5. Non homogeneous fluxes.

$$\begin{cases} \partial_t f + a(x,v) \cdot \nabla_x f = g \\ \operatorname{div}_x a = 0 \end{cases}$$

Typically we consider here

$$a(x,v) = v + b(x,v)$$

with b small.

This can be intended

to more general $a(x,v)$ with $\nabla_v a$ is non degenerate and $\nabla_x a \ll \nabla_v a$.

One has the following theorem:

Th: Define

$$\delta = \| (1 + |\xi| + |\zeta|) \hat{b} \|_{L^1}$$

Assume $f \in L^\infty_t L^p_{x,v}$, $p \geq 2$

$g \in L^1_t L^q_{x,v}$, $\frac{1}{p} + \frac{1}{q} \leq 1$.

Then $\rho_\varphi \in L^2_t H^s_x$

for any $s < \frac{1}{2} - \delta$.

<u>Sketch of the proof:</u>

Define: $\hat{m}(\xi, \zeta)$

$$= \frac{\xi}{|\xi|} \cdot \frac{\zeta}{(1+|\zeta|^2)^{\sigma/2}}$$

for $\underline{\sigma < 1}$.

Note that:

$$\xi \cdot \nabla_\zeta \hat{m} \geq (1-\theta) \frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}}$$

$\longrightarrow$ Gain 1 derivative in x at the cost of 1 derivative in ζ.

$\leq |\zeta|^{1-\theta}$

But:

$$\frac{d}{dt} \frac{1}{2} \int \boxed{\hat{m}} \, |\hat{f}|^2 \, d\xi \, d\zeta$$

$$\geq (1-\theta) \int \frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}} |\hat{f}|^2 \, d\xi \, d\zeta$$

$$+ \boxed{\operatorname{Im} \int \hat{m}\, \xi \cdot \hat{b} * \hat{f}\, \hat{f}^{*} \, d\xi \, d\zeta}$$

$$+ \operatorname{Re} \int \hat{m}\, \hat{f}^{*} g \,.$$

$$= \frac{1}{2} \operatorname{Im} \int \boxed{\left(\hat{m}(\xi, \zeta) - \hat{m}(\xi', \zeta') \right)} \, \xi \cdot \hat{b}(\xi - \xi', \zeta - \zeta') \, \hat{f}(\xi', \zeta') \, \hat{f}^{*}(\xi, \zeta) \, d\xi \, d\zeta \,.$$

$$\leq \delta \int \left(|\zeta|^{1-\theta} + \underbrace{\frac{|\xi|}{\max(1, |\zeta|^{\theta})}} \right) |\hat{f}|^{2} \, d\xi \, d\zeta \,.$$

Hence:

$$(1-\theta-2\delta)\int \frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}} |\hat f|^2 \, d\xi \, d\zeta \, dt$$

$$\leq \int \left(1+\boxed{|\zeta|^{1-\theta}}\right)\left(|\hat f(t=0)|^2 + |\hat f(t=T)|^2\right) d\xi \, d\zeta$$

$$+ \delta \, \|f\|^2_{L^2_{t,x} \boxed{H_v^{(1-\theta)/2}}}$$

$$+ \mathrm{Re} \int \boxed{\hat m} \, \hat f^* g \, d\xi \, d\zeta .$$

$$\sim |\zeta|^{1-\theta}$$

To conclude, we need to regularize in v.

For this, consider

$$\hat{f}_r = \hat{f} \frac{1}{(1+|\xi|+|\zeta|)^r}$$

for some r to be chosen with $r < 1$.

Write the equation on $\hat{f}_r$.

Apply the previous to $\hat{f}_r$.

Deduce the regularity of $\hat{f}$ by losing r derivative from $\hat{f}_r$.

Great illustration of the method: It shows the advantage of such quantitative type of approach.

Still many open questions:

- What if b is not smooth?
- Other potential applications: Eqs with noise or stochastic terms.

6. An example of future work

Consider the Vlasov Eq:

$$\partial_t f + v \cdot \nabla_x f + K * \rho \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f$$

$$\rho = \int f(t,x,v)\, dv.$$

We consider repulsive kernels:

$$K = -\nabla V(|x|), \quad V' \leq 0.$$

Moreover V' has the precise behavior: $\frac{1}{C} \frac{1}{|x|^\alpha} \leq |V'| \leq \frac{C}{|x|^\alpha}.$

We have the following:

<u>Prop</u>: For any $\theta \in (0,1)$

$$\int_0^\infty \int \frac{|v-w|^2}{|x-y|^\theta} f(t,x,v)\, f(t,y,w)\, dv\, dw\, dx\, dy\, dt$$

$$+ \frac{1}{C_0} \int_0^\infty \int \frac{1}{|x-z|^{\alpha-1}} \frac{\rho(t,x)\rho(t,y)\rho(t,z)}{|x-y|^\theta + |z-y|^\theta}\, dx\, dy\, dz\, dt$$

$$\leq C \sup_t \int \left(1+|x|^2+|v|^2\right) f\, dx\, dv$$

Sketch of the proof:

Define: $L(x, v) = v \cdot \dfrac{x}{|x|^\theta}$.

so $v \cdot \nabla_x L \geq (1 - \theta) \dfrac{|v|^2}{|x|^\theta}$.

Now calculate:

$$\frac{d}{dt} \frac{1}{2} \int f \, L * f \, dx = \ldots$$

Corollaries:

1. If $\alpha < 1$, then we have the mean-field limit from:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum K(X_i - X_j) \end{cases}$$

with the only condition:

$$\frac{1}{N} \sum_i \left(|X_i^0|^2 + |V_i^0|^2 \right) < +\infty$$

Recovers H.D., but with diffusion.

2. Long term dispersion in Vlasov-Poisson, recovering B.D., without any regularity on the solution.

$$\rightarrow \int_0^\infty \int \frac{|v-w|^2}{|x-y|^\theta} f\, f(y,w)\, dt$$

$$\leq \sup_t \int (1+|v|^2+\ldots)\, f\, dx\, dv.$$