TECHNISCHE
UNIVERSITÄT
WIEN

Modal based hypocoercivity methods on the torus & real line with application to Goldstein-Taylor models

Anton ARNOLD

with J. Dolbeault, A. Einav, C. Schmeiser, B. Signorello, T. Wöhrer

CIRM, March 2021

Outline:

1. hypocoercive ODEs: sharp decay estimates
2. Goldstein-Taylor model (= 2 velocity BGK model) on torus
3. Goldstein-Taylor model on $\mathbb{R}$

Long-time decay for nonsymmetric ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $\bar{x}^T \mathbf{C}_H x \geq \kappa \|x\|^2\ \forall x$ (for some $\kappa > 0$).

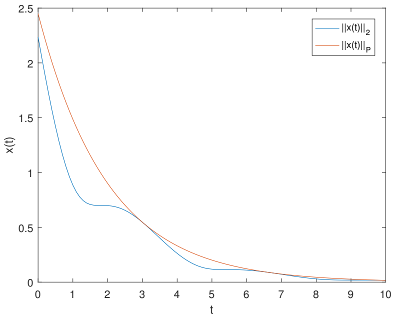
Long-time decay for nonsymmetric ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $\bar{x}^T \mathbf{C}_H x \geq \kappa \|x\|^2\ \forall x$ (for some $\kappa > 0$).

ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$ for (1).

- $\mathbf{C}$ not coercive $\Rightarrow$ no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of modified norm $\|x(t)\|_{\mathbf{P}} := \sqrt{\bar{x}^T\, \mathbf{P}\, x}$; $\mathbf{P} := [2\ 1; 1\ 2]$



How to find $\mathbf{P} > 0$ / the Lyapunov functional?

hypocoercive ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n$$

<u>Definition:</u> $\mathbf{C}$ is *hypocoercive* (= positive stable) if $\exists\, \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \qquad j = 1, ..., n.$$

If all eigenvalues of $\mathbf{C}$ are non-defective:

$$\exists c \geq 1 : \qquad \|x(t)\|_2 \leq c\|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

• always: $\mu \geq \kappa := \max_x \dfrac{\bar{x}^T \mathbf{C}_H x}{\|x\|^2}$ (i.e. spectral gap $\geq$ coercivity)

hypocoercive ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n$$

Definition: $\mathbf{C}$ is *hypocoercive* (= positive stable) if $\exists\, \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \qquad j = 1, ..., n.$$

If all eigenvalues of $\mathbf{C}$ are non-defective:

$$\exists c \geq 1: \qquad \|x(t)\|_2 \leq c\|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

• always: $\mu \geq \kappa := \max_x \dfrac{\bar{x}^T \mathbf{C}_H x}{\|x\|^2}$ (i.e. spectral gap $\geq$ coercivity)

Conditions for hypocoercivity:

1. $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \in \mathbb{C}^{n\times n}$; $\mathbf{C}_1^* = -\mathbf{C}_1$, $\mathbf{C}_2^* = \mathbf{C}_2 \geq 0$ (w.l.o.g.)
2. No (non-trivial) subspace of $\ker \mathbf{C}_2$ is invariant under $\mathbf{C}_1$

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov matrix inequality

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$.

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective* *(i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P} \geq 2\mu \mathbf{P}$.

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov matrix inequality

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$.

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{PC} + \mathbf{C}^\top \mathbf{P} \geq 2\mu \mathbf{P}$.
2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{PC} + \mathbf{C}^\top \mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$.

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov matrix inequality

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$ *.*

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C})\,|\,\Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0\ : \quad \mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P} \geq 2\mu\mathbf{P}$ *.*
2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0\ : \quad \mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$ *.*

Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top \ ; \qquad z_j \ \ldots \text{ eigenvectors of } \mathbf{C}^\top$$

□

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- For complex $\mathbf{C}$: $\mathbf{P} > 0$ Hermitian with $\mathbf{P}\mathbf{C} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$.

Long-time decay of $\mathbf{P}$–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, **C** real):

  Let $\|x\|_P^2 := x^T\,\mathbf{P}\,x$.

$$\frac{d}{dt}\|x\|_{\mathbf{P}}^2 \quad = \quad -x^T\big(\underbrace{\mathbf{PC}+\mathbf{C}^T\mathbf{P}}_{\geq 2\mu\mathbf{P}}\big)x \leq -2\mu\|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \quad \leq \quad \|x(0)\|_{\mathbf{P}}\, e^{-\mu t}, \quad t \geq 0.$$

Long-time decay of **P**–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, **C** real):

  Let $\|x\|_P^2 := x^T\,\mathbf{P}\,x$.

$$\frac{d}{dt}\|x\|_{\mathbf{P}}^2 = -x^T\big(\underbrace{\mathbf{PC}+\mathbf{C}^T\mathbf{P}}_{\geq 2\mu\mathbf{P}}\big)x \leq -2\mu\|x\|_{\mathbf{P}}^2$$

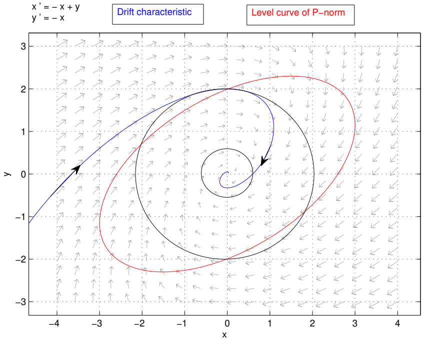
$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \leq \|x(0)\|_{\mathbf{P}}\, e^{-\mu t}, \quad t \geq 0.$$

- **P**–norm can be used for entropy/energy methods of kinetic equations (e.g. relaxation/BGK, Fokker-Planck)

Decay of P–norm (continued)

ex: $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2-axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x^T \mathbf{P}\, x}$; $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

  $\rightarrow$ uniform decay with sharp rate $\frac{1}{2}$

Outline:

1. hypocoercive ODEs
2. Goldstein-Taylor model on torus
3. Goldstein-Taylor model on $\mathbb{R}$

Goldstein-Taylor model

$$\begin{cases} \partial_t f_+(x,t) + \partial_x f_+(x,t) = \dfrac{\sigma(x)}{2}(f_- - f_+), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) = -\dfrac{\sigma(x)}{2}(f_- - f_+), \end{cases} \qquad t > 0,\ x \in \mathbb{T} \text{ or } \mathbb{R}$$

- as BGK-toy model
- original applications: turbulent fluid motion, telegrapher's equation

Goldstein-Taylor model

$$\begin{cases} \partial_t f_+(x,t) + \partial_x f_+(x,t) = \dfrac{\sigma(x)}{2}(f_- - f_+), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) = -\dfrac{\sigma(x)}{2}(f_- - f_+), \end{cases} \qquad t > 0,\ x \in \mathbb{T} \text{ or } \mathbb{R}$$

- as BGK-toy model
- original applications: turbulent fluid motion, telegrapher's equation

- mass density $u(x,t) := f_+ + f_-$, flux density $v(x,t) := f_+ - f_-$

$$\begin{cases} \partial_t u + \partial_x v &= 0, \\ \partial_t v + \partial_x u &= -\sigma(x) v \end{cases}$$

- goal (on $\mathbb{T}$):
  sharp decay estimates $u(t) \to u_\infty = 2f_\infty := (f_{+,0} + f_{-,0})_{avg}$,
  $v(t) \to v_\infty = 0$; ultimately for $\sigma(x)$

Modal decomposition of (u, v) on torus for $\sigma = const$:

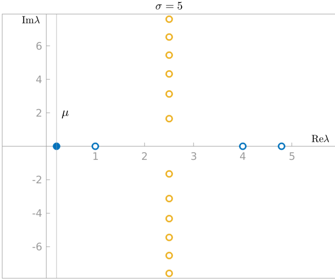
$$\frac{d}{dt}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} = -\begin{pmatrix}0 & ik\\ ik & \sigma\end{pmatrix}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} =: -\mathbf{C}_k\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix}, \qquad k \in \mathbb{Z}.$$

$\widehat{u}(0,t) = \widehat{u}_0, \quad \widehat{v}(0,t) = \widehat{v}_0 e^{-\sigma t}$

Modal decomposition of (u, v) on torus for $\sigma = const$:

$$\frac{d}{dt}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} = -\begin{pmatrix}0 & ik\\ ik & \sigma\end{pmatrix}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} =: -\mathbf{C}_k\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix}, \qquad k \in \mathbb{Z}.$$

$\widehat{u}(0,t) = \widehat{u}_0, \quad \widehat{v}(0,t) = \widehat{v}_0 e^{-\sigma t}$



Case I: $0 < |k| < \frac{\sigma}{2}$: $\lambda_{\mp,k} := \frac{\sigma}{2} \mp \sqrt{\frac{\sigma^2}{4} - k^2}$,

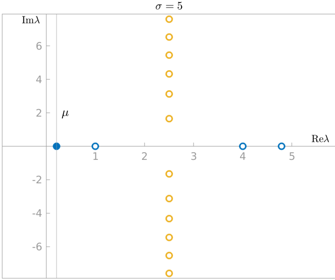
$$\mathbf{P}_k^{(I)} := \begin{pmatrix}1 & -\frac{2ki}{\sigma}\\ \frac{2ki}{\sigma} & 1\end{pmatrix} \sim \begin{pmatrix}1 & -\frac{2}{\sigma}\partial_x\\ \frac{2}{\sigma}\partial_x & 1\end{pmatrix}$$

Case II: $|k| = \frac{\sigma}{2}$ ... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$

Modal decomposition of (u, v) on torus for $\sigma = const$:

$$\frac{d}{dt}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} = -\begin{pmatrix}0 & ik\\ ik & \sigma\end{pmatrix}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} =: -\mathbf{C}_k\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix}, \qquad k \in \mathbb{Z}.$$

$\widehat{u}(0,t) = \widehat{u}_0, \quad \widehat{v}(0,t) = \widehat{v}_0 e^{-\sigma t}$



Case I: $0 < |k| < \frac{\sigma}{2}$: $\lambda_{\mp,k} := \frac{\sigma}{2} \mp \sqrt{\frac{\sigma^2}{4} - k^2}$,

$$\mathbf{P}_k^{(I)} := \begin{pmatrix}1 & -\frac{2ki}{\sigma}\\ \frac{2ki}{\sigma} & 1\end{pmatrix} \sim \begin{pmatrix}1 & -\frac{2}{\sigma}\partial_x\\ \frac{2}{\sigma}\partial_x & 1\end{pmatrix}$$

Case II: $|k| = \frac{\sigma}{2}$ ... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$

Case III: $|k| > \frac{\sigma}{2}$: $\Re\lambda_{\pm,k} := \frac{\sigma}{2}$, $\mathbf{P}_k^{(III)} := \begin{pmatrix}1 & -\frac{i\sigma}{2k}\\ \frac{i\sigma}{2k} & 1\end{pmatrix} \sim \begin{pmatrix}1 & \frac{\sigma}{2}\partial_x^{-1}\\ -\frac{\sigma}{2}\partial_x^{-1} & 1\end{pmatrix}$

Modal Lyapunov functional

<u>Situation A:</u> $0 < \sigma < 2$: all modes $k \neq 0$ in Case III

$$\mathcal{E}(\widehat{u},\widehat{v}) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \left\| \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} \right\|^2_{\mathbf{P}_k^{(III)}} + \left\| \begin{pmatrix} \widehat{u}(0) \\ \widehat{v}(0) \end{pmatrix} \right\|^2_2$$

$$= \underbrace{\|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right)dx}_{=:E_\sigma(\tilde{u},v)}, \quad \tilde{u}(x) := u(x) - u_{avg}$$

Modal Lyapunov functional

Situation A: $0 < \sigma < 2$: all modes $k \neq 0$ in Case III

$$\begin{aligned}\mathcal{E}(\widehat{u},\widehat{v}) \;&:=\; \sum_{k\in\mathbb{Z}\setminus\{0\}} \left\| \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} \right\|^2_{\mathbf{P}_k^{(III)}} + \left\| \begin{pmatrix} \widehat{u}(0) \\ \widehat{v}(0) \end{pmatrix} \right\|^2_2 \\ &=\; \underbrace{\|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right)dx}_{=:E_\sigma(\tilde{u},v)}, \quad \tilde{u}(x) := u(x) - u_{avg}\end{aligned}$$

Situation B: $\sigma = 2$: defective

Modal Lyapunov functional

Situation A: $0 < \sigma < 2$: all modes $k \neq 0$ in Case III

$$\mathcal{E}(\widehat{u}, \widehat{v}) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\| \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} \right\|^2_{\mathbf{P}_k^{(III)}} + \left\| \begin{pmatrix} \widehat{u}(0) \\ \widehat{v}(0) \end{pmatrix} \right\|^2_2$$

$$= \underbrace{\|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi} \int_0^{2\pi} \Re \left(\partial_x^{-1} \tilde{u} \overline{v} \right) dx}_{=: E_\sigma(\tilde{u}, v)}, \quad \tilde{u}(x) := u(x) - u_{avg}$$

Situation B: $\sigma = 2$: defective

Situation C: $\sigma > 2$: Low modes in Case I determine global decay rate, high modes in Case III.

$\Rightarrow$ Mixture would yield a (complicated) ΨDO with non-smooth symbol.

Mixed modes in Situation C ($\sigma > 2$)

remedy:

Only modes $k = \pm 1$ determine global decay rate $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$.
$\Rightarrow$ Higher modes gives some leeway to adjust the decay estimate.
$\Rightarrow$ Suboptimal norms $\|.\|_{\mathbf{P}_k}$ can be used for $|k| \geq 2$:

Lemma 2

Let $\sigma > 2$: $\mathbf{P}_k := \begin{pmatrix} 1 & -\frac{2i}{k\sigma} \\ \frac{2i}{k\sigma} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{2}{\sigma}\partial_x^{-1} \\ -\frac{2}{\sigma}\partial_x^{-1} & 1 \end{pmatrix}$ *satisfies*

$$\mathbf{C}_k^* \mathbf{P}_k + \mathbf{P}_k \mathbf{C}_k \geq 2\mu \mathbf{P}_k \quad \forall k \neq 0\,. \tag{2}$$

$\Rightarrow \{\mathbf{P}_k\}_{k\neq 0}$ is "good enough" for optimal decay, since $\mathbf{P}_1 = \mathbf{P}_1^{(I)}$, all modes satisfy (2) with uniform rate μ.
also: $\{\mathbf{P}_k\} \sim$ simple ΨDO.

Decay of spatial Lyapunov functional – with symbol $\mathbf{P}_k$

$$E_\sigma(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx\,, \quad \tilde{u}(x) := u(x) - u_{\mathrm{avg}}$$

Proposition 1

i) *If* $0 < \sigma < 2$: $E_\sigma\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_\sigma\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\sigma t}$.

ii) *If* $\sigma > 2$: $E_{\frac{4}{\sigma}}\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_{\frac{4}{\sigma}}\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\left(\sigma - \sqrt{\sigma^2 - 4}\right)t}$.

Decay of spatial Lyapunov functional – with symbol $\mathbf{P}_k$

$$E_\sigma(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx\,, \quad \tilde{u}(x) := u(x) - u_{\mathrm{avg}}$$

Proposition 1

i) *If* $0 < \sigma < 2$*:* $E_\sigma\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_\sigma\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\sigma t}$.

ii) *If* $\sigma > 2$*:* $E_{\frac{4}{\sigma}}\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_{\frac{4}{\sigma}}\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\left(\sigma - \sqrt{\sigma^2 - 4}\right)t}$.

Lemma 3

For $0 < \theta < 2$*:*

$$\left(1 - \frac{\theta}{2}\right)\left(\|f\|_2^2 + \|g\|_2^2\right) \le E_\theta(f, g) \le \left(1 + \frac{\theta}{2}\right)\left(\|f\|_2^2 + \|g\|_2^2\right) .$$

Decay of spatial Lyapunov functional – with symbol $\mathbf{P}_k$

$$E_\sigma(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx\,, \quad \tilde{u}(x) := u(x) - u_{\mathrm{avg}}$$

Proposition 1

i) *If* $0 < \sigma < 2$*:* $E_\sigma\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_\sigma\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\sigma t}$.

ii) *If* $\sigma > 2$*:* $E_{\frac{4}{\sigma}}\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \le E_{\frac{4}{\sigma}}\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\left(\sigma - \sqrt{\sigma^2-4}\right)t}$.

Lemma 3

For $0 < \theta < 2$*:*

$$\left(1 - \frac{\theta}{2}\right)\left(\|f\|_2^2 + \|g\|_2^2\right) \le E_\theta(f, g) \le \left(1 + \frac{\theta}{2}\right)\left(\|f\|_2^2 + \|g\|_2^2\right)\ .$$

Remark: The sharp rate from Proposition 1 can also be recovered from [Dolbeault-Mouhot-Schmeiser 2015] when optimizing a parameter $\delta(k)$.

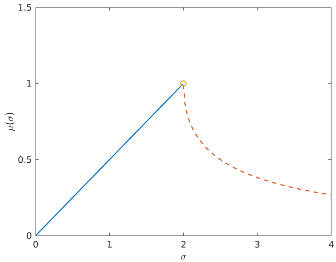
$\Rightarrow$ Decay in L^2 with sharp rate $\mu(\sigma)$:

Theorem 1 (AA-Einav-Signorello-Wöhrer, JSP 2021)

For $\sigma = const \neq 2$:

$$\left\| \begin{pmatrix} f_+(t) \\ f_-(t) \end{pmatrix} - \begin{pmatrix} f_\infty \\ f_\infty \end{pmatrix} \right\|_2 \leq C_\sigma \left\| \begin{pmatrix} f_{+,0} \\ f_{-,0} \end{pmatrix} - \begin{pmatrix} f_\infty \\ f_\infty \end{pmatrix} \right\|_2 e^{-\mu(\sigma)t} \;, t \geq 0 \;;$$

$$C_\sigma := \begin{cases} \sqrt{\frac{2+\sigma}{2-\sigma}}, & 0 < \sigma < 2 \\ \sqrt{\frac{\sigma+2}{\sigma-2}}, & \sigma > 2 \end{cases}, \qquad \mu(\sigma) := \begin{cases} \frac{\sigma}{2}, & 0 < \sigma < 2 \\ \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}, & \sigma > 2 \end{cases}$$



Extension to $\sigma(x)$ on torus

- modal decomposition unfeasible
- But $E_\theta(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\theta}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx$ is still a Lyapunov functional:

Theorem 2 (AA-Einav-Signorello-Wöhrer, JSP 2021)

For $0 < \sigma_{\min} \leq \sigma_{\max} < \infty$, *let* $\theta^* := \min\left(\sigma_{\min}, \frac{4}{\sigma_{\max}}\right)$. *Then*

$$E_{\theta^*}\left(u(t) - u_{\text{avg}}, v(t)\right) \leq E_{\theta^*}\left(u_0 - u_{\text{avg}}, v_0\right) e^{-\alpha^* t}, \quad t \geq 0$$

for an explicit $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$.

Extension to $\sigma(x)$ on torus

- modal decomposition unfeasible
- But $E_\theta(\tilde{u}, v) := \|\tilde{u}\|_2^2 + \|v\|_2^2 - \frac{\theta}{2\pi} \int_0^{2\pi} \Re\left(\partial_x^{-1} \tilde{u} \overline{v}\right) dx$ is still a Lyapunov functional:

Theorem 2 (AA-Einav-Signorello-Wöhrer, JSP 2021)

For $0 < \sigma_{\min} \le \sigma_{\max} < \infty$, *let* $\theta^* := \min\left(\sigma_{\min}, \frac{4}{\sigma_{\max}}\right)$. *Then*

$$E_{\theta^*}\left(u(t) - u_{\text{avg}}, v(t)\right) \le E_{\theta^*}\left(u_0 - u_{\text{avg}}, v_0\right) e^{-\alpha^* t}, \quad t \ge 0$$

for an explicit $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$.

- pros: Strategy is applicable to BGK models with n velocities; $n = 3$ in [AESW '21].
- cons: Decay rate α^* is not optimal.
  Sharp rate for $\sigma(x)$ in [Bernard-Salvarani 2013] is based on equivalence to telegrapher's equation; but only for $n = 2$.

Outline:

1. hypocoercive ODEs
2. Goldstein-Taylor model on torus
3. Goldstein-Taylor model on $\mathbb{R}$

Goldstein-Taylor on $\mathbb{R}$ with $\sigma \equiv 1$: modal decomposition

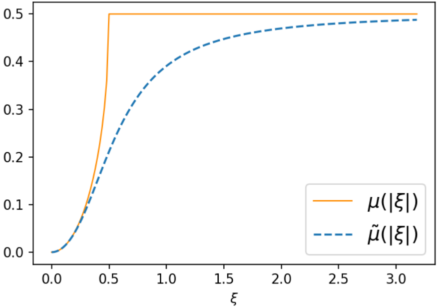
$$\begin{cases} \partial_t f_+(x,t) + \partial_x f_+(x,t) = \dfrac{\sigma}{2}(f_- - f_+), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) = -\dfrac{\sigma}{2}(f_- - f_+), \end{cases} \qquad t > 0,\ x \in \mathbb{R}$$

$$\frac{d}{dt}\begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} = -\begin{pmatrix} 0 & i\xi \\ i\xi & 1 \end{pmatrix}\begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} =: -\mathbf{C}_\xi \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix}, \qquad \xi \in \mathbb{R}.$$

Goldstein-Taylor on $\mathbb{R}$

3 problems:

- Decay rate $\mu(\xi)$ of small modes vanishes at $\xi = 0$.
  $\Rightarrow$ Global decay is only algebraic.



- Mixed mode–behavior, like Situation C on torus $\Rightarrow$ non-smooth ΨDO
- Defective case $|\xi| = \frac{\sigma}{2}$ is always present, cannot be skipped.
  $\Rightarrow$ Global approximation by $\tilde{\mu}(\xi)$ is one possible remedy.

Approximated modal decay function

- $\tilde{\mathbf{P}}(\xi) := \begin{pmatrix} 1 & -\frac{2\,i\,\xi}{1+4\,\xi^2} \\ \frac{2\,i\,\xi}{1+4\,\xi^2} & 1 \end{pmatrix}$ is a good approximation of the optimal scaling matrices $\mathbf{P}_\xi^{(I)}$ as $\xi \to 0$ and of $\mathbf{P}_\xi^{(III)}$ as $|\xi| \to \infty$.
- $\Rightarrow$ sharp modal spectral gap is replaced by $\tilde{\mu}(\xi) = \frac{1}{2} - \frac{1}{2\sqrt{1+4\,\xi^2\,(1+4\,\xi^2)}}$, avoids the defective case $|\xi| = \frac{\sigma}{2}$ at the kink of $\mu(\xi)$.

Approximated modal decay function

- $\tilde{\mathbf{P}}(\xi) := \begin{pmatrix} 1 & -\frac{2\,i\,\xi}{1+4\,\xi^2} \\ \frac{2\,i\,\xi}{1+4\,\xi^2} & 1 \end{pmatrix}$ is a good approximation of the optimal scaling matrices $\mathbf{P}_\xi^{(I)}$ as $\xi \to 0$ and of $\mathbf{P}_\xi^{(III)}$ as $|\xi| \to \infty$.
- $\Rightarrow$ sharp modal spectral gap is replaced by $\tilde{\mu}(\xi) = \frac{1}{2} - \frac{1}{2\sqrt{1+4\,\xi^2\,(1+4\,\xi^2)}}$, avoids the defective case $|\xi| = \frac{\sigma}{2}$ at the kink of $\mu(\xi)$.
- global Lyapunov functional on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(\widehat{u},\widehat{v}) := \int_{\mathbb{R}} \left\| \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} \right\|^2_{\tilde{\mathbf{P}}(\xi)} d\xi = \|u\|^2 + \|v\|^2 - 4 \int_{\mathbb{R}} u\,\partial_x \left(1 - 4\,\partial_x^2\right)^{-1} v\,dx$$

- Decay proof uses modal decay estimates for $\xi \neq 0$:

$$\left\| \begin{pmatrix} \widehat{u}(\xi,t) \\ \widehat{v}(\xi,t) \end{pmatrix} \right\|_2^2 \le \operatorname{cond}(\tilde{\mathbf{P}}(\xi))\, e^{-2\,\tilde{\mu}(\xi)\,t} \left\| \begin{pmatrix} \widehat{u}(\xi,0) \\ \widehat{v}(\xi,0) \end{pmatrix} \right\|_2^2 , \quad t \ge 0\,.$$

algebraic decay

Theorem 3 (AA-Dolbeault-Schmeiser-Wöhrer 2021)

Let $y := \binom{u}{v}$. *Then:*

$$\|y(x,t)\|^2_{\mathrm{L}^2(\mathbb{R})}$$
$$\leq \inf_{0<R\leq\frac{\sqrt{5}-1}{4}} \left(\frac{1+2\,R+4\,R^2}{1-2\,R+4\,R^2} \min\left\{2R, \sqrt{\frac{\pi}{2t}}\right\} \|y_0\|^2_{\mathrm{L}^1(\mathbb{R})} + 3e^{-2\tilde{\mu}(R)t} \|y_0\|^2_{\mathrm{L}^2(\mathbb{R})} \right)$$

- High modes (with $|\xi| > R$) decay exponentially in $L^2(\mathbb{R})$.
- Low modes (with $|\xi| < R$) decay algebraically, using $\|\widehat{y}(.,0)\|^2_{\mathrm{L}^\infty(\mathbb{R}_\xi)}$.

Conclusion

- Modal decomposition to obtain sharp decay estimates for modal ODEs
- Modal scaling matrices $\mathbf{P}_k$ hint to pseudo-differential Lyapunov functionals to be used with non-constant coefficients.
- Hypocoercivity on $\mathbb{R}$: splitting of low and high modes $\rightarrow$ algebraic decay.

Conclusion

- Modal decomposition to obtain sharp decay estimates for modal ODEs
- Modal scaling matrices $\mathbf{P}_k$ hint to pseudo-differential Lyapunov functionals to be used with non-constant coefficients.
- Hypocoercivity on $\mathbb{R}$: splitting of low and high modes $\rightarrow$ algebraic decay.

References

- A. Arnold, A. Einav, B. Signorello, T. Wöhrer: Large-time convergence of the non-homogeneous Goldstein-Taylor equation, J. Stat. Phys. (2021)
- A. Arnold, J. Dolbeault, C. Schmeiser, T. Wöhrer: Sharpening of decay rates in Fourier based hypocoercivity methods, to appear in INdAM proceedings (2021)