Plan of the lecture:

1. An introduction to averaging lemmas, and some classical results.

2. An example of application: Scalar conservation laws.

3. The basic case for the new method.

4. The L^p case: Combining with renormalization techniques.

5. Averaging lemmas in spatially inhomogeneous settings.

6. An example of commutator method in real variables and applications to mean-field limits.

1. Introduction

Consider a kinetic equation:

$$\partial_t f + a(v) \cdot \nabla_x f = (-\Delta_x)^{\theta} (-\Delta_v)^{\theta'} g.$$

on some $f(t, x, v)$.

Includes : - mean-field models : $(-\Delta_x)^{\sigma}(-\Delta_v)^{\sigma'} g$

$$= \mathrm{div}_v\left(K * \rho\, f\right)$$

$$\rho = \int f\, dv.$$

- collision models

$$(-\Delta_x)^{\sigma}(-\Delta_v)^{\sigma'} g$$

$$= Q[f]$$

Q : collision kernel.

- Kinetic formulation:

r.h.s. $\partial_v m$,

m = measure.

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As a 1st order transport eq., there cannot be any regularizing effect in general: For M. F,

$$f(t,x,v) = f^0\big(X(t,x,v),\; V(t,x,v)\big).$$

But averages in v can be smoother:

$$\rho_\varphi = \int f(t,x,v)\,\varphi(v)\,dv$$

enjoys a gain of regularity.

• G. L. P. S.

$$\partial_t f + v \cdot \nabla_x f = g$$

If $f \in L^2_{t,x,v\,\mathrm{loc}}$, $g \in L^2_{t,x,v\,\mathrm{loc}}$

then $\varsigma_\varphi \in H^{1/2}_{t,x,\mathrm{loc}}$.

But $\varsigma_\varphi \notin L^\infty_t H^{1/2}_x$

which is not compatible with time reversibility.

And $\zeta^0_\varphi \in H^{1/2}_x$ cannot be propagated by the eq.

In L^2, one can use Fourier transform: $f' = \mathcal{F}_{t,x} f$

$$g' = \mathcal{F}_{t,x}\, g$$

$$i\tau f' + i v \cdot \xi f' = g'.$$

with τ, ξ dual var. for t, x.

So $f' = \dfrac{g'}{i\tau + i v.\xi}$.

Perform interpolation in Fourier: $\cdot$ If $\tau + v.\xi \neq 0$ use the formula.

$\cdot$ If $\tau + v.\xi = 0$, use $f \in L^2$.

- D.L.M. (+ B., B.)

$$\partial_t f + v.\nabla_x f = (-\Delta_x)^{\theta} (-\Delta_v)^{\theta'} g$$

then if $f \in L^p_{t,x,v}$, $g \in L^q_{t,x,v}$

we have that $\zeta \in W^{s,r}_{t,x}$

for some $\underline{s > 0}$, $r \in [p, q]$

or $[q, p]$

iff $\theta < \frac{1}{2}$, $p > 1$.

Moreover $s < \frac{1}{2}$ only if

$p = q = 2$, $\theta = \theta' = 0$.

- D. L. : Weak solutions to Vlasov-Maxwell system.
- D. L. : Renormalized solutions to Boltzmann.
- G. S. for f equi-integrable in v.
- See also A. S., A. M. ...

2. Kinetic formulations for scalar conservation laws

S.C.L: $\quad u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$.

$$\partial_t u + \operatorname{div}_x \big(A(u) \big) = 0,$$

where $A : \mathbb{R} \to \mathbb{R}^d$.

Ex: Burgers in 1D:

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0.$$

In 1D, for convex fluxes,

u becomes BV_x for all $t>0$

(see O.).

$\to$ L.P.T, consider

$$f(t,x,\underset{\in \mathbb{R}}{v}) = \begin{cases} \mathbb{1}_{\{0 \le v \le u(t,x)\}} & u>0 \\ -\mathbb{1}_{\{u(t,x) \le v \le 0\}} & u<0. \end{cases}$$

If u is a smooth solution:

$$\partial_t f + a(v) \cdot \nabla_x f = 0$$

with $a = A'$.

u is a entropy solution

iff $\partial_t f + a(v) \cdot \nabla_x f = \partial_v m$

with m a non-negative

measure.

Recovers Kruzhkov theory (see P.).

We can use averaging lemma

to deduce some regularity

on u.

Assume that $u \geq 0$, $u \in L^\infty$,

and $\left| \left\{ v \,/\, \left| \tau + a(v) \cdot \frac{\xi}{|\xi|} \right| \leq \varepsilon \right\} \right|$

$\leq C \cdot \varepsilon$.

Then $u \in B_r^{1/3, 3}$.

(initially $W^{s,3}$, $\forall s < \frac{1}{3}$, ..., G. P. S.).

Only uses m is a measure.

3. Our new method: the basic idea.

$$\partial_t f + v \cdot \nabla_x f = g.$$

where $f \in L^\infty_t L^p_{x,v}$

$g \in L^1_t L^q_{x,v}$

with $\frac{1}{p} + \frac{1}{q} = 1$ (≤ 1 because all is local).

Find some $K(x,v)$, and look at:

$$\frac{d}{dt} \frac{1}{2} \int f \; K * f \; dx\, dv$$

$$= \int (v-w) \cdot \nabla_x K(x-y, v-w) \, f(x,v) \, f(y,w)$$

$$+ \int f \; K * g.$$

Assume f, g have compact support.

$$\int f \, K * f \leq C \|f\|^2_{L^\infty_t L^2_{x,v}}$$

if $\hat{K} = \mathcal{F}_{x,v} K \in L^\infty$.

$$\int f \, K * g \leq C \|f\|_{L^\infty_t L^p_{x,v}} \|g\|_{L^1_t L^q_{x,v}}$$

if K is Calderon-Zygmund.

$$\widehat{v \cdot \nabla_x K} = \xi \cdot \nabla_\zeta \hat{K}.$$

Take: $\hat{K} = \dfrac{\xi}{|\xi|} \cdot \dfrac{\zeta}{(1+|\zeta|^2)^{1/2}}$.

$$\xi \cdot \nabla_\zeta \hat{K} \geqslant \frac{|\xi|}{(1+|\zeta|^2)^{3/2}}.$$

Hence:

$$\boxed{\begin{aligned}&\int |\hat{f}|^2 \frac{|\xi|}{(1+|\zeta|^2)^{1/2}}\, d\xi\, d\zeta\, \boxed{dt} \\ &\leq C \left(\|f\|^2_{L^\infty_t L^2_{x,v}} + \|f\|_{L^\infty_t L^p}\right.\end{aligned}}$$

$$\|g\|_{L^1_t L^q} \Big) .$$

$q < 2$. .

Rks:

1. This is a dispersive effect in Fourier:

$$\widehat{v \cdot \nabla_x} = - \xi \cdot \nabla_\xi$$

2. We obtain a sort

of dissipative estimate that is compatible with time reversibility:

$$\frac{d}{dt} \frac{1}{2} \int f \, K * f = \int f \, K * g$$

$+$ positive.

But $K(x, -v) = -K(x, v)$

(still $K(-x, -v) = K(x, v)$).

3. Very simple proof of the classical averaging in $H^{1/2}$.

But if $f \in L^\infty_t L^p_{x,v}$ with $p \geq 2$ $(p < \infty)$

and $g \in L^1_t L^q_{x,v}$, $\frac{1}{p} + \frac{1}{q} \leq 1$

then locally in x, v,

$$f \in L^2_t H^{1/2}_x H^{-1}_v \ldots$$

4. Compatible with some

hydrodynamic limits: The estimate works for

$$\varepsilon \partial_t f + v \cdot \nabla_x f = g.$$

(see M. T.)

4. The more general L^p case.

Consider now the more general:

$$\partial_t f + a(v) \cdot \nabla_x f = (-\Delta_v)^{\frac{\alpha}{2}} g,$$

for $\alpha \geq 0$, $f \in L^\infty_t L^p_{x,v}$, $g \in L^1_t L^q_{x,v}$ with not necessarily $\frac{1}{p} + \frac{1}{q} \leq 1$.

Main issue: Bound

$$\int f \, K * \left((-\Delta_v)^{\alpha/2} g \right) ?$$

$\rightarrow$ Because of $(-\Delta_v)^{\alpha/2}$

$\rightarrow$ Because maybe

$$\frac{1}{p} + \frac{1}{q} > 1.$$

<u>Th.</u>: Assume that:

- a is one-to-one with $\det Da^{-1} \in L^\infty$ (could be extended to L^γ)
- f and g are as above

Then for any smooth φ,

$$\boxed{S_\varphi \in L^2_t H^{s}_x} \text{ with :}$$

$$s = \min\left(1, \frac{1-\mu+\nu}{\alpha+1+\mu-\nu}\right)\cdot\frac{1-\nu}{2}-\frac{\mu}{2}.$$

$$\mu = \max\left(0, d\left(\frac{1}{p}-\frac{1}{q}-1\right)\right)$$

$$\nu = \max\left(0, d\left(\frac{2}{p}-1\right)\right).$$

<u>Rk</u> : 1. The formulas are simpler if $p \geq 2$, as

then $\nu = 0$. If $\frac{1}{p} + \frac{1}{q} \leq 1$

then one also has $\mu = 0$, and

$$s = \frac{1}{2} \frac{1}{\alpha + 1}.$$

This is similar to A. L.

as they are studying the

case where $f \in L^\infty_t L^p_x W^{\gamma,p}_v$,

$g \in L^1_t L^q_x W^{\gamma',q}_v$ with

possibly $\gamma, \gamma' \geq 0$.

→ Hypoellipticity.

But with $\frac{1}{p} + \frac{1}{q} = 1$.

2. Still compatible with hydrodynamic limits...

<u>One example of application:</u>

<u>Measure-valued solutions to SCL</u>

Introduced by DiPerna, they offer a sort of "statistical" description of possible oscillations in scalar conservation laws.

<u>Def</u>: $u(t,x)$ is a measure-valued solution iff $u = \int f \, dv$, $f(t,x,v)$ with

$$\partial_t f + a(v)\cdot\nabla_x f = \partial_v m$$

for m a bounded measure.

⚠ We do not impose that

$$f = \mathbb{1}_{\{0 \le v \le u(t,x)\}}$$

for $u \geq 0$. Instead f can be anything between 0 and 1.

With previous averaging lemmas, if a is not degenerate in the appropriate sense, then $u \in B^{1/5}_{5/3,\infty}$ (if $u \in L^\infty$, f has compact support).

The new result implies that $u \in L^2_t H^s_x$ for any $s < \frac{1}{4}$.

Sketch of the proof (very rough):

The main idea is to use some renormalization.

Denote by $\tilde{f} = \mathcal{F}_x f$

$\tilde{g} = \mathcal{F}_x g$

Then:

$$\partial_t \tilde{f} + i v . \xi \tilde{f} = (-\Delta_v)^{\alpha/2} \tilde{g} .$$

Define:

$$\phi(\xi, v) = \left(2^{k\theta}\right)^{d} \phi_0\left(\frac{v}{2^{k\theta}}\right)$$

when $2^{k-1} \leq |\xi| < 2^{k}$.

Now look at $F = \tilde{f} *_v \phi$

and apply the method to F.

$$\partial_t F + i\, v \cdot \xi\, F = (-\Delta_v)^{\alpha/2} G$$

$+$ Commutator.

And we optimize in θ.

5. Inhomogeneous settings.

Consider now :

$$\partial_t f + a(x, v) \cdot \nabla_v f = g.$$

For simplicity, we do not consider extra derivatives in g, and assume duality :

$f \in L^\infty_t L^p_x$, $g \in L^1_t L^q_x$

with $p \geq 2$, $\frac{1}{p} + \frac{1}{q} \leq 1$.

Finally $\mathrm{div}_x\, a(x, v) = 0$.

For S.C.L, one may want to have spatial inhomogeneities: Initially the flux $A(x, u)$ depends explicitly on x.

Then naturally $a = a(x, v)$

as $a = \frac{\partial}{\partial u} A$.

A first question would be how to get well-posedness for a that is singular in x.

Instead here, we want to know what remains of the gain in regularity provided by averaging lemmas.

So far nothing...