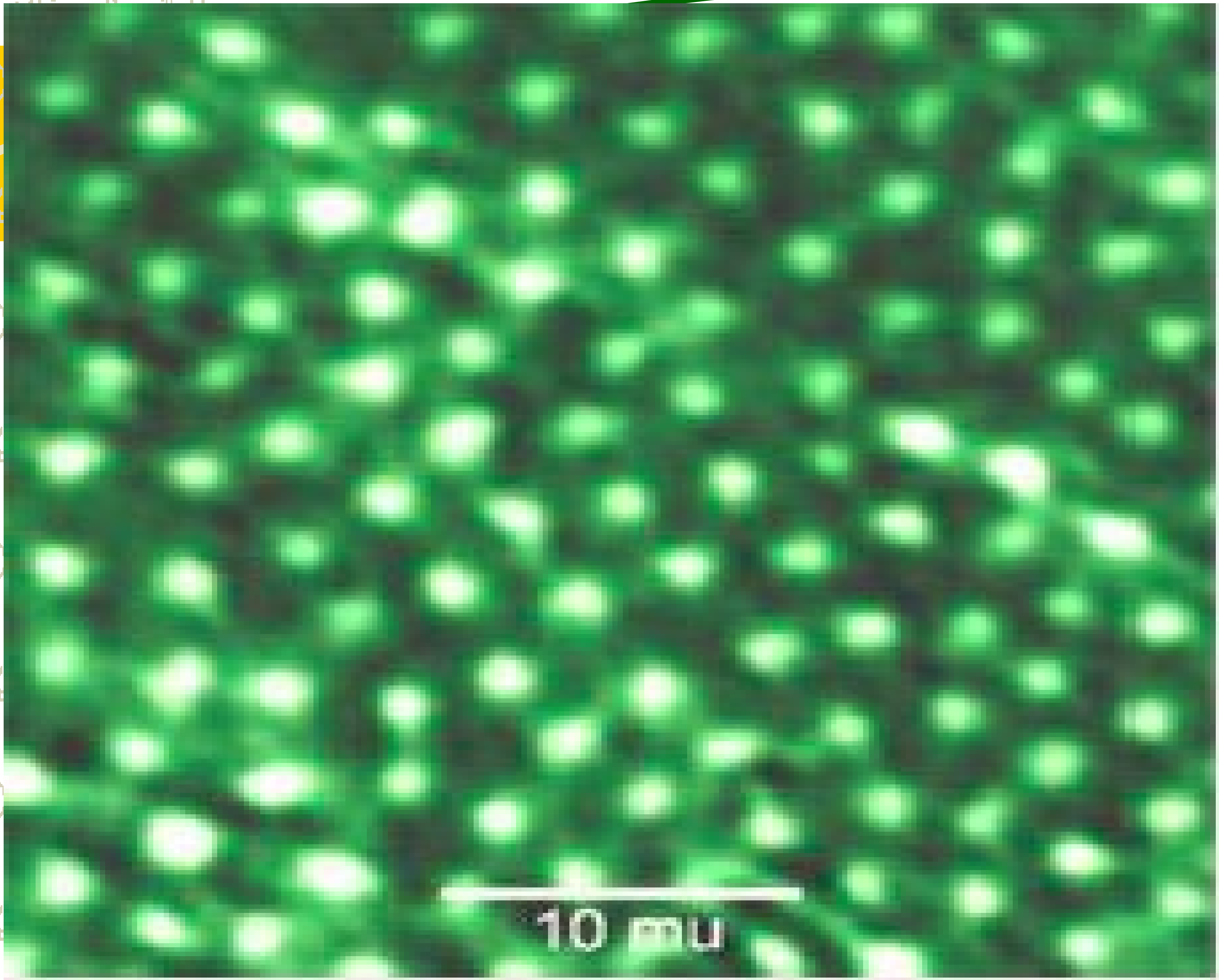




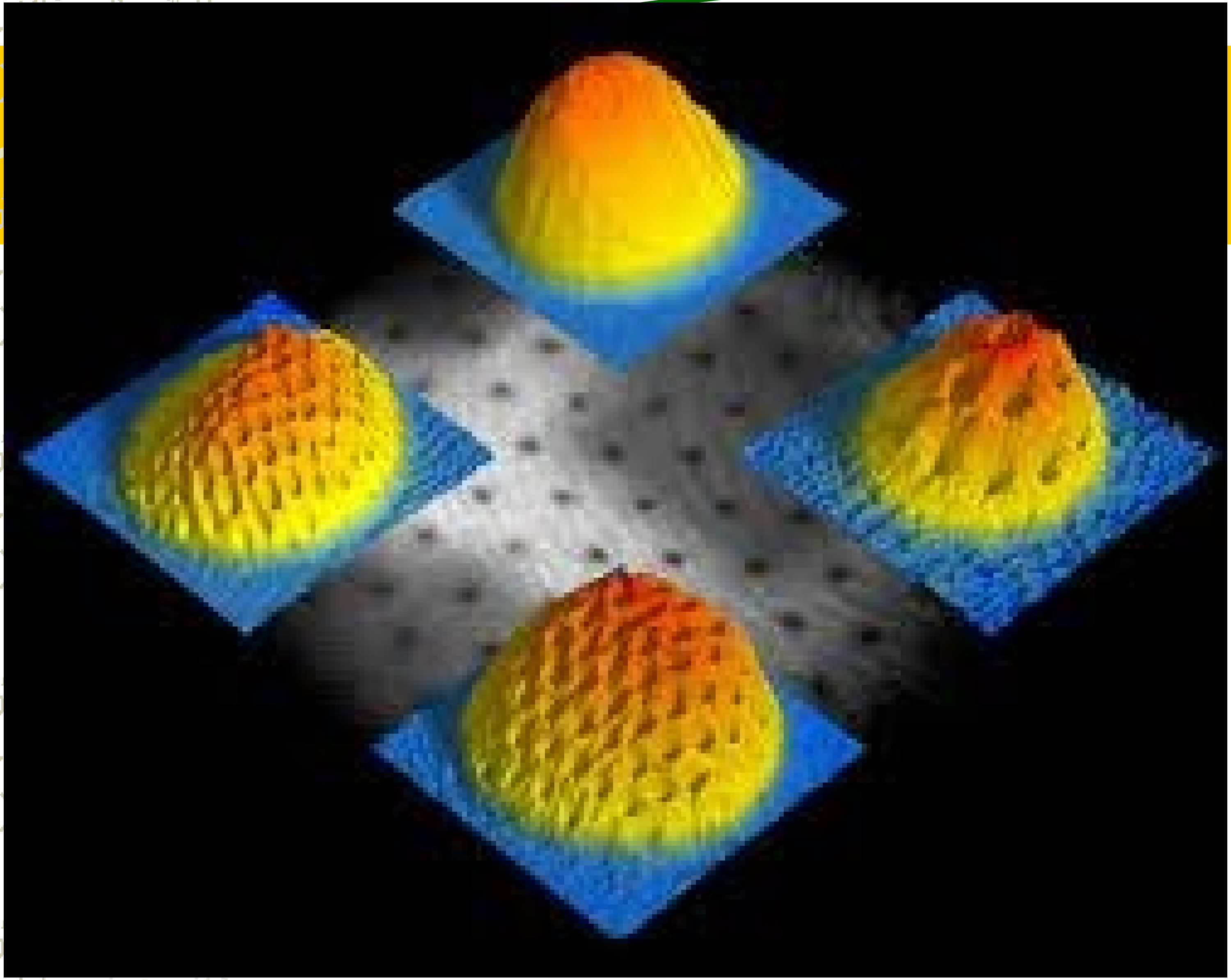




Quantized Vortex in liquid Helium 3

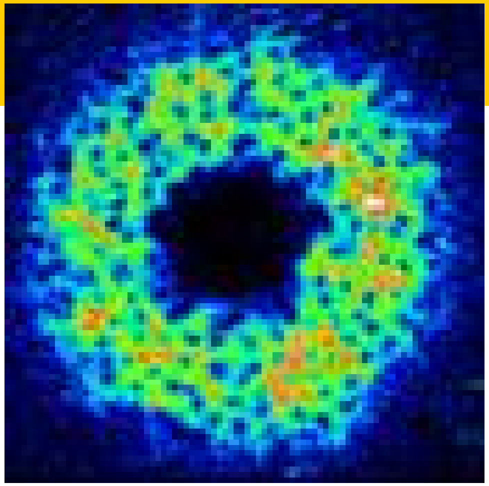


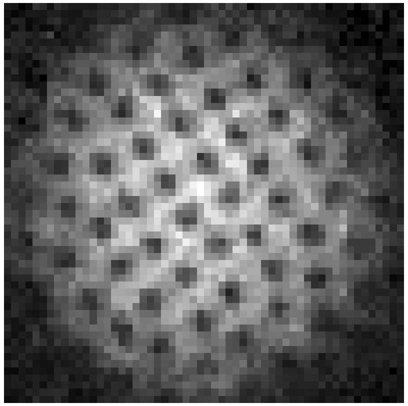
Quantized Vortex in a type-II superconductor



Quantized Vortex in Bose-Einstein condensation (BEC)

Motivation





- Quantized vortex existing in
  - Superconductors:
    - Ginzburg-Landau equations (GLE)
  - Liquid helium:
    - Two-fluid model –coupled Euler equations
    - Gross-Pitaevskii equation (GPE)
  - Bose-Einstein condensation (BEC):
    - Nonlinear Schroedinger equation (NLSE) or GPE
  - Nonlinear optics & propagation of laser beams
    - Nonlinear Schroedinger equation (NLSE)
    - Nonlinear wave/Klein-Gordon equation (NLWE/NKGE)

Mathematical models

- Ginzburg-Landau equation (GLE):Superconductivity, nonlinear heat flow, etc.

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\Omega\subseteq\mathbb{R}^2 \quad \overset{\text{complex GLE}}{\Rightarrow} \quad (\alpha+i\beta)\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$

- Nonlinear Schrodinger equation (NLSE): nonlinear optics, BEC, superfluidity

$$-i\,\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\Omega\subseteq\mathbb{R}^2$$

- Nonlinear wave equation (NLWE): wave motion; cosmology

$$\psi_{tt} = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\Omega\subseteq\mathbb{R}^2$$

- Nolinear Klein-Gordon equation: cosmology, relativistic QM, ...

$$\psi_{tt}+\psi = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi \overset{\psi=\phi e^{it}}{\Rightarrow} 2i\phi_t + \phi_{tt} = \Delta\phi + \frac{1}{\varepsilon^2}\left(1-|\phi|^2\right)\phi, \quad \vec{x}\in\Omega\subseteq\mathbb{R}^2$$

Here:

ψ : complex-valued wave function or order parameter,

$\varepsilon > 0$: dimensionless constant

Mathematical models

- `Free Energy' or Lyapunov functional:

$$E(\psi) = \int_{\Omega} \left[|\nabla \psi|^2 + \frac{1}{2\varepsilon^2}\left(1-|\psi|^2\right)^2 \right] d\vec{x},$$

$$-i\,\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$

$$\Downarrow$$

$$-i\frac{\partial\psi}{\partial t} = -\frac{\delta E(\psi)}{\delta\psi^*}$$

- Dispersive system (NLSE/GPE):
  - Energy conservation: $E(\psi) \equiv E(\psi_0), \quad t \ge 0$
  - Density conservation: $\int_{\mathbb{R}^2} \left(|\psi(\vec{x},t)|^2 - |\psi_0(\vec{x})|^2 \right) d\vec{x} \equiv 0, \quad t \ge 0$
  - Admits particle like solutions: solitons, kinks & vortices
- Dissipative system (GLE):

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$

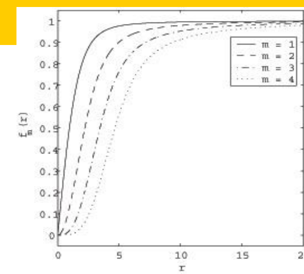
$$\Downarrow$$

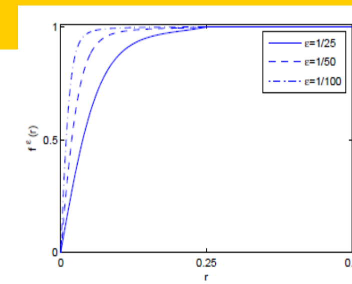
$$\frac{\partial\psi}{\partial t} = -\frac{\delta E(\psi)}{\delta\psi^*}$$

  - Energy diminishing, etc.

Single (or central) vortex state — `bright-tail’

$$\Omega=\mathbb{R}^2 \Rightarrow \psi_t=\Delta\psi+\frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\mathbb{R}^2$$

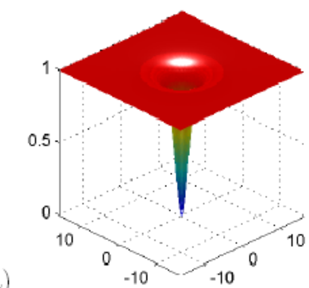




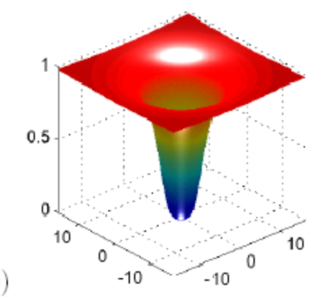
- **Ansatz** $\psi(\vec{x})=\phi_n(\vec{x})=f_n(r)\,e^{in\theta}$

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)-\frac{n^2}{r^2}+\frac{1}{\varepsilon^2}(1-f_n^2(r))\right]f_n(r)=0, \quad 0<r$$

$$f_n(0)=0, \qquad f_n(+\infty)=1.$$



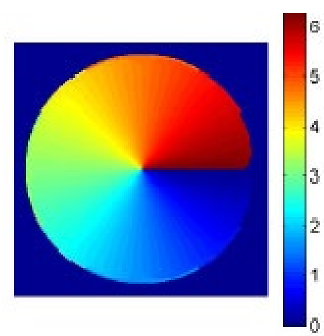
a)

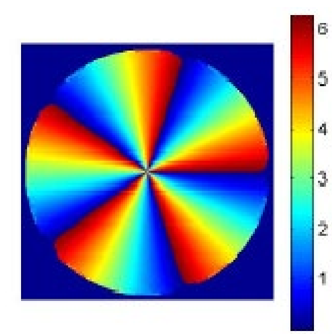


b)

- **Asymptotic results** –core size $O(\varepsilon)$

$$f_n(r)\approx\begin{cases} a\left(\dfrac{r}{\varepsilon}\right)^{|n|}+O\left((r/\varepsilon)^{|n|+2}\right), & 0\le r\le O(\varepsilon) \\ 1-n^2\varepsilon^2/2r^2+O(\varepsilon^4/r^4), & r\gg\varepsilon. \end{cases}$$





Stability of vortices

- Physical observables – density & velocity

$$\psi = \sqrt{\rho}\, e^{iS} \Rightarrow \rho = |\psi|^2, \quad \vec{v} = \nabla S = \frac{1}{\rho}\mathrm{Im}(\psi^* \nabla \psi), \quad J = \rho \vec{v} = \mathrm{Im}(\psi^* \nabla \psi)$$

- Results (Bao, Du & Zhang, Eur. J. Appl. Math., 07')
  - For GLE or NLWE with initial data perturbed
    - n=1: velocity density
    - N=3: velocity density

$$\psi_t = \Delta \psi + \frac{1}{\varepsilon^2}\left(1 - |\psi|^2\right)\psi$$

$$\psi(\vec{x},0) = \psi_0(\vec{x}) = \phi_n^h(\vec{x}) + \text{noise}$$

  - For NLSE with external potential perturbed
    - n=1: velocity density
    - N=3: velocity density

$$-i\,\psi_t = \Delta \psi + \frac{1}{\varepsilon^2}\left(1 + W(\vec{x},t) - |\psi|^2\right)\psi$$

$$\psi(\vec{x},0) = \psi_0(\vec{x}) = \phi_n^h(\vec{x}) + \text{noise}$$

- Results

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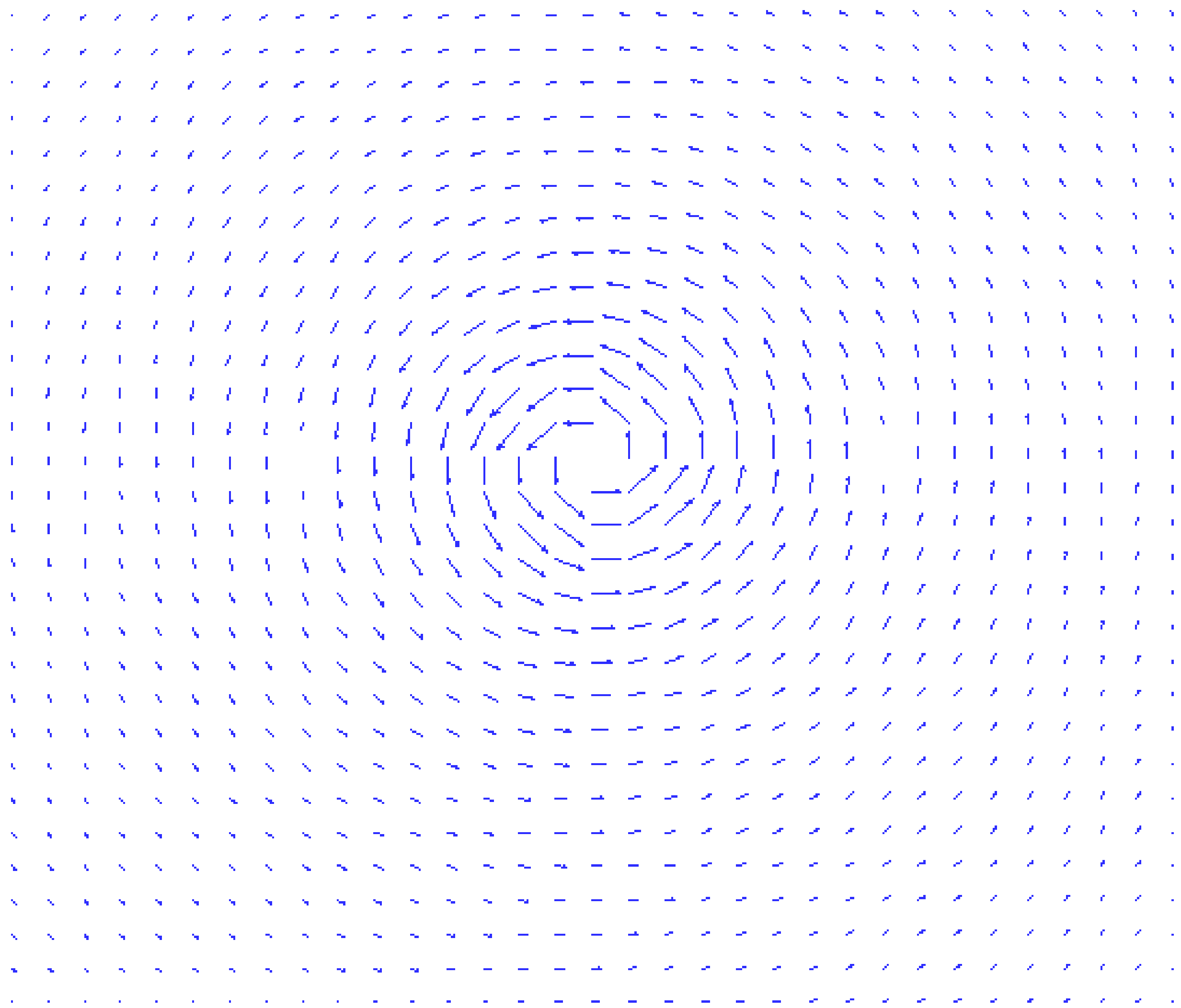

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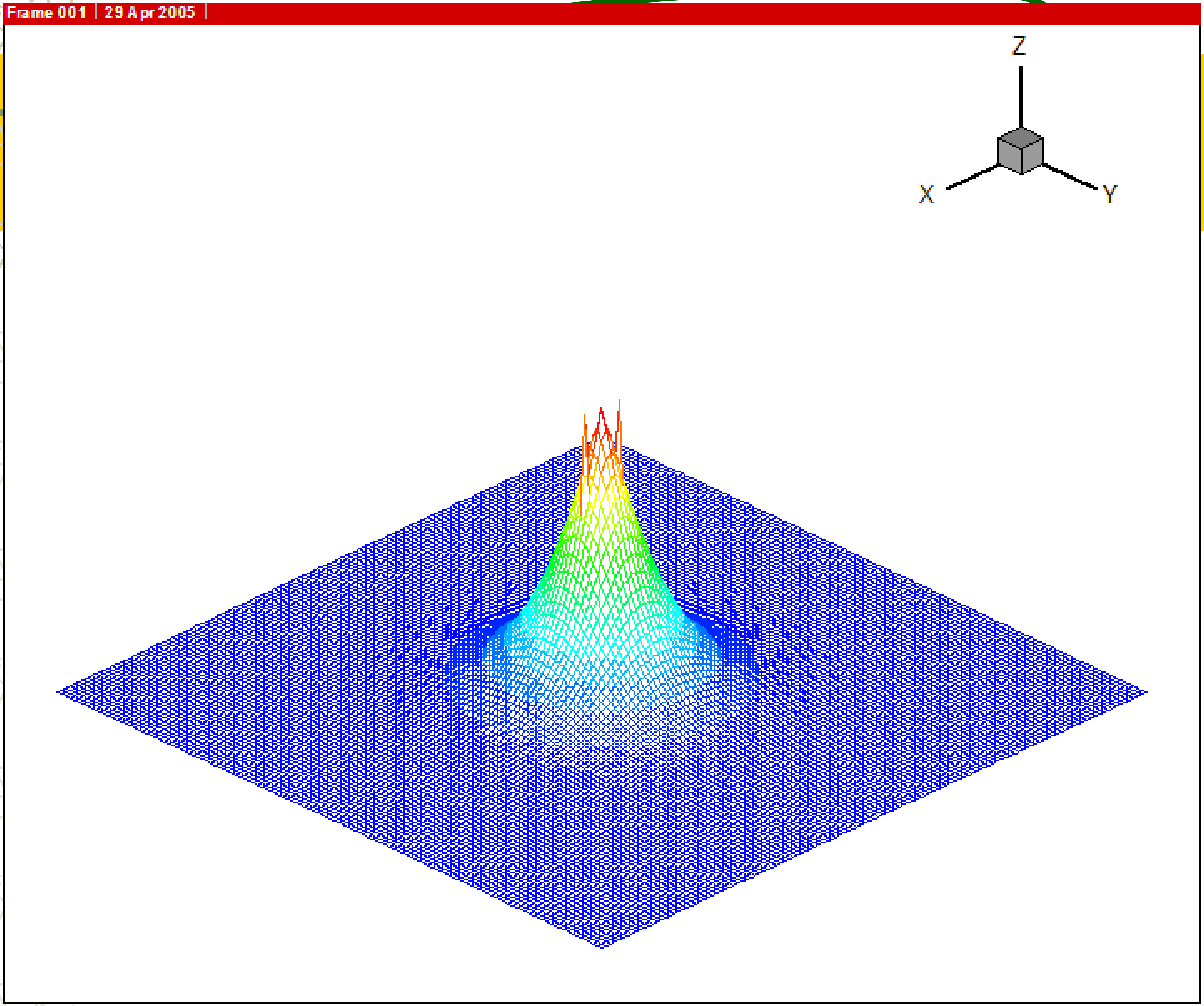

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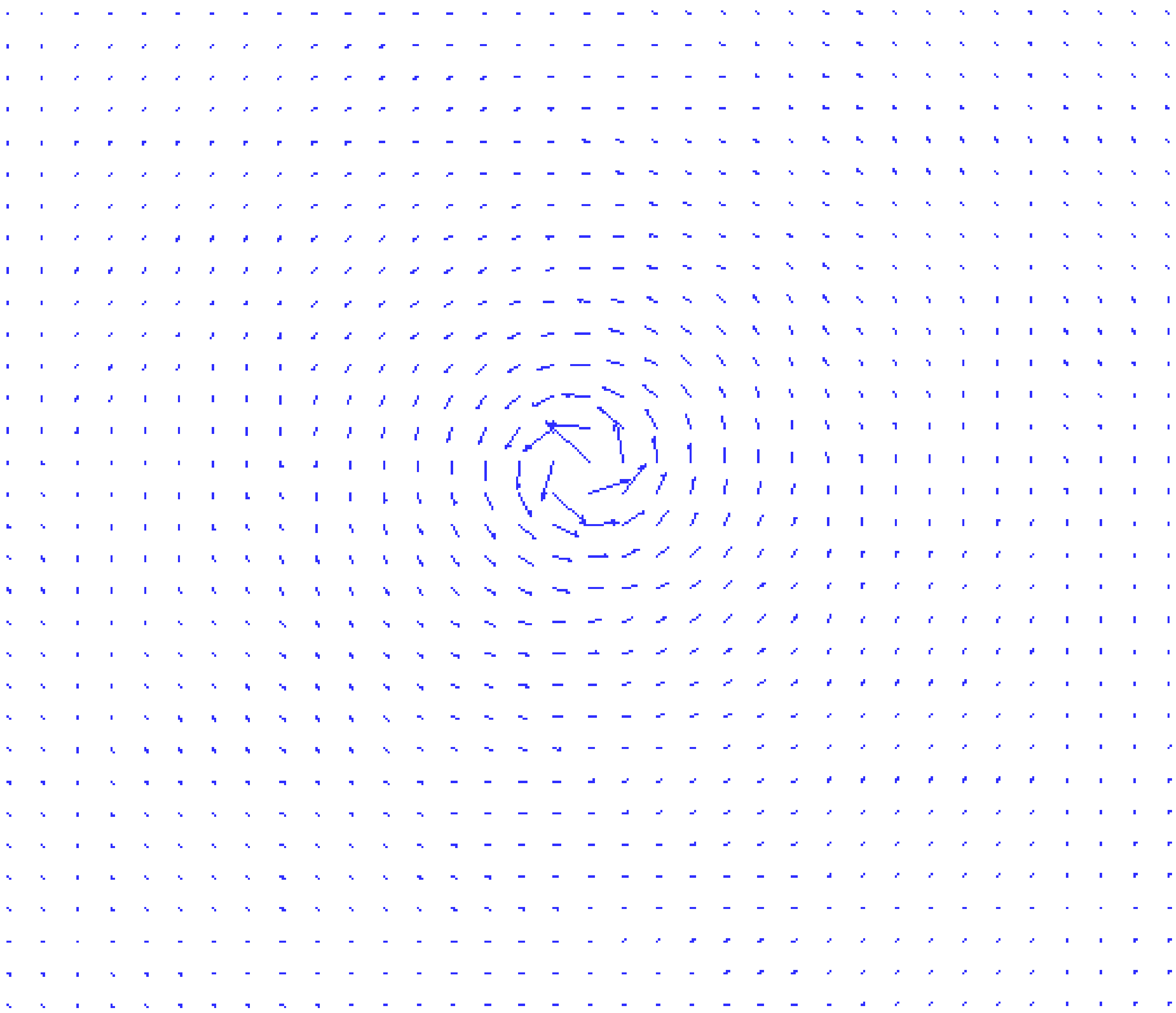

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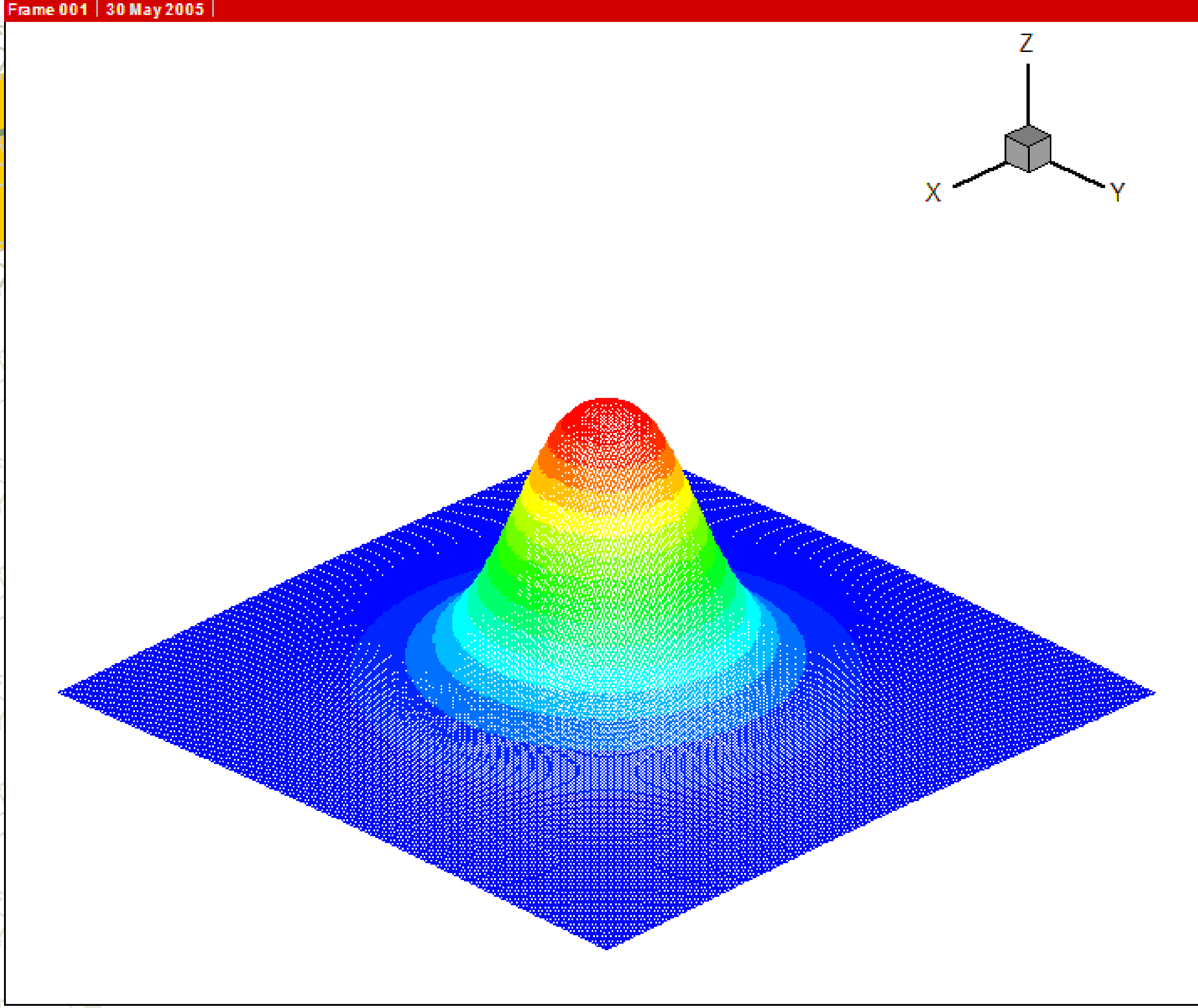

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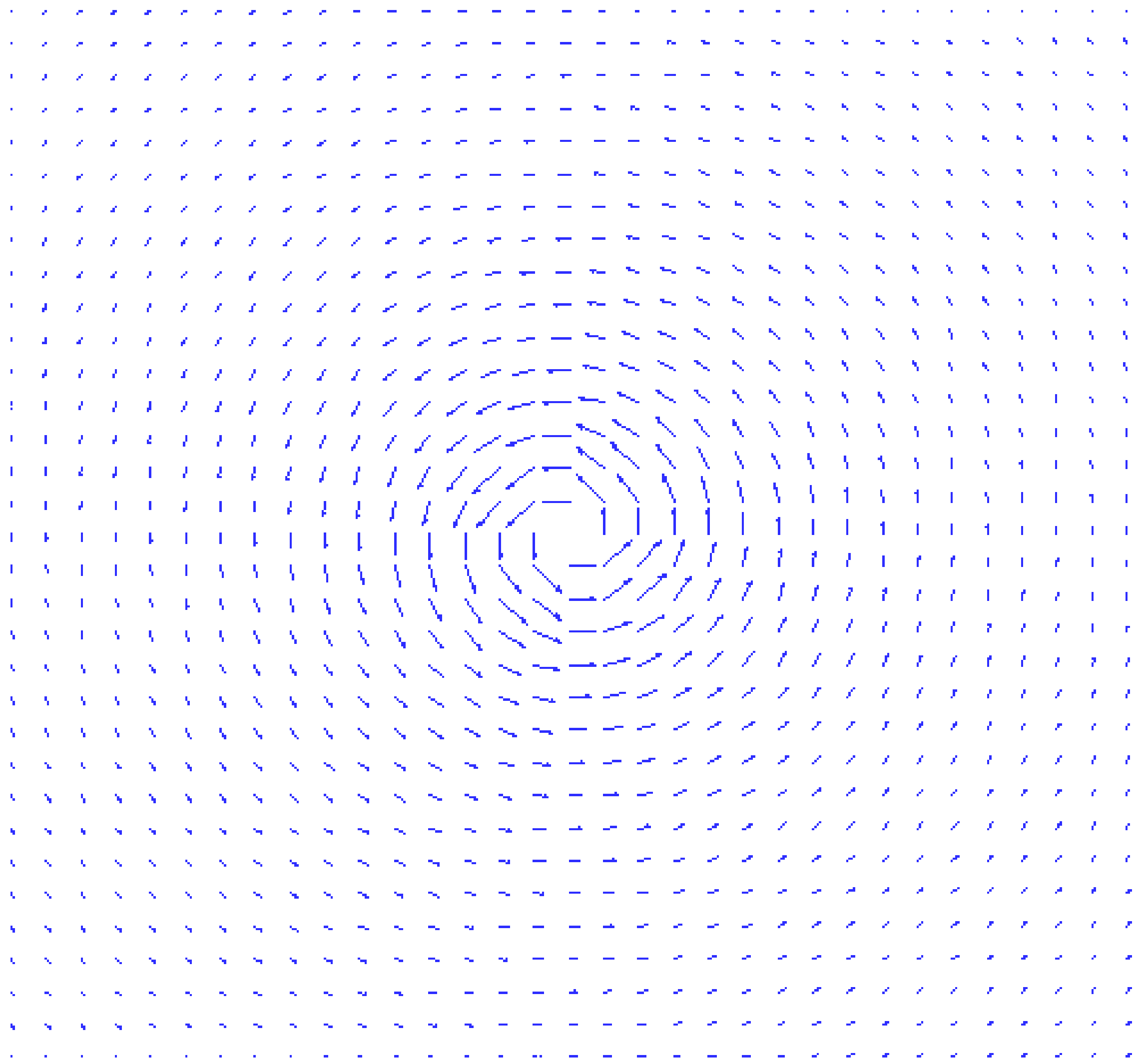

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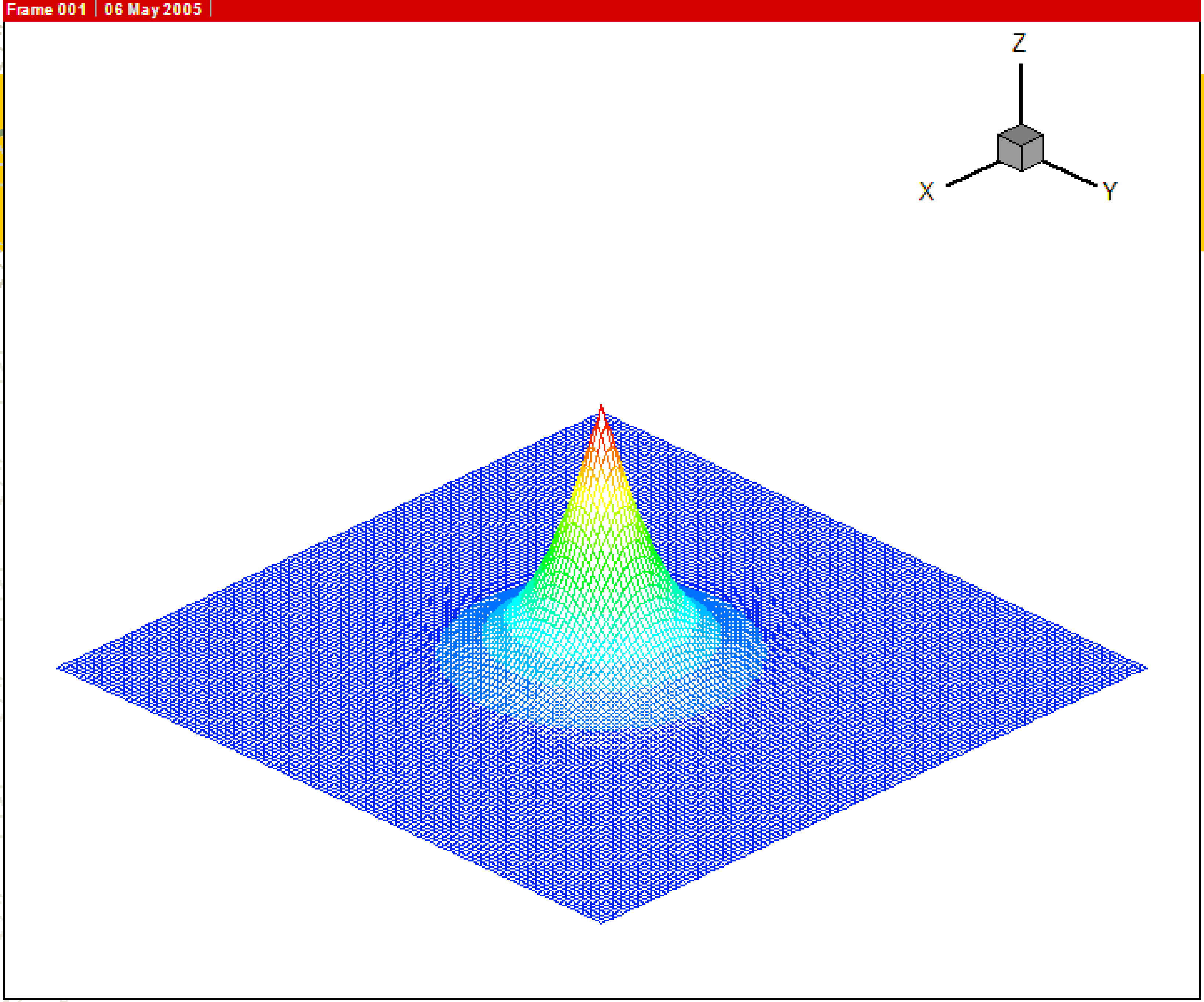

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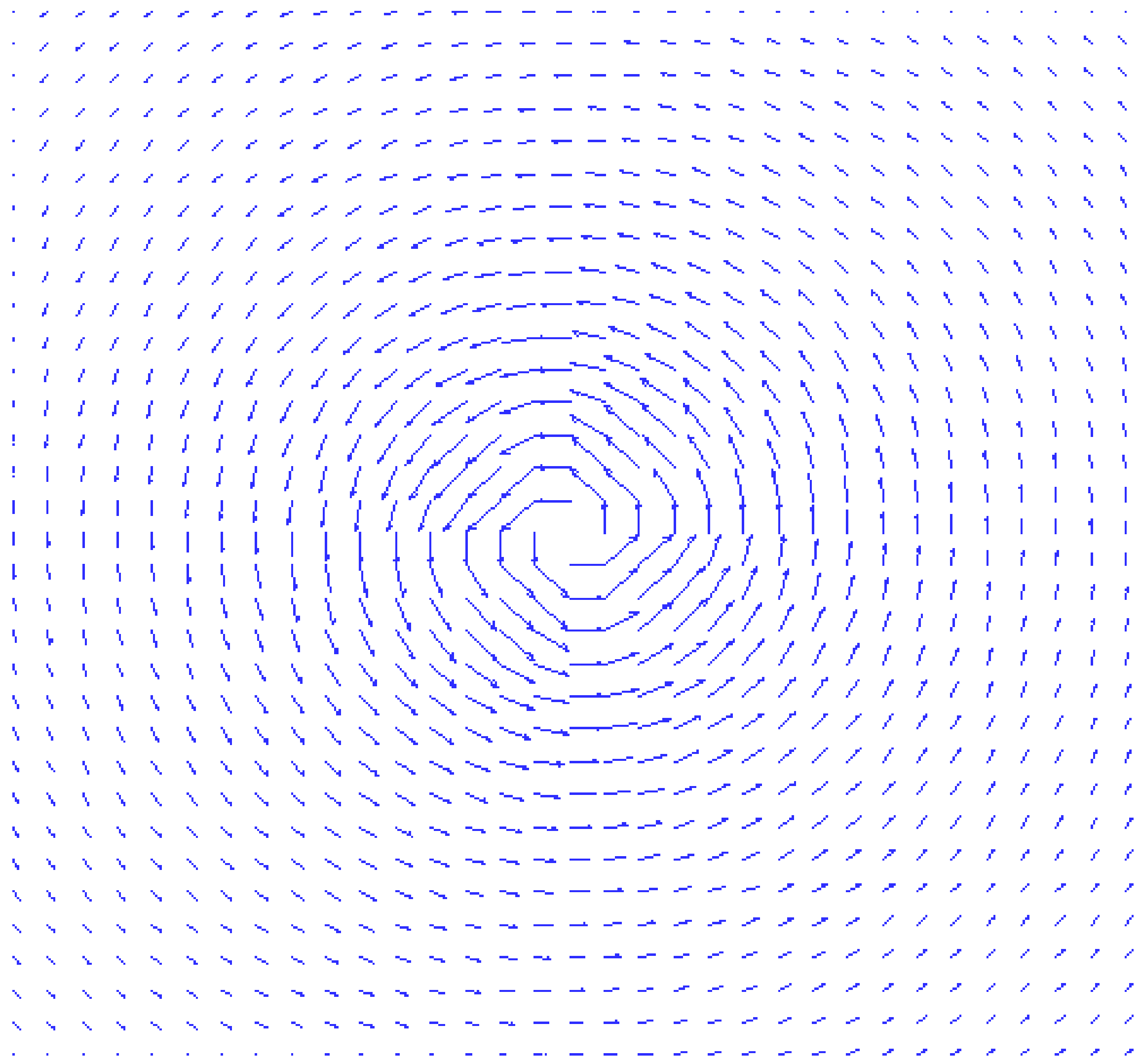

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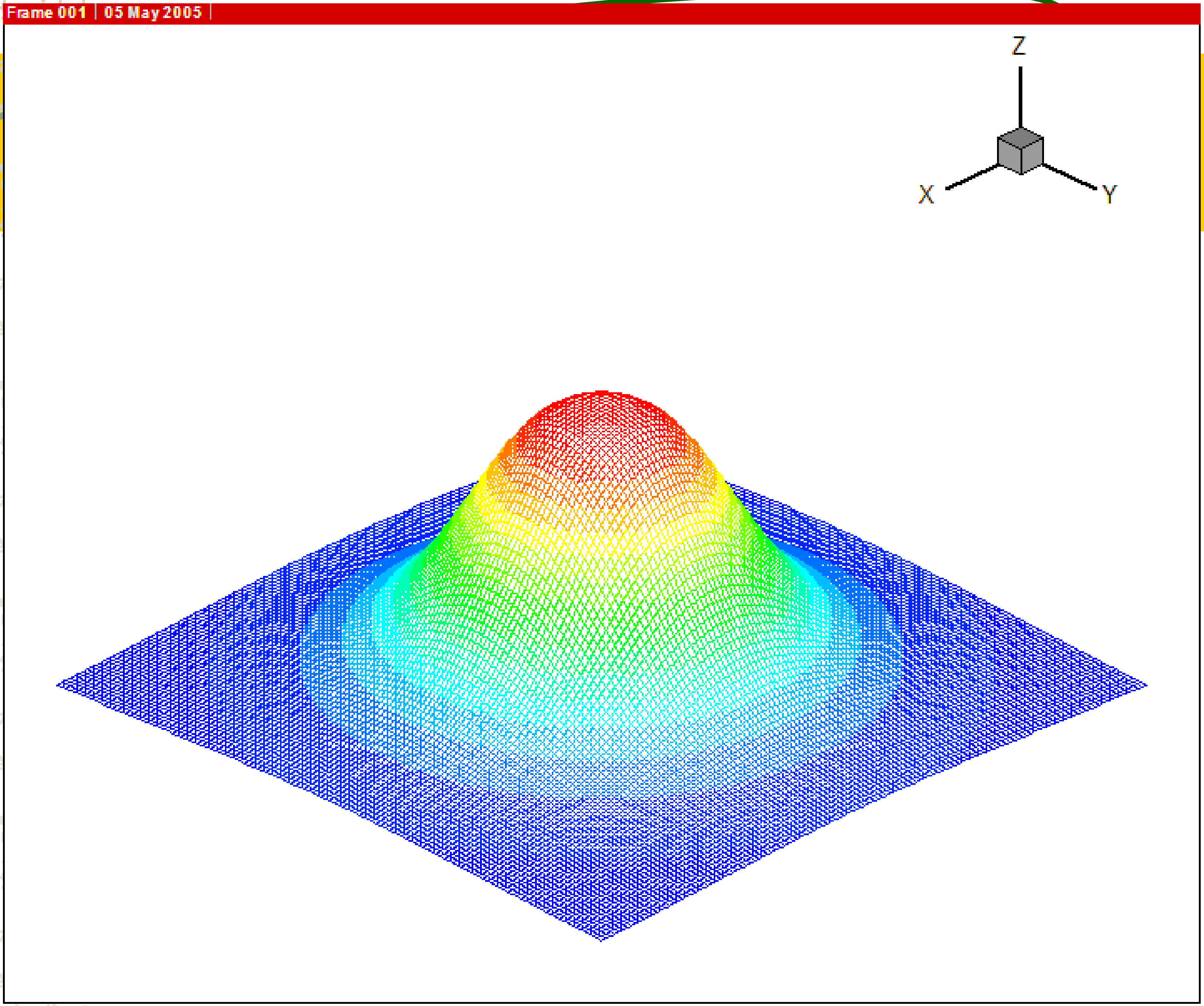

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Summary of Stability

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$

- For GLE or NLWE/NKGE:
  - under perturbation either in initial data or in external potential
    - $|n|=1$: dynamically stable,
    - $|n|>1$: dynamically unstable. Splitting pattern depends on perturbation, when $t \gg 1$, it becomes n well-separated vortices with index 1 or –1.
- For NLSE or GPE:

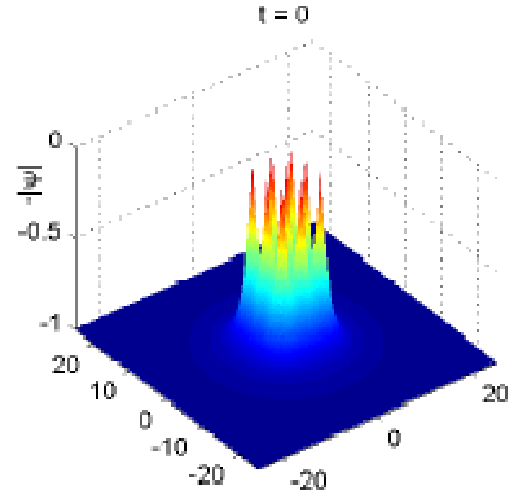
$$-i\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$

  - under perturbation in initial data
    - dynamically stable: Angular momentum expectation is conserved!!
  - under perturbation in external potential
    - $|n|=1$: dynamically stable,
    - $|n|>1$: dynamically unstable. Vortex centers can NOT move out of core size.

Vortex interaction

- Initial setup:

$$\psi_0(\vec{x}) = e^{ih(\vec{x})}\prod_{j=1}^{N}\phi_{m_j}(\vec{x}-\vec{x}_j^0), \quad \text{with } m_j = \pm 1$$



- Reduced dynamical laws
  - Take ansatz

$$\psi(\vec{x},t) = \prod_{j=1}^{N}\phi_{m_j}(\vec{x}-\vec{x}_j(t)) + \text{ high order terms}$$

  - Plug into GLE or GPE or NLWE
  - Use matched asymptotic techniques when
  - Obtain ODE system for the dynamics of vortex centers $\varepsilon \rightarrow 0$
  - Prove rigorously

Reduced dynamic laws (RDL)

$$X := (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) \Rightarrow W(X) = -\sum_{1\le j\ne k\le N} m_j m_k \ln|\vec{x}_j - \vec{x}_k| = -\ln \prod_{1\le j\ne k\le N} |\vec{x}_j - \vec{x}_k|^{m_j m_k}$$

- For GLE– Neu, 90', W. E, 91', F. Lin, 95'--, Bethual, ….

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi \Rightarrow \quad \kappa \vec{v}_j(t) := \kappa \frac{d\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l\ne j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{|\vec{x}_j(t)-\vec{x}_l(t)|^2} := -\nabla_{\vec{x}_j} W(X), \qquad t \ge 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \qquad 1 \le j \le N.$$

- For NLSE/GPE – Neu, 90', F. Lin&Xin, 95'--, …

$$\tilde{W}(X) = -\sum_{1\le j\ne k\le N} m_k \ln|\vec{x}_j - \vec{x}_k| = -\ln \prod_{1\le j\ne k\le N} |\vec{x}_j - \vec{x}_k|^{m_k}$$

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi \Rightarrow \quad J\vec{v}_j(t) := J\frac{d\vec{x}_j(t)}{dt} = 2\sum_{l=1, l\ne j}^{N} m_l \frac{(\vec{x}_j(t)-\vec{x}_l(t))}{|\vec{x}_j(t)-\vec{x}_l(t)|^2} := -\nabla_{\vec{x}_j}\tilde{W}(X) \qquad t \ge 0,$$

$$\vec{x}_j(0) = \vec{x}_j^0 \qquad 1 \le j \le N; \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- For NLWE – Neu, 90', …

$$\psi_{tt} = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi \Rightarrow \quad \kappa\frac{d^2\vec{x}_j(t)}{dt^2} = 2m_j \sum_{l=1, l\ne j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{|\vec{x}_j(t)-\vec{x}_l(t)|^2} := -\nabla_{\vec{x}_j} W(X), \qquad t \ge 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad \vec{x}_j'(0) = \vec{v}_j^0, \quad 1 \le j \le N.$$

Existing mathematical results

- For GLE (or NLHE): $$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$
  - Pairs of vortices with like (opposite) index undergo a replusive (attractive) interaction: *Neu 90*, *Pismen & Rubinstein 91, W. E, 94,* Bethual, etc.
  - *Energy concentrated at vortices in 2D & filaments in 3D: Lin 95--*
  - *Vortices are attracted by impurities: Chapman & Richardson 97, Jian 01*
  - *Collective dynamics of RDL: Xu, Bao & Shi – DCDS-B,18'* $$\frac{d\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{|\vec{x}_j(t)-\vec{x}_l(t)|^2}$$
- For NLSE: $$-i\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi$$
  - Vortices behave like point vortices in ideal fluid: *Neu 90*
  - Obeys classical Kirchhoff law for fluid point vortices: *Lin & Xin 98*
  - Equations for vortex dynamics & radiation: *Ovchinnikov& Sigal 98--; Jerrard, Bethual, Fibich&Gavish 08', etc.*
  - *Global existence???*

$$J\frac{d\vec{x}_j(t)}{dt} = 2\sum_{l=1, l\neq j}^{N} m_l \frac{(\vec{x}_j(t)-\vec{x}_l(t))}{|\vec{x}_j(t)-\vec{x}_l(t)|^2} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Collective dynamics of RDL for GLE

$$\frac{d\,\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j} W(X),$$

$$\vec{x}_j(0) = \vec{x}_j^0, \qquad 1 \le j \le N.$$

- Mass center $\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t)$

Lemma The mass center of the N vortices for GLE is conserved

$$\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t) \equiv \overline{x}(0) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j^0$$

- Energy decreasing

$$W(X(t_2)) \le W(X(t_1)) \le W(X(0)), \qquad t_2 \ge t_1 \ge 0$$

- Some dynamical properties

$$\vec{x}_j^0 \to \vec{x}_j^0 + \vec{x}^0 \ (1 \le j \le N) \Rightarrow \vec{x}_j(t) \to \vec{x}_j(t) + \vec{x}^0$$

$$\vec{x}_j^0 \to \alpha\vec{x}_j^0 \ (1 \le j \le N) \Rightarrow \vec{x}_j(t) \to \alpha\vec{x}_j(t/\alpha^2)$$

$$Q(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{x}_j^0 \to Q(\theta)\vec{x}_j^0 \ (1 \le j \le N) \Rightarrow \vec{x}_j(t) \to Q(\theta)\vec{x}_j(t)$$

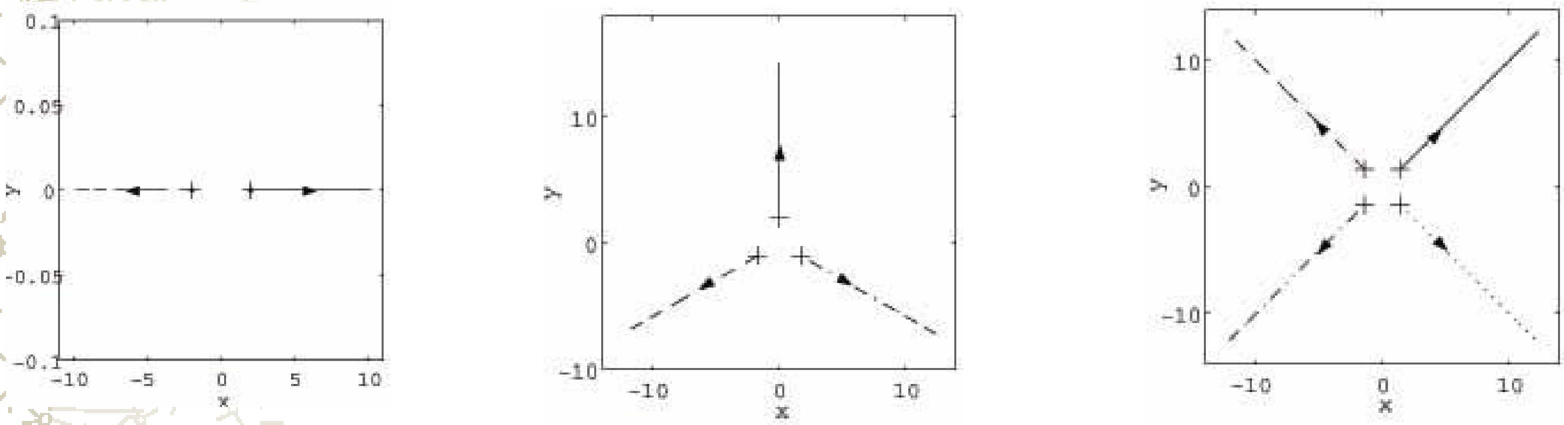
Analytical Solutions of GLE– case I

- $N(\geq 2)$ like vortices on a circle (Bao, Du & Zhang, SIAP, 07')

$$\vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N}\right) \quad , \quad \sin\left(\frac{2j\pi}{N}\right)\right), \qquad m_j = m_0 = \pm 1, \qquad 1 \leq j \leq N$$

- Analytical solutions for GLE

$$\vec{x}_j(t) = \sqrt{a^2 + 2(N-1)t}\left(\cos\left(\frac{2j\pi}{N}\right) \quad , \quad \sin\left(\frac{2j\pi}{N}\right)\right)$$



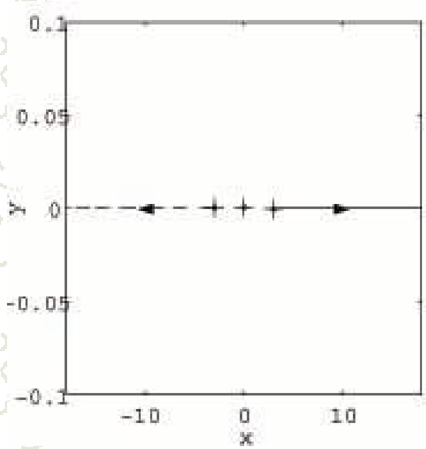
Analytical Solutions of GLE--Case II

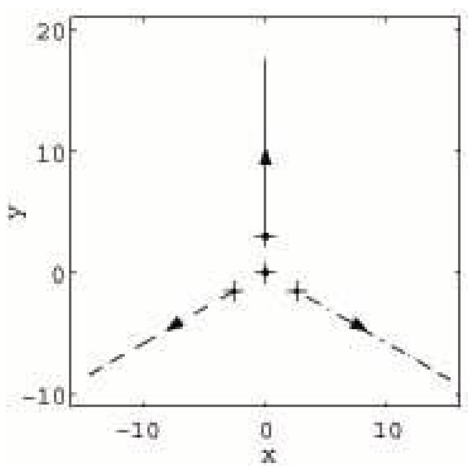
- $N\,(\geq 3)$ like vortices on a circle and its center (Bao,Du&Zhang, SIAP, 07')

$$\vec{x}_N^0 = \vec{0}; \qquad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right)\right), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + 2Nt}\left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$





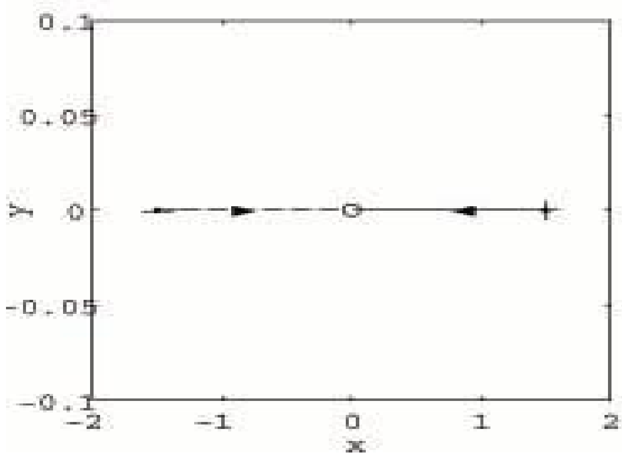
Analytical Solutions of GLE– Case III

- Two opposite vortices (Bao, Du & Zhang, SIAP, 07')

$$\vec{x}_1^0 = -\vec{x}_2^0 = a\left(\cos(\theta_0) \quad , \quad \sin(\theta_0)\right), \qquad m_1 = -m_2 = 1$$

- Analytical solutions for GLE

$$\vec{x}_1(t) = -\vec{x}_2(t) = \sqrt{a^2 - 2t}\ \left(\cos(\theta_0) \quad , \quad \sin(\theta_0)\right)$$



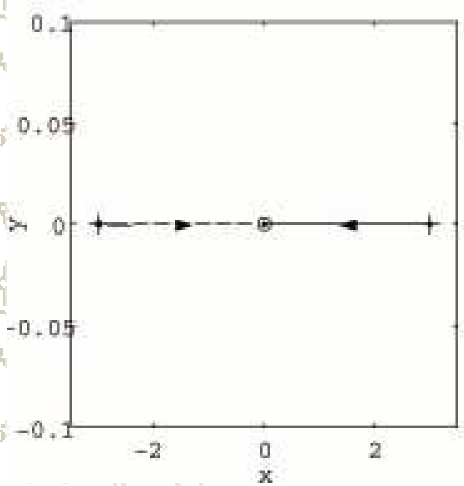
Analytical Solutions of GLE--Case IV

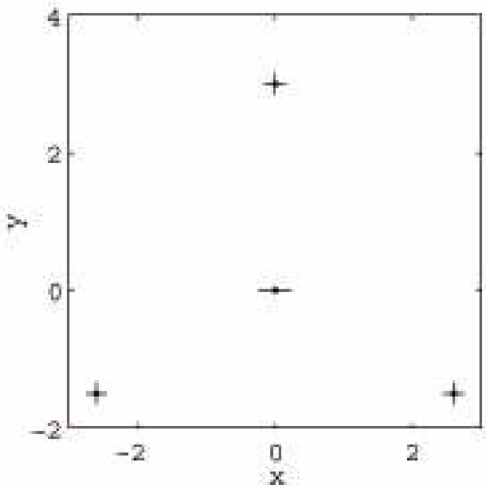
- $N(\geq 3)$ opposite vortices on a circle and its center

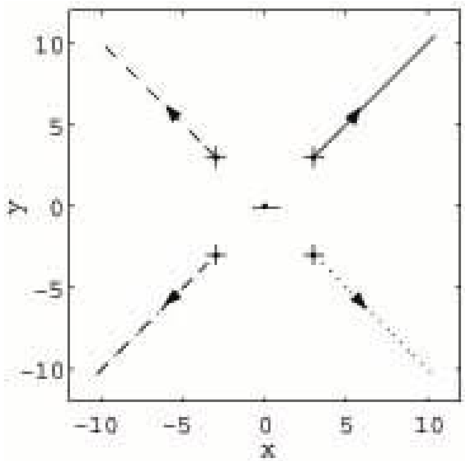
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \quad \sin\left(\frac{2j\pi}{N-1}\right)\right) \quad (+), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-4)}{\kappa}t}\left(\cos\left(\frac{2j\pi}{N-1}\right), \quad \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$







Non-Autonomous First Integrals

$$\frac{d\,\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1,l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j} W(X)$$

- Introduce

$$0 \le N^+ \le N, 0 \le N^- \le N, N^+ + N^- = N, M_0 := \frac{1}{2}\left[\left(N^+ - N^-\right)^2 - N\right]$$

- Three first integrals – Xu, Bao, Shi, DCDS-B, 18'

$$\begin{aligned} H_1(\mathbf{X},t) &= -4NM_0t + \sum_{1\le j<l\le N} |\mathbf{x}_j - \mathbf{x}_l|^2, \\ H_2(\mathbf{X},t) &= -4M_0t + \sum_{j=1}^{N} |\mathbf{x}_j|^2, \quad \mathbf{X} := (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \\ H_3(\mathbf{X},t) &= -4(N-2)M_0t + \sum_{1\le j<l\le N} |\mathbf{x}_j + \mathbf{x}_l|^2. \end{aligned} \Longrightarrow$$

$$\begin{aligned} H_1(\mathbf{X}(t),t) &\equiv H_1^0 := \sum_{1\le j<l\le N} |\mathbf{x}_j^0 - \mathbf{x}_l^0|^2, \\ H_2(\mathbf{X}(t),t) &\equiv H_2^0 := \sum_{j=1}^{N} |\mathbf{x}_j^0|^2, \qquad t \ge 0, \\ H_3(\mathbf{X}(t),t) &\equiv H_3^0 := \sum_{1\le j<l\le N} |\mathbf{x}_j^0 + \mathbf{x}_l^0|^2. \end{aligned}$$

Global Existence with Same Winding Number

$$\frac{d\vec{x}_j(t)}{dt} = 2\sum_{l=1,l\neq j}^{N} \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j} W(X)$$

- Global existence -- Xu, Bao, Shi, DCDS-B, 18'
  - Thm Suppose the N vortices have the same winding number, there is no finite time collision, i.e. global existence. In addition, at least two vortices move to infinity when t goes to infinity.

$$d_{jl}(t) = |\mathbf{x}_j(t) - \mathbf{x}_l(t)| \Longrightarrow \begin{array}{l} d_{\min}(t) = \min\limits_{1\le j<l\le N} d_{jl}(t), \\ D_{\min}(t) = d^2_{\min}(t), \quad 1 \le j \ne l \le N \\ D_{jl}(t) = d^2_{jl}(t). \end{array}$$

- Distances – Xu, Bao, Shi, DCDS-B, 18'

Theorem 2.3. *Suppose $2 \le N \le 4$ and the N vortices have the same winding number, i.e. $m_j = m_0$ for $1 \le j \le N$ in (1.1), then $d_{\min}(t)$ and $D_{\min}(t)$ are monotonically increasing functions.*

Theorem 2.2. *Suppose the N vortices have the same winding number, i.e. $m_j = m_0$ for $1 \le j \le N$ in (1.1), and the initial data $\mathbf{X}^0$ in (1.2) is collinear, then $d_{\min}(t)$ and $D_{\min}(t)$ are monotonically increasing functions.*

Long Time Dynamics when N=3

- Collinear case

Lemma 3.1. *If the initial data* $\mathbf{X}^0 \in \mathbb{R}_*^{2\times 3}$ *in (1.2) with* $N = 3$ *is collinear, then one vortex moves to the mass center* $\bar{\mathbf{x}}^0$ *and the other two vortices repel with each other and move outwards to far field along the line when* $t \to +\infty$.

- Non-collinear case

Theorem 3.1. *Assume the initial data* $\mathbf{X}^0 \in \mathbb{R}_*^{2\times 3}$ *in (1.2) with* $N = 3$ *is not collinear, then there exists a unit vector* $\mathbf{e} \in \mathbb{R}^2$ *such that*

$$\lim_{t\to+\infty} d_S(t) := \inf_{\mathbf{X}\in S_{\mathbf{e}}^3(\bar{\mathbf{x}}^0)} \|\mathbf{X}(t) - \mathbf{X}\|_2 = 0. \qquad (3.4)$$

Orbital Stability of Self-similar Solution

Definition

Definition 3.1. For the self-similar solution $\tilde{\mathbf{X}}(t) = \sqrt{r_0^2 + 2(N-1)t}\,\tilde{\mathbf{X}}^0$ with $\tilde{\mathbf{X}}^0 = (\tilde{\mathbf{x}}_1^0, \ldots, \tilde{\mathbf{x}}_N^0) \in S_{\mathbf{e}}^N(\mathbf{0}, \theta_0)$ and $r_0 = |\tilde{\mathbf{x}}_1^0|$ of the ODEs (1.1) with $m_1 = \ldots = m_N$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, when the initial data $\mathbf{X}^0$ in (1.2) satisfies $\|\mathbf{X}^0 - \tilde{\mathbf{X}}^0\|_2 < \delta$, the solution $\mathbf{X}(t)$ of the ODEs (1.1) with (1.2) satisfies

$$\sup_{t\geq 0} \inf_{r>0,\ \theta\in[0,2\pi)} \left\| \mathbf{X}(t) - \bar{\mathbf{x}}^0 - rQ(\theta)\tilde{\mathbf{X}}(t) \right\| < \varepsilon,$$

then the self-similar solution $\tilde{\mathbf{X}}(t)$ is called as orbitally stable.

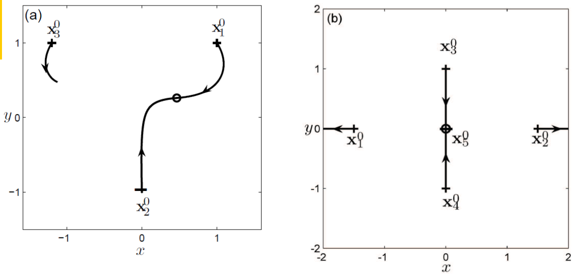
Orbital stability

Theorem 3.2. *For any $\theta_0 \in \mathbb{R}$ and $\tilde{\mathbf{X}}^0 = (\tilde{\mathbf{x}}_1^0, \tilde{\mathbf{x}}_2^0, \tilde{\mathbf{x}}_3^0) \in S_{\mathbf{e}}^3(\mathbf{0}, \theta_0)$, the solution $\tilde{\mathbf{X}}(t) = \sqrt{4t + r_0^2}\,\tilde{\mathbf{X}}^0$ with $r_0 = |\tilde{\mathbf{x}}_1^0|$ of the ODEs (1.1) with $N = 3$ and $m_1 = m_2 = m_3$ is orbitally stable.*

Finite Time Collision

$$\frac{d\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1,l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j} W(X)$$

- Typical collision patterns



Theorem 2.4. *Suppose the N vortices have opposite winding numbers, i.e. $|N^+ - N^-| < N$, we have*

(i) If $M_0 < 0$, finite time collision happens, i.e. $0 < T_{\max} < +\infty$, and there exists a collision cluster among the N vortices. In addition, $T_{\max} \leq T_a := -\frac{H_1^0}{4NM_0}$.

(ii) If $M_0 = 0$, then the solution of (1.1) is bounded, i.e.

$$M_0 := \frac{1}{2}\left[\left(N^+ - N^-\right)^2 - N\right]$$

$$|\mathbf{x}_j(t)| \leq \sqrt{H_2^0} = \sqrt{\sum_{j=1}^{N} |\mathbf{x}_j^0|^2}, \qquad t \geq 0, \qquad 1 \leq j \leq N. \qquad (2.29)$$

(iii) If $M_0 > 0$ and there is no finite time collision, i.e. $T_{\max} = +\infty$, then at least two vortices move to infinity as $t \to +\infty$.

(iv) Let $I \subseteq \{1, 2, \ldots, N\}$ be a set with M $(2 \leq M \leq N)$ elements. If the collective winding number of I defined as $M_1 := \frac{1}{2}\sum_{j,l \in I, j \neq l} m_j m_l \geq 0$, then the set of vortices $\{\mathbf{x}_j(t) \mid j \in I\}$ cannot be a collision cluster among the N vortices for $0 \leq t \leq T_{\max}$.

Finite Time Collision

$$\frac{d\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1,l\neq j}^{N} m_l \frac{\vec{x}_j(t)-\vec{x}_l(t)}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j} W(X)$$

- Necessary condition for a collision

Proposition 2.1. *If the N vortices be a collision cluster at $0 < T_{\max} < +\infty$ under a given initial data $\mathbf{X}^0 \in \mathbb{R}_*^{2\times N}$, then we have*

$$M_0 < 0, \qquad H_1^0 = N H_2^0, \qquad H_3^0 = (N-2) H_2^0. \qquad (2.31)$$

$$M_0 := \frac{1}{2}\left[\left(N^+ - N^-\right)^2 - N\right]$$

- Necessary condition for an equilibrium

Proposition 2.2. *If the ODEs (1.1) admits an equilibrium solution, then $N \geq 4$ is a square of an integer, i.e. $N = (N^+ - N^-)^2$ and*

$$1 \leq N^+ = \frac{1}{2}\left(N \pm \sqrt{N}\right) < N, \qquad 1 \leq N^- = N - N^+ < N. \qquad (2.35)$$

Collision Patterns when N=3

$$m_1 = m_3 = 1, \quad m_2 = -1$$

Theorem 3.3. *For any given initial data* $\mathbf{X}^0 \in \mathbb{R}_*^{2\times 3}$ *in* (1.2) *with* $N = 3$, *we have*

(i) If $|\mathbf{x}_1^0 - \mathbf{x}_2^0| = |\mathbf{x}_2^0 - \mathbf{x}_3^0|$, *then the three vortices be a collision cluster and they will collide at* $\bar{\mathbf{x}}^0$ *when* $t \to T_{\max}^- = \frac{H_1^0}{12}$ *with* $H_1^0 = \sum_{1\le j<l\le 3} |\mathbf{x}_j^0 - \mathbf{x}_l^0|^2$.

(ii) If $|\mathbf{x}_1^0 - \mathbf{x}_2^0| < |\mathbf{x}_2^0 - \mathbf{x}_3^0|$, *then only* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *form a collision cluster, and respectively, if* $|\mathbf{x}_1^0 - \mathbf{x}_2^0| > |\mathbf{x}_2^0 - \mathbf{x}_3^0|$, *then only* $\mathbf{x}_2$ *and* $\mathbf{x}_3$ *form a collision cluster. Moreover, the collision time* $0 < T_{\max} < \frac{H_1^0}{12}$.

Collective dynamics of RDL for NLSE

$$J\frac{d\,\vec{x}_j(t)}{dt} = 2\sum_{l=1,l\neq j}^{N} m_l \frac{(\vec{x}_j(t)-\vec{x}_l(t))}{\left|\vec{x}_j(t)-\vec{x}_l(t)\right|^2} := -\nabla_{\vec{x}_j}\tilde{W}(X) \qquad t\geq 0,$$

$$\vec{x}_j(0)=\vec{x}_j^0 \qquad 1\leq j\leq N; \qquad J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

- Signed mass center $\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\,\vec{x}_j(t)$

Lemma The signed mass center of the N vortices for NLSE is conserve

$$\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\,\vec{x}_j(t) \equiv \tilde{x}(0) := \frac{1}{N}\sum_{j=1}^{N} m_j\,\vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} m_j\,\vec{x}_j^0$$

- Energy conservation $\tilde{W}(X) = -\sum_{1\leq j\neq k\leq N} m_k \ln\left|\vec{x}_j-\vec{x}_k\right| = -\ln\prod_{1\leq j\neq k\leq N}\left|\vec{x}_j-\vec{x}_k\right|^{m_k}$

$$\tilde{W}(X(t)) \equiv \tilde{W}(X(0)), \qquad t\geq 0$$

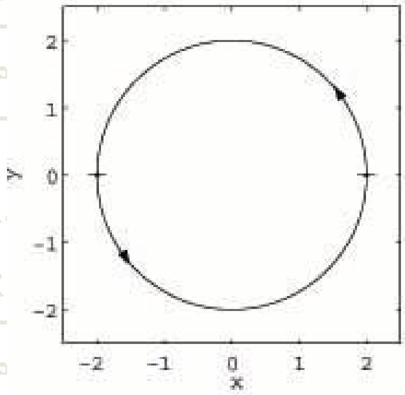
Analytical Solutions of NLSE– Case I

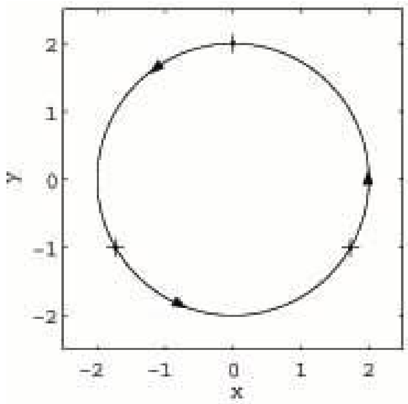
- $N(\geq 2)$ like vortices on a circle (Bao, Du & Zhang, SIAP, 07')

$$\vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N}\right) \quad , \quad \sin\left(\frac{2j\pi}{N}\right)\right), \qquad m_j = m_0 = \pm 1, \qquad 1 \leq j \leq N$$

- Analytical solutions for GLE

$$\vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right) \quad , \quad \sin\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right)\right)$$





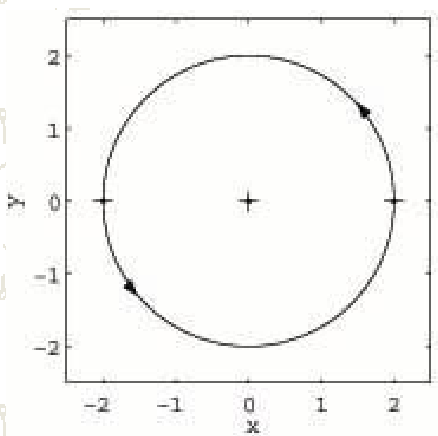
Analytical Solutions of NLSE--Case II

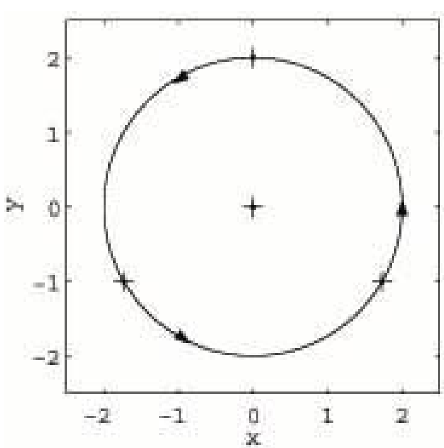
- $N(\geq 3)$ like vortices on a circle and its center(Bao,Du&Zhang, SIAP, 07')

$$\vec{x}_N^0 = \vec{0}; \qquad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right)\right), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right), \sin\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right)\right)$$





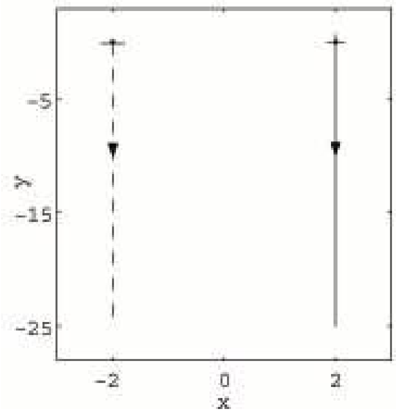
Analytical Solutions of NLSE--Case III

- Two opposite vortices (Bao, Du & Zhang, SIAP, 07')

$$\vec{x}_1^0 = -\vec{x}_2^0 = a\big(\cos(\theta_0) \quad , \quad \sin(\theta_0)\big), \qquad m_1 = -m_2 = 1$$

- Analytical solutions for GLE

$$\vec{x}_j(t) = \vec{x}_j^0 + \frac{t}{a}\big(-\sin(\theta_0) \quad , \quad \cos(\theta_0)\big), \qquad j = 1, 2$$



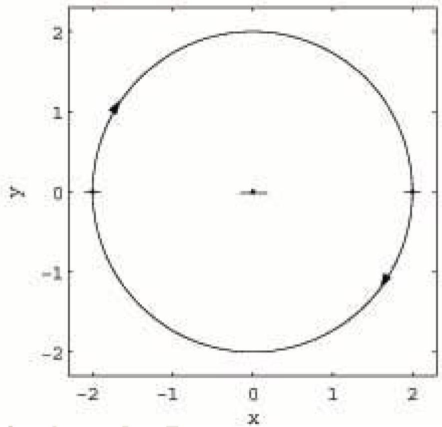
Analytical Solutions of NLSE--Case IV

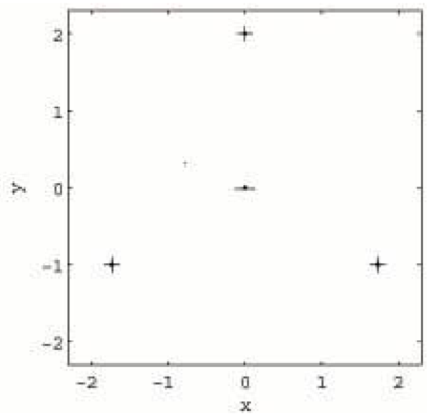
- $N(\geq 3)$ opposite vortices on a circle and its center

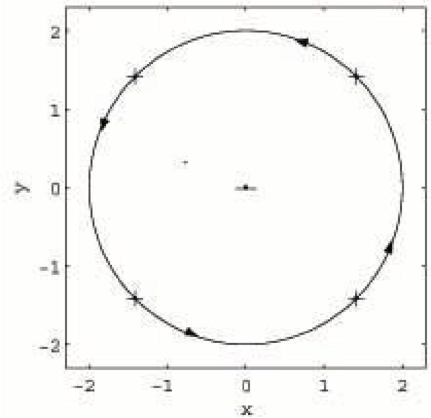
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \quad \sin\left(\frac{2j\pi}{N-1}\right)\right) \quad (+), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right), \quad \sin\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right)\right)$$







GLE or NLSE on Bounded Domains

- GLE: $$\lambda_\varepsilon \partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\Omega, \quad \text{with} \quad \lambda_\varepsilon = \frac{1}{|\ln\varepsilon|}$$
- GPE: $$i\,\partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\left(1-|\psi|^2\right)\psi, \quad \vec{x}\in\Omega,$$
- Initial condition (IC): $\psi(\vec{x},0)=\psi_0(\vec{x})=e^{ih(\vec{x})}\prod_{j=1}^{N}\phi_{n_j}(\vec{x}-\vec{x}_j^0)$
- Boundary condition (BC) --- Dirichlet vs Neumann

$$\psi(\vec{x},t)=g(\vec{x})=e^{i\omega(\vec{x})} \quad \text{or} \quad \partial_{\vec{n}}\psi(\vec{x},t)=0$$

- Three scaling: $d_\Omega \gg d_{\min}^0 \gg \varepsilon$
- Two different limits – S. Serfaty, ARMA, 11' & JAMS 16;
  - Fix N, $\varepsilon \to 0$
  - First $\varepsilon \to 0$, then $N \to +\infty$

Reduced dynamic laws (RDL)

$$X := X(t) = (\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_N(t))^T$$

- GLE with Dirichlet BC for superconductivity— Neu 90'; Lin 96-98; Jerrard & Soner 98'; Weinstein & Xin 96'; Rubinstein & Sternberg 95'; Serfaty 05'; Chapman, Du & Gunzburger 96, Jian & Song 01' , …..

$$\frac{d\,\vec{x}_j(t)}{dt} = 2m_j\left[\nabla_{\vec{x}} R(\vec{x}, X)\Big|_{\vec{x}=\vec{x}_j(t)} + \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{\left|\vec{x}_j(t) - \vec{x}_l(t)\right|^2}\right], \quad \vec{x}_j(0) = \vec{x}_j^0, \quad 1 \le j \le N;$$

- GPE with Dirichlet BC for superfluidity-- Neu 90'; Lin 98; Collinder & Jerrard 99'; Jerrard & Spirn 08'; Lin & Xin 99'; Rubinstein & Sternberg 95'; Serfaty 05'; Ovchinnikov & Sigal 98' ; …..

$$\frac{d\,\vec{x}_j(t)}{dt} = 2J\left[\nabla_{\vec{x}} R(\vec{x}, X)\Big|_{\vec{x}=\vec{x}_j(t)} + \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{\left|\vec{x}_j(t) - \vec{x}_l(t)\right|^2}\right], \quad \vec{x}_j(0) = \vec{x}_j^0, \quad 1 \le j \le N;$$

- With

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta R(\vec{x}, X) = 0, \ \vec{x} \in \Omega; \quad \partial_{\vec{n}} R(\vec{x}, X) = \partial_{\vec{n}} \omega(\vec{x}) - \partial_{\vec{n}} \sum_{l=1}^{N} m_l \ln\left|\vec{x} - \vec{x}_l\right|, \ \vec{x} \in \partial\Omega;$$

Reduced dynamic laws (RDL)

$$X := X(t) = (\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_N(t))^T$$

- GLE with Neumann BC for superconductivity— Neu 90'; Lin 96-98; Jerrard & Soner 98'; Weinstein & Xin 96'; Rubinstein & Sternberg 95'; Serfaty 05'; Chapman, Du & Gunzburger 96, Jian & Song 01' , …..

$$\frac{d}{dt}\mathbf{x}_j(t) = 2n_j \left[J\nabla_{\mathbf{x}}\tilde{Q}(\mathbf{x};X)\,|_{\mathbf{x}=\mathbf{x}_j(t)} + \sum_{l=1\&l\neq j}^{M} n_l \frac{\mathbf{x}_j(t)-\mathbf{x}_l(t)}{|\mathbf{x}_j(t)-\mathbf{x}_l(t)|^2} \right]$$

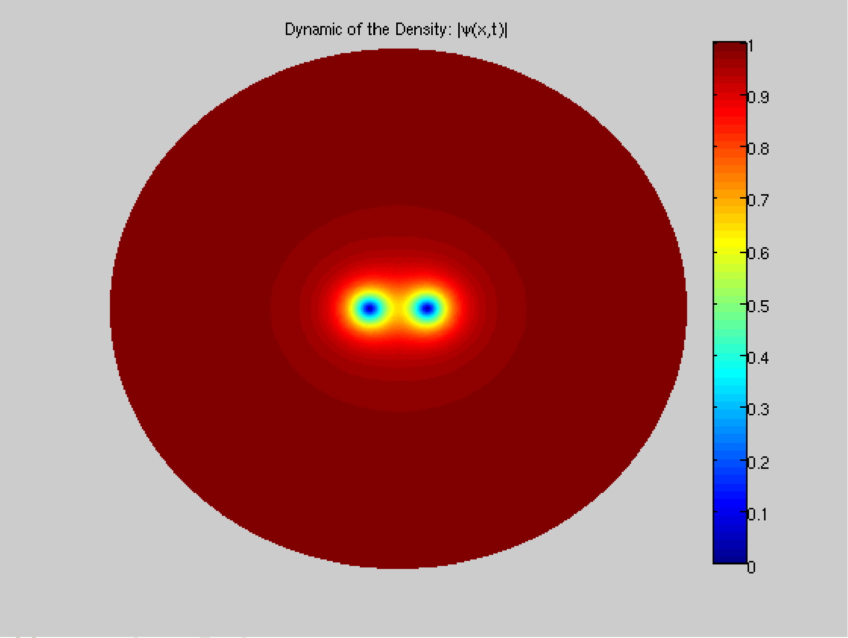
- GPE with Neumann BC for superfluidity-- Neu 90'; Lin 98; Collinder & Jerrard 99'; Jerrard & Spirn 08'; Lin & Xin 99'; Rubinstein & Sternberg 95'; Serfaty 05'; Ovchinnikov & Sigal 98' ; …..

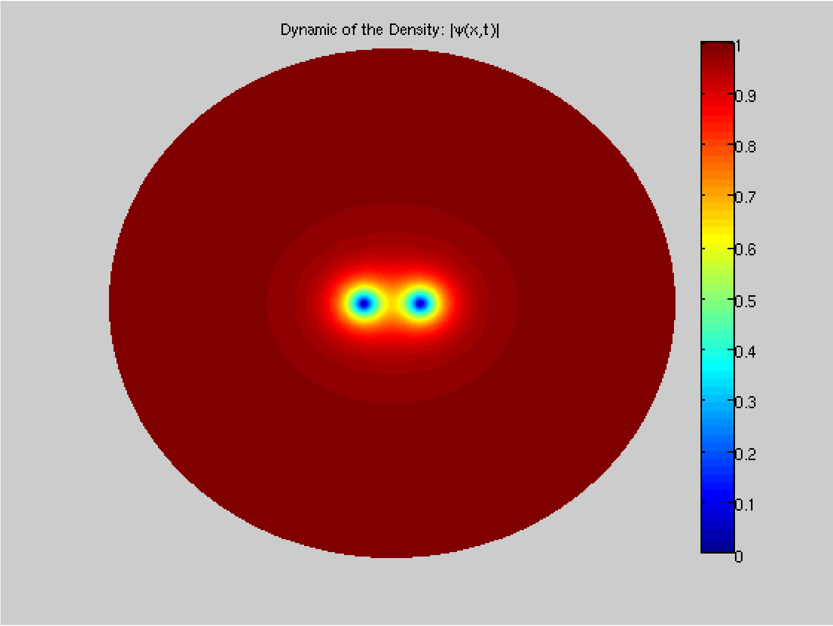
$$\dot{\mathbf{x}}_j(t) = 2J \left[J\nabla_{\mathbf{x}}\tilde{Q}(\mathbf{x};X)\,|_{\mathbf{x}=\mathbf{x}_j(t)} + \sum_{l=1\&l\neq j}^{M} n_l \frac{\mathbf{x}_j(t)-\mathbf{x}_l(t)}{|\mathbf{x}_j(t)-\mathbf{x}_l(t)|^2} \right]$$

  - With

$$\Delta\tilde{Q}(\vec{x},X) = 0,\ \vec{x}\in\Omega; \quad \partial_{\vec{n}}\tilde{Q}(\vec{x},X) = -\partial_{\vec{n}}\sum_{l=1}^{N} m_l\theta(\vec{x}-\vec{x}_l),\ \vec{x}\in\partial\Omega; \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Vortex-pair with Dirichlet BC

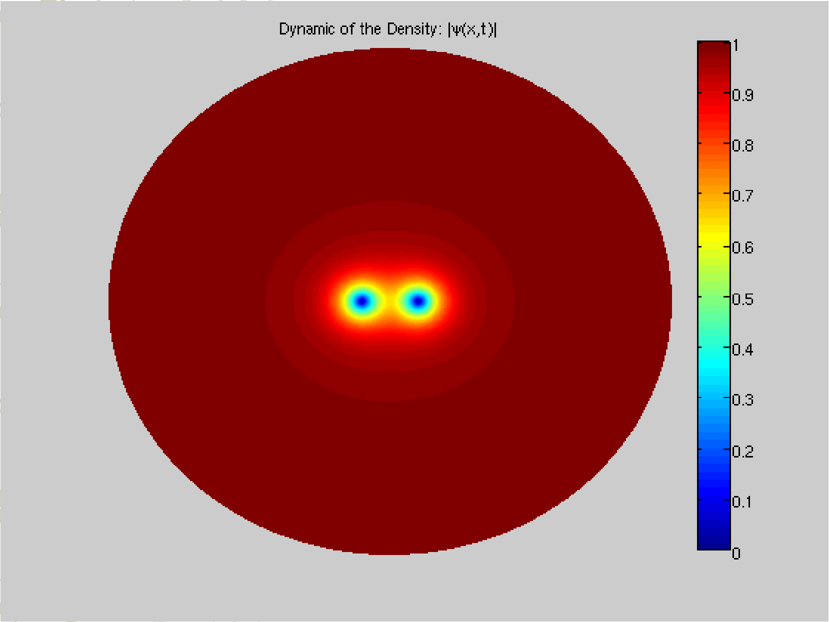


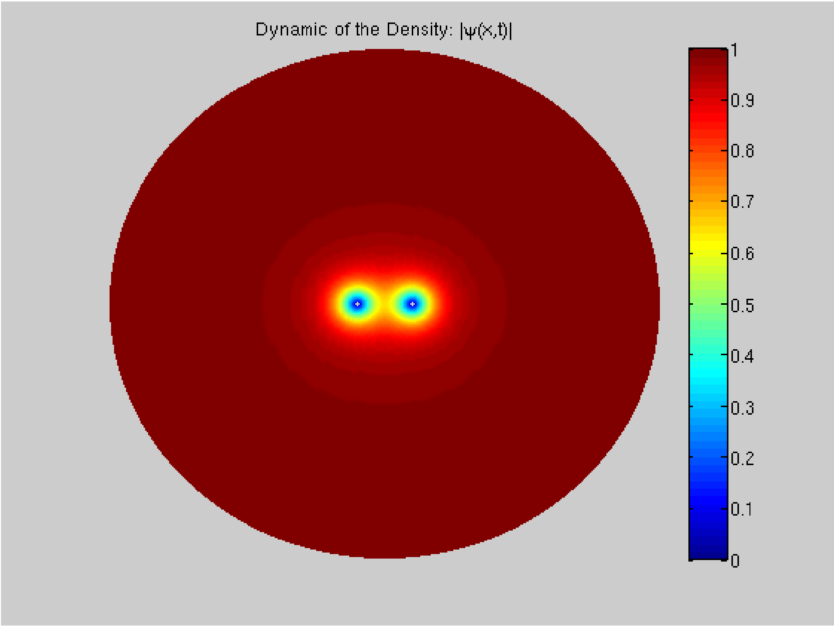


GPE for superfluidity

GLE for superconductivity

Vortex-dipole (vortex-antivortex) with Dirichlet BC

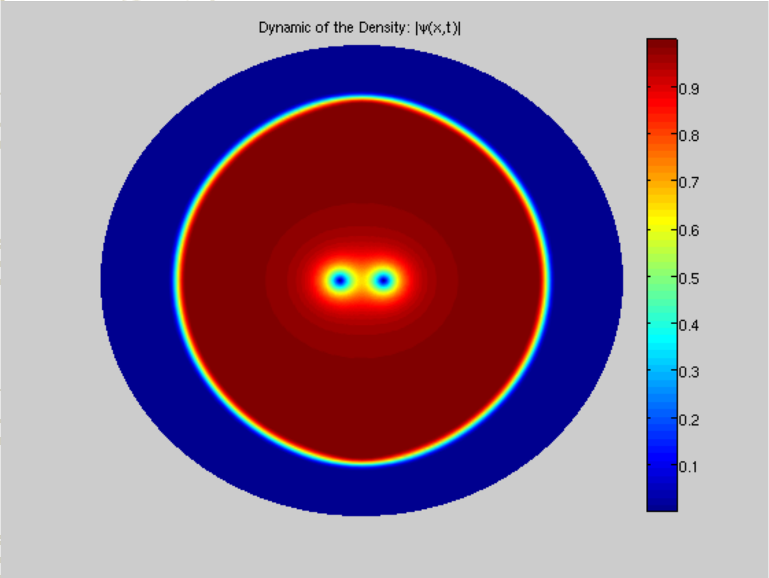


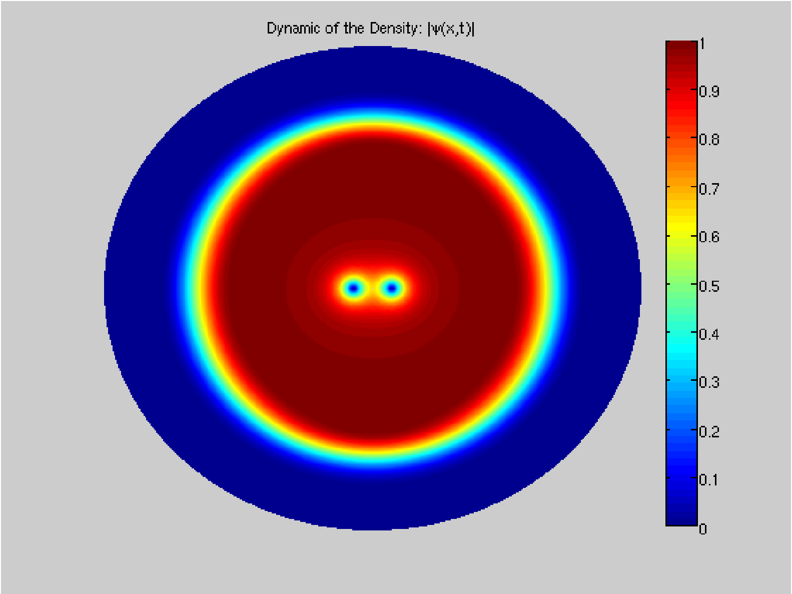


GPE for superfluidity

GLE for superconductivity

Vortex-pair with Neumann BC

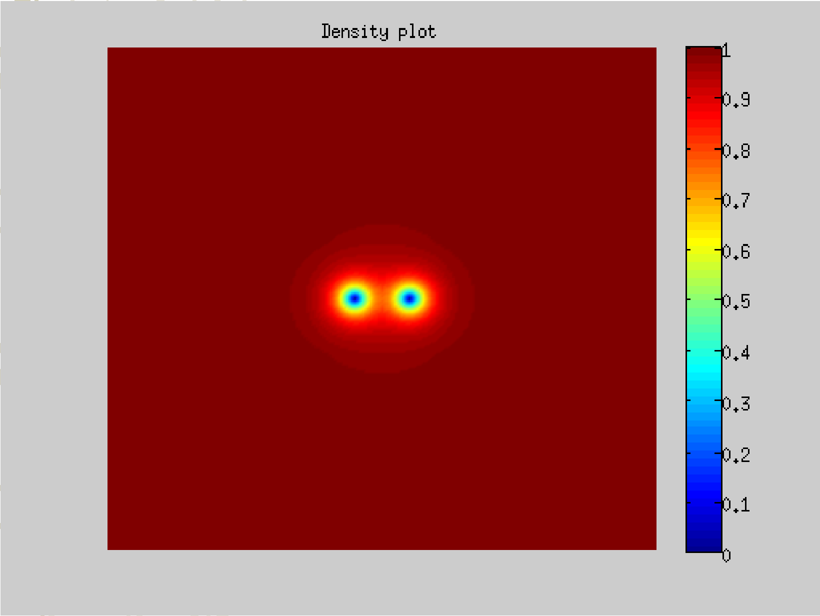


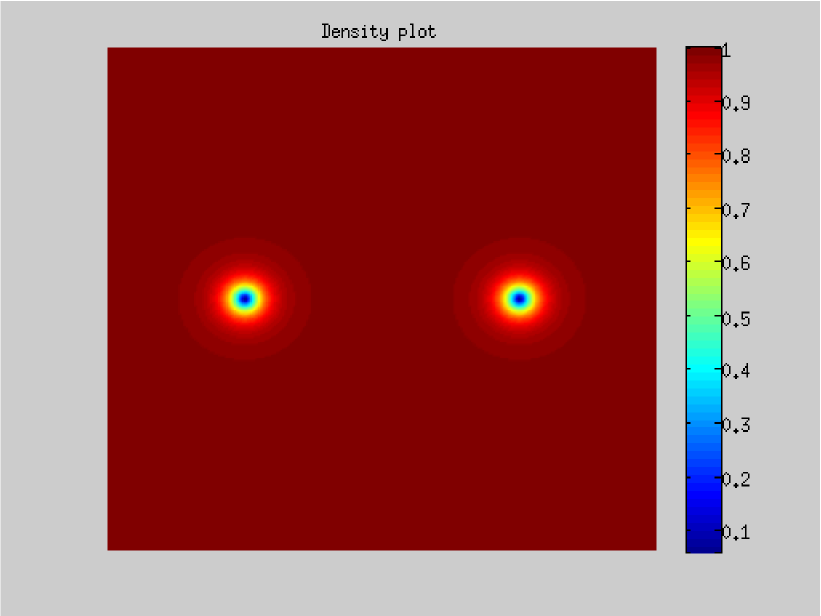


GPE for superfluidity

GLE for superconductivity

Vortex-pair of GPE with Diriechlet BC

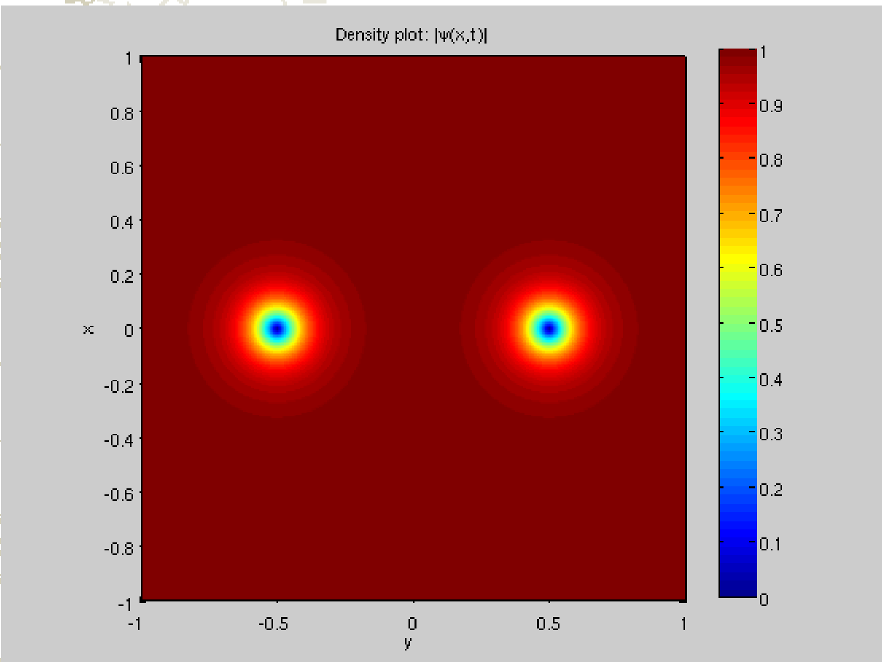




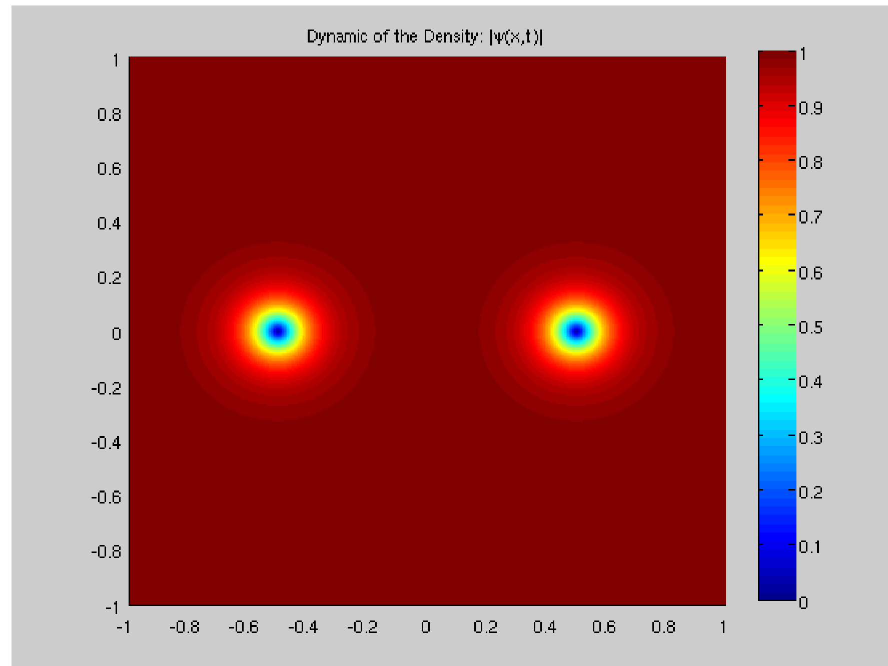
$$h(\vec{x}) \equiv 0$$

$$h(\vec{x}) = x + y$$

Vortex-dipole of GPE with Dirichlet BC

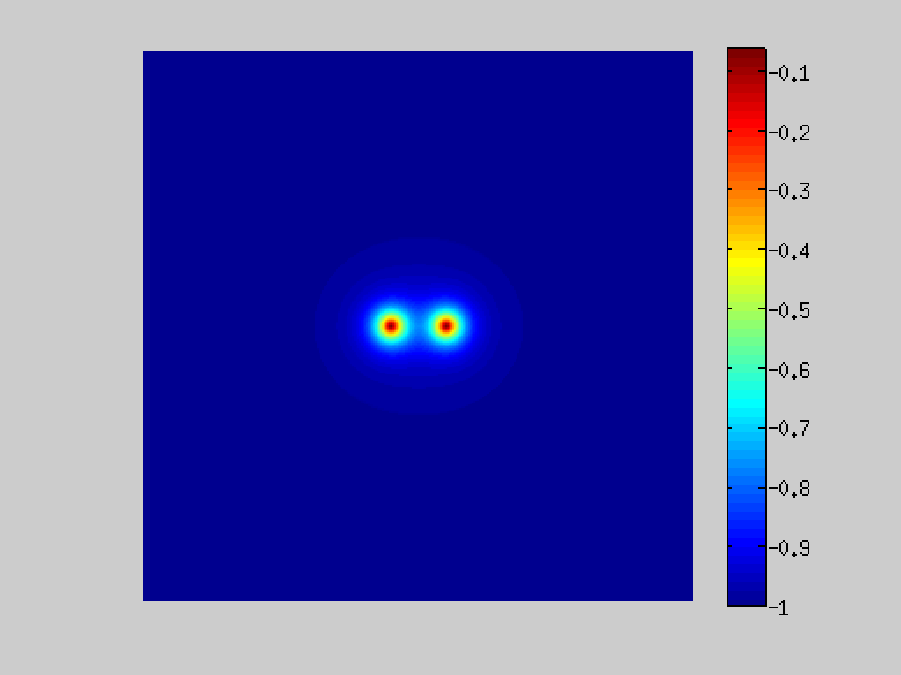


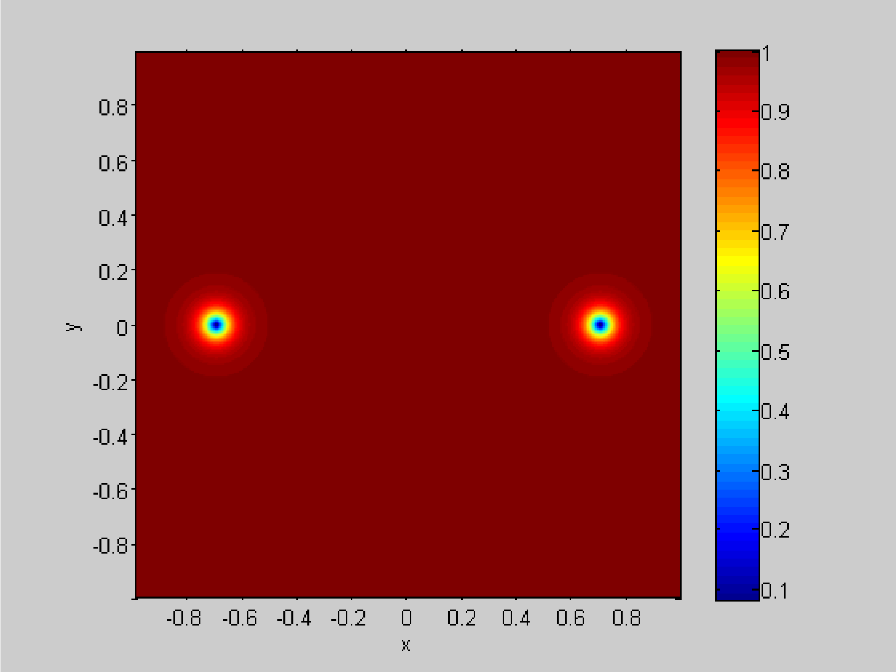
$$h(\vec{x}) \equiv 0$$



$$h(\vec{x}) = x + y$$

Vortex-dipole of GPE with Neumann BC

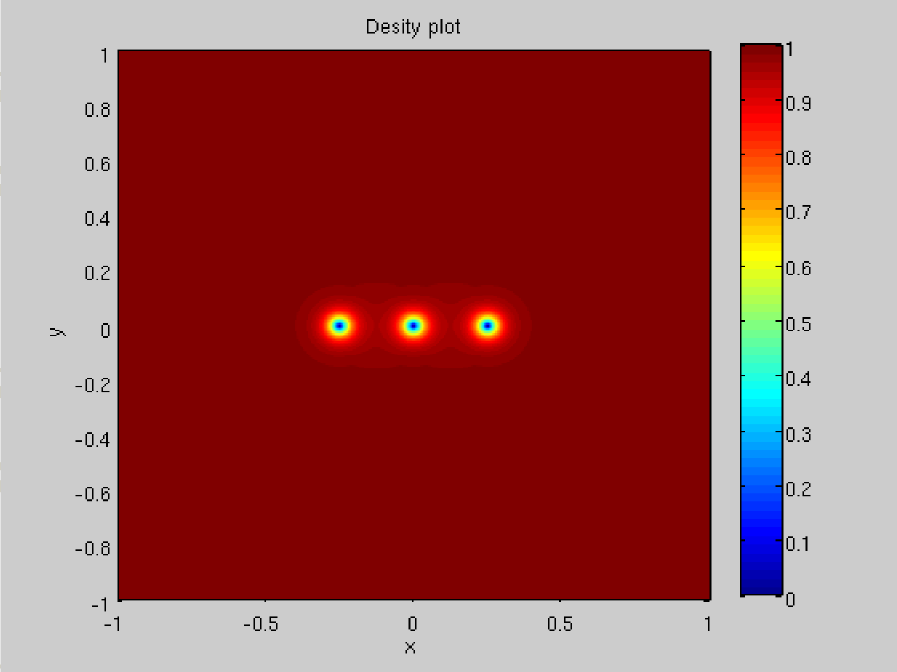


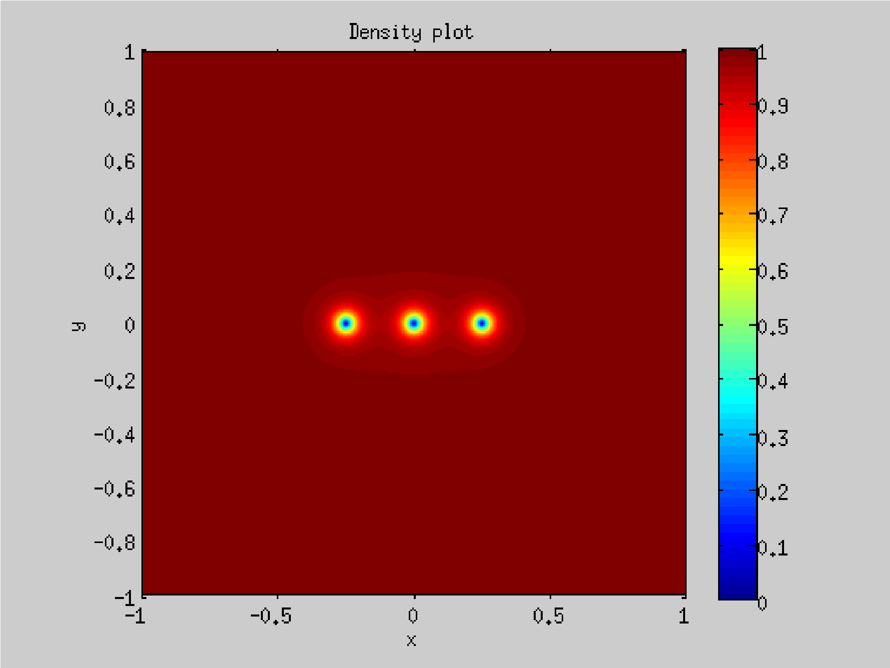


Initial close to each other

Initial far away from each other

Vortex-lattice (3 vortices) of GPE with Dirichlet BC

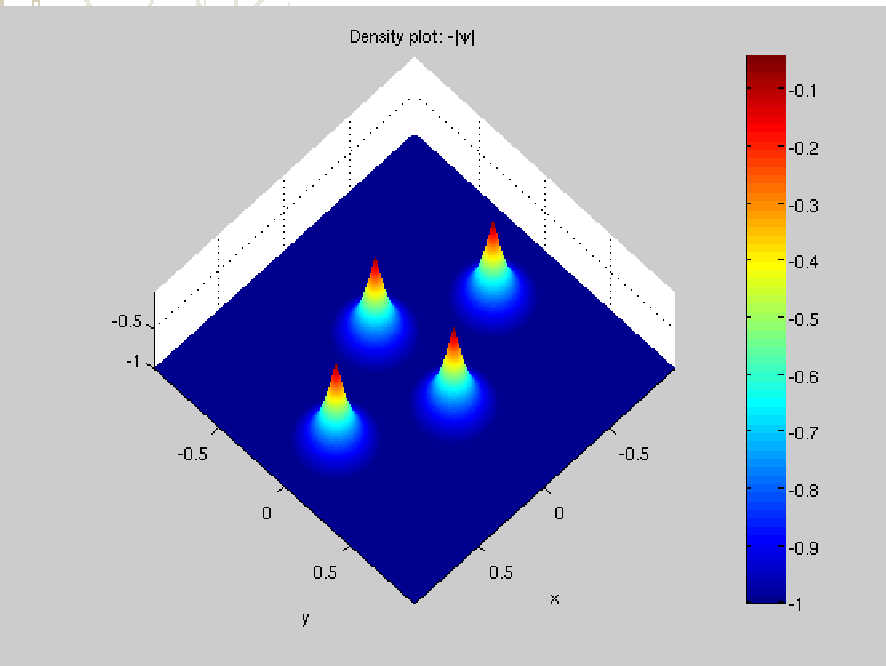


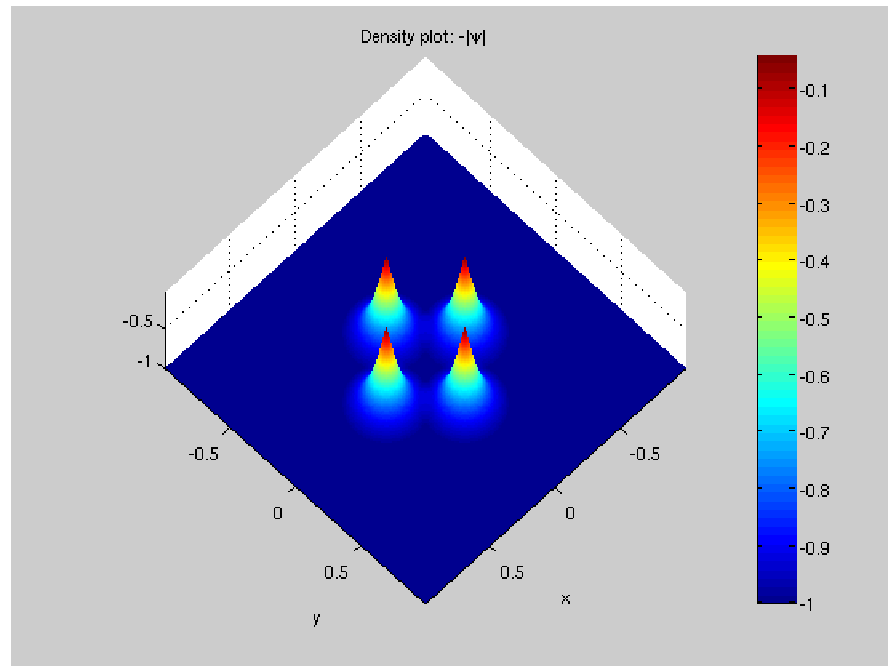


3 with same winding

2 with winding 1 & 1 with -1

Vortex-lattice (4 vortices) of GPE with Neumann BC

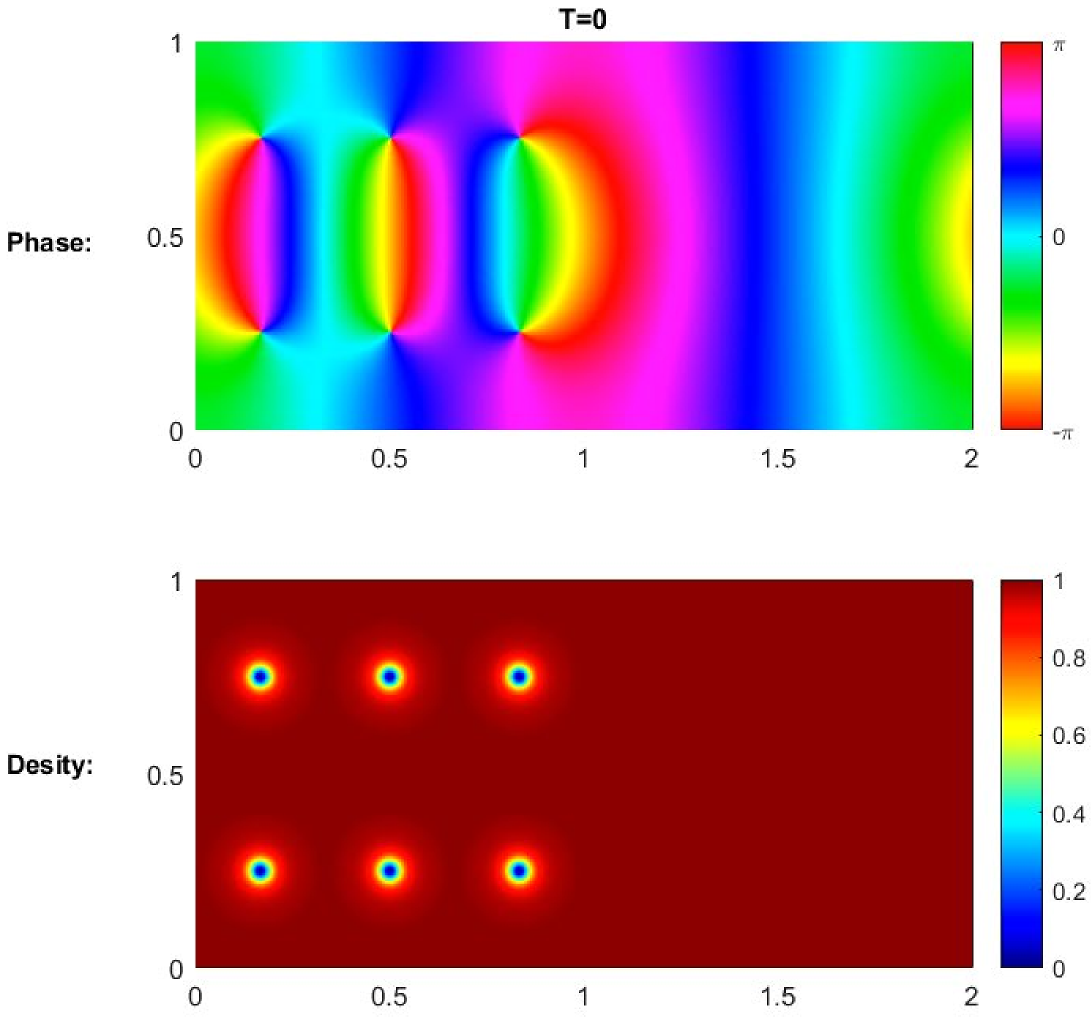




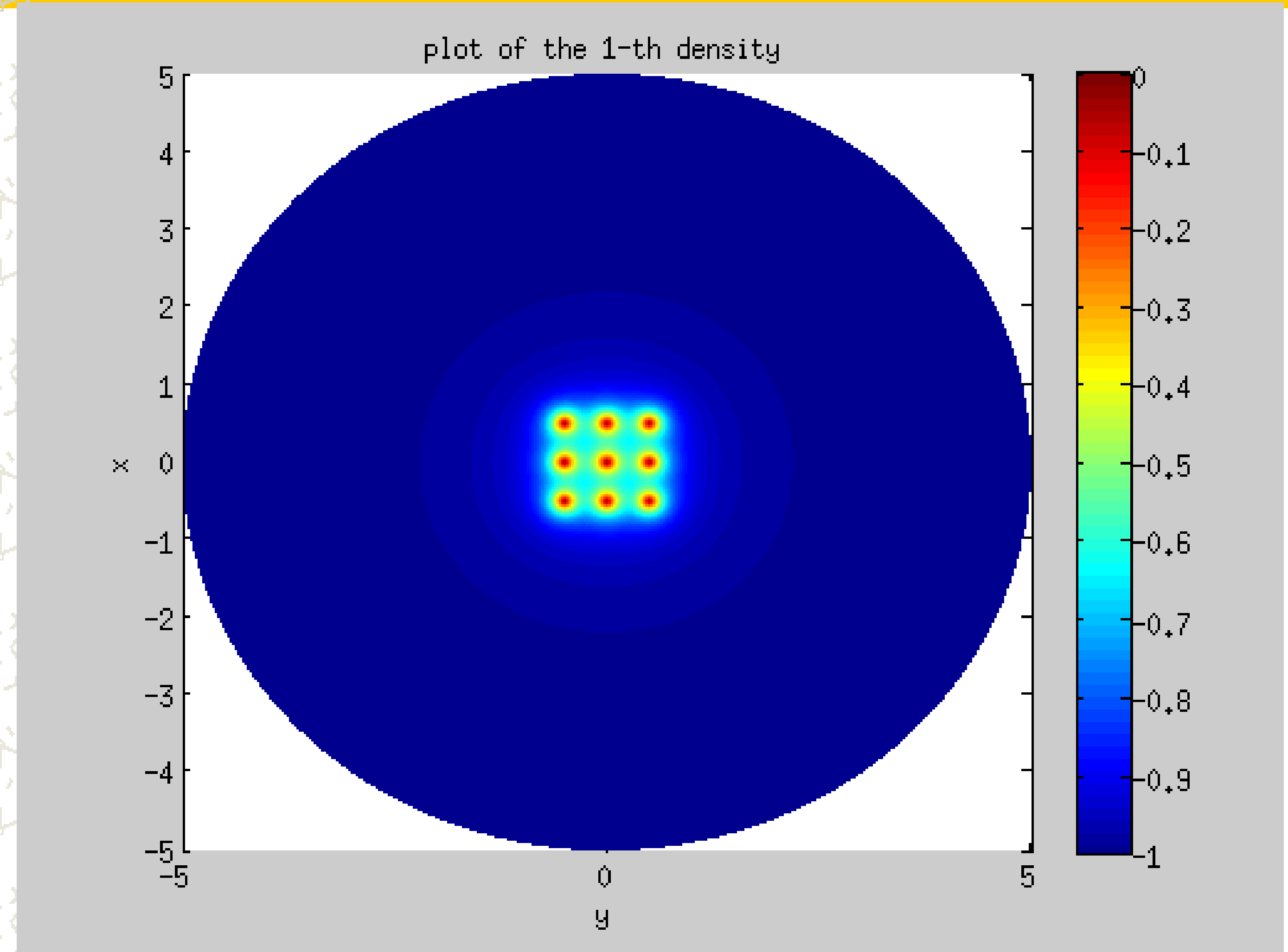
4 with same winding

2 with winding 1 & 2 with -1

Leapfrog of 3 Vortex Dipoles of NLSE under PBC



Vortex-lattice (9 vortices) of GPE with Dirichlet BC



Impurities Bound by Vortex lattice

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Hubbard Model for Atomic Impurities Bound by the Vortex Lattice of a Rotating Bose-Einstein Condensate

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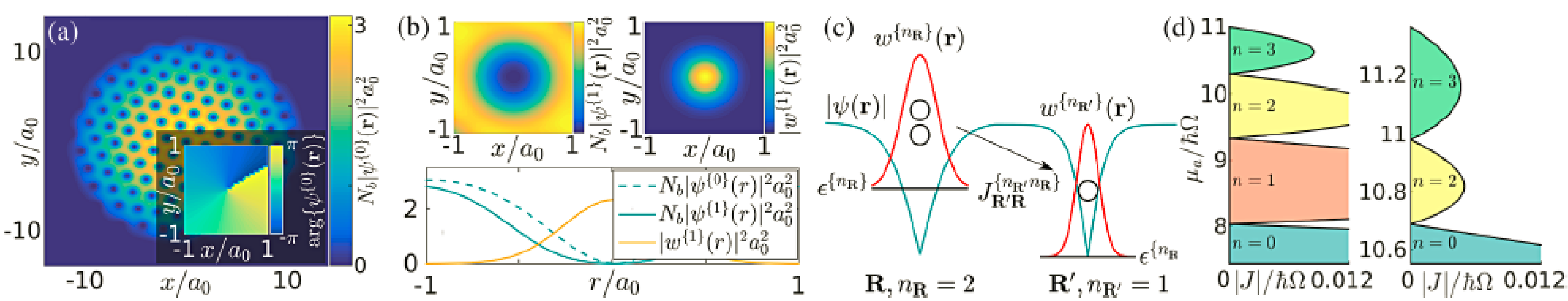
(Received 1 January 2016; published 14 June 2016)

We investigate cold bosonic impurity atoms trapped in a vortex lattice formed by condensed bosons of another species. We describe the dynamics of the impurities by a bosonic Hubbard model containing occupation-dependent parameters to capture the effects of strong impurity-impurity interactions. These



Vortex patterns and the critical rotational frequency in rotating dipolar Bose-Einstein condensates

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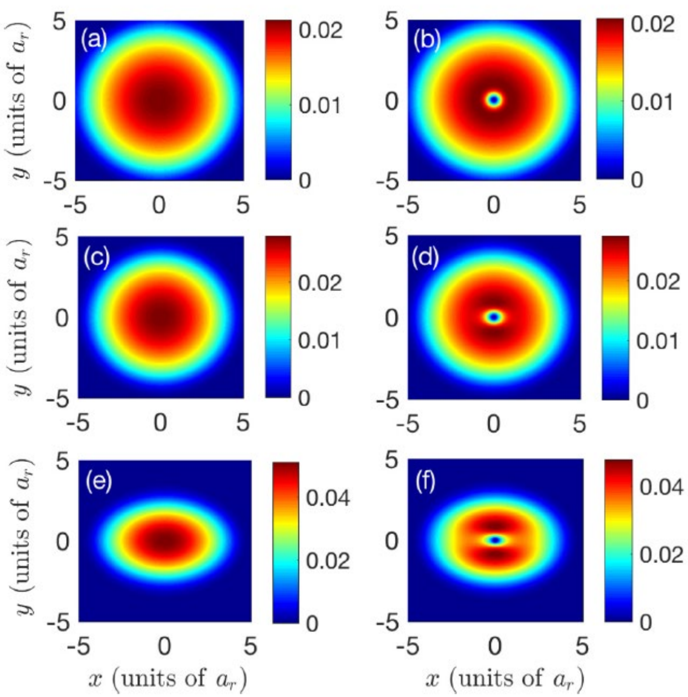

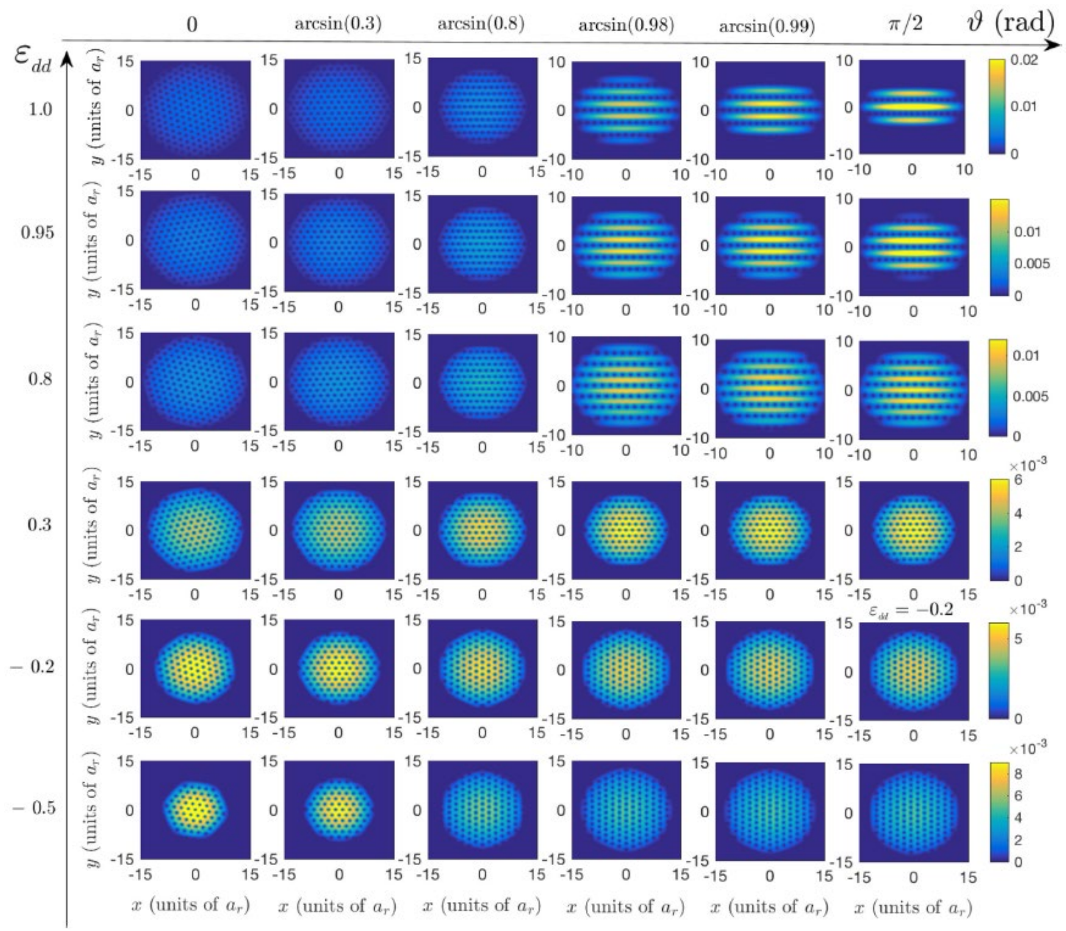

Quantum Turbulence in 3D

- YouTube movie

https://www.youtube.com/watch?v=uk5DpF4vnFs