On the cubic-quintic Schrödinger equation

Rémi Carles

CNRS & Univ Rennes

Based on a joint work with
Christof Sparber (Univ. Illinois)

Cubic Schrödinger equation in 2D

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = \lambda |u|^2 u, \quad x \in \mathbb{R}^d,$$

with $\lambda \in \mathbb{R}$.

$\rightsquigarrow$ Appears in various physical contexts: optics, superfluids, BEC, etc.

$\rightsquigarrow$ Often, cubic nonlinearity stems from Taylor expansion: $f(|u|^2)u$.

Conserved quantities:

$$\text{Mass:} \quad M = \|u(t)\|^2_{L^2(\mathbb{R}^d)},$$

$$\text{Angular momentum:} \quad J = \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx,$$

$$\text{Energy:} \quad E = \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \|u(t)\|^4_{L^4(\mathbb{R}^d)}.$$

$\rightsquigarrow$ The sign of λ plays a role at the level of the energy. . . but not only.

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Well-posedness

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = \lambda |u|^2 u, \quad x \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}.$$

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Impose $u_{|t=0} = u_0$.

- $d = 1$: $u_0 \in L^2 \rightsquigarrow u \in C(\mathbb{R}; L^2)$, higher regularity propagated (Tsutsumi 1987).
- $d = 2$: $u_0 \in L^2, \lambda > 0 \rightsquigarrow u \in C(\mathbb{R}; L^2)$, higher regularity propagated (Dodson 2015).
- $d = 3$: $u_0 \in H^1$, $\lambda > 0 \rightsquigarrow u \in C(\mathbb{R}; H^1)$, higher regularity propagated (Ginibre & Velo 1979).

If $\lambda < 0$ and $d \geqslant 2$, finite time blow-up is possible.

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Finite time blow-up

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = -|u|^2 u, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0.$$
$$E = \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \|u(t)\|^4_{L^4(\mathbb{R}^d)}.$$

Theorem (Zhakharov 1972, Glassey 1977)

Suppose $d \geqslant 2$ and $u_0 \in H^1 \cap \mathcal{F}(H^1)$. If $E < 0$, then

$$\exists T_\pm > 0, \quad \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm T_\pm}{\longrightarrow} \infty.$$

Proof.

The map $t \mapsto \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx$ is C^2 as long as u is H^1, and

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx \leqslant 2E.$$

Finite time blow up (continued)

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Gagliardo-Nirenberg:

$$\|u\|^4_{L^4(\mathbb{R}^d)} \leqslant C\|u\|^{4-d}_{L^2(\mathbb{R}^d)}\|\nabla u\|^d_{L^2(\mathbb{R}^d)}.$$

$\rightsquigarrow$ No blow-up if $d = 1$.

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Best constant? M. Weinstein 1983,

$$\|u\|^4_{L^4(\mathbb{R}^2)} \leqslant \left(\frac{\|u\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}} \right)^2 \|\nabla u\|^2_{L^2(\mathbb{R}^2)},$$

where Q is the unique positive, radial solution to

$$-\frac{1}{2}\Delta Q + Q = Q^3, \quad x \in \mathbb{R}^2.$$

$\rightsquigarrow$ If $\|u_0\|_{L^2} < \|Q\|_{L^2}$, GWP.
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Special solution $u(t,x) = e^{i\omega t}\phi(x)$:

$$-\frac{1}{2}\Delta\phi + \omega\phi = |\phi|^2\phi.$$

A priori estimates (Pohozaev identities):

$$\frac{1}{2}\|\nabla\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - \|\phi\|_{L^4}^4 = 0 \quad (\text{multiplier } \bar{\phi}),$$
$$\omega\|\phi\|_{L^2}^2 = \frac{1}{2}\|\phi\|_{L^4}^4 \quad (\text{multiplier } x\cdot\nabla\bar{\phi}).$$

$\rightsquigarrow$ Nec. $\omega > 0$. Conversely, $\exists\ H^1$ solution if $\omega > 0$, with exponential decay.

$\rightsquigarrow$ Any solution satisfies $E(\phi) = 0$: instability by blow-up.

Solitary waves

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⇝ Defocusing quintic term: stabilize 2D and 3D solitons (optics, BEC)..?

⇝ GWP in $H^1(\mathbb{R}^d)$, for $d \leqslant 3$ (X. Zhang 2006 for $d = 3$).

⇝ **Caution:** two notions of orbital stability!

⇝ In 1D, explicit solitary waves, for $0 < \omega < \frac{3}{16}$,

$$\phi(x) = 2\sqrt{\frac{\omega}{1 + \sqrt{1 - \frac{16\omega}{3}}\cosh\left(2x\sqrt{2\omega}\right)}}.$$

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2D cubic-quintic case: small mass dispersion

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = -|u|^2 u + |u|^4 u, \quad x \in \mathbb{R}^2, \quad u_{|t=0} = u_0.$$

Theorem

Let $d = 2$ and $u_0 \in \Sigma = \left\{ f \in H^1(\mathbb{R}^2),\ x \mapsto x u_0(x) \in L^2(\mathbb{R}^2) \right\}$. If $\|u_0\|_{L^2} \leqslant \|Q\|_{L^2}$, then u is asymptotically linear,

$$\exists u_\pm \in \Sigma, \quad \|e^{-i\frac{t}{2}\Delta} u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

⇝ Not surprising if $\|u_0\|_{L^2} < \|Q\|_{L^2}$: X. Cheng 2019, in $H^1(\mathbb{R}^2)$.

⇝ Hint: virial computation.

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2} |x|^2 |u(t,x)|^2 dx = 2E(u) + \frac{4}{3}\|u(t)\|^6_{L^6(\mathbb{R}^2)} \geqslant 2E(u_0) \geqslant \frac{2}{3}\|u_0\|^6_{L^6(\mathbb{R}^2)}.$$

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Dispersion: Strichartz estimates

2D admissible pairs: $\frac{2}{q} + \frac{2}{r} = 1, \quad 2 < q \leqslant \infty.$

$$\|e^{i\frac{t}{2}\Delta} f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^2))} \leqslant C_q \|f\|_{L^2(\mathbb{R}^2)},$$

$$\left\| \int_0^t e^{i\frac{t-s}{2}\Delta} F(s) ds \right\|_{L^{q_1}(I;L^{r_1}(\mathbb{R}^2))} \leqslant C_{q_1,q_2} \|F\|_{L^{q_2'}(I;L^{r_2'}(\mathbb{R}^2))}.$$

LWP & GWP: we know that $u \in L^q_{\mathrm{loc}}(\mathbb{R}; L^r(\mathbb{R}^2))$.
Asymptotically linear behavior: prove $u \in L^q(\mathbb{R}; L^r(\mathbb{R}^2))$.
$\rightsquigarrow$ Classically obtained thanks to:

- Bootstrap argument (small data).
- A priori estimates:
  - Pseudo-conformal conservation law.
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Pseudo-conformal conservation law

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\Delta u=-|u|^2u+|u|^4u,\quad x\in\mathbb{R}^2,\quad u_{|t=0}=u_0.$$

$$\frac{d}{dt}\left(\frac{1}{2}\|\underbrace{(x+it\nabla)u}_{=:J(t)u}\|_{L^2}^2-\frac{t^2}{2}\|u\|_{L^4}^4+\frac{t^2}{3}\|u\|_{L^6}^6\right)=-\frac{2t}{3}\|u\|_{L^6}^6.$$

Standard factorization: $J(t)u=it\,e^{i|x|^2/(2t)}\nabla\left(ue^{-i|x|^2/(2t)}\right)$.

Sharp Gagliardo–Nirenberg inequality:

$$\|u(t)\|_{L^4(\mathbb{R}^2)}^4\leqslant\frac{1}{t^2}\left(\frac{\|u(t)\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}}\right)^2\|(x+it\nabla)u\|_{L^2(\mathbb{R}^2)}^2.$$

$\rightsquigarrow$ If $\|u_0\|_{L^2}^2<\|Q\|_{L^2}^2$, then $Ju\in L_t^\infty L_x^2$: OK.

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$$\frac{d}{dt}\left(\frac{1}{2}\|\underbrace{(x+it\nabla)u}_{=:J(t)u}\|_{L^2}^2 - \frac{t^2}{2}\|u\|_{L^4}^4 + \frac{t^2}{3}\|u\|_{L^6}^6\right) = -\frac{2t}{3}\|u\|_{L^6}^6.$$

Standard factorization: $J(t)u = it\, e^{i|x|^2/(2t)}\nabla\left(ue^{-i|x|^2/(2t)}\right)$.
Sharp Gagliardo–Nirenberg inequality:

$$\|u(t)\|_{L^4(\mathbb{R}^2)}^4 \leqslant \frac{1}{t^2}\left(\frac{\|u(t)\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}}\right)^2 \|(x+it\nabla)u\|_{L^2(\mathbb{R}^2)}^2.$$

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We infer $\|u\|_{L^6}^6\lesssim\frac{1}{1+t^2}$: we cannot assert $u\in L_t^3L_x^6$ (Strichartz).

Remark

$i\frac{\partial u}{\partial t}+\frac{1}{2}\Delta u=-|u|^2u+|u|^3u$: $\|u\|_{L^5}^5\lesssim\frac{1}{1+t^2}\rightsquigarrow u\in L_t^{10/3}L_x^5$!

$\rightsquigarrow$ Way out: conformal transform and rigidity properties for the mass-critical blow-up.

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Conformal transform

$$\psi(t,x) = \frac{1}{t} u\left(\frac{-1}{t}, \frac{x}{t}\right) e^{i|x|^2/(2t)}, \quad t \neq 0.$$

Problem at infinite time for u = problem at $t = 0$ for ψ.

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By assumption, $\|\psi(t)\|_{L^2} = \|u_0\|_{L^2} = \|Q\|_{L^2}$. It suffices to show $Ju \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2))$ (stronger property than what we need). We argue by contradiction: Suppose on the contrary that

$$\|J(t_n)u\|_{L^2(\mathbb{R}^2)} \underset{n\to\infty}{\longrightarrow} \infty \quad \text{for some } t_n \to \infty.$$

This is equivalent to:

$$\|\nabla\psi(\tau_n, \cdot)\|_{L^2(\mathbb{R}^2)} = \left\| J\left(\frac{-1}{\tau_n}\right) u \right\|_{L^2(\mathbb{R}^2)} \underset{n\to\infty}{\longrightarrow} \infty, \quad \tau_n := \frac{-1}{t_n} \underset{n\to\infty}{\longrightarrow} 0^-.$$

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Rigidity

Based on the approach of Keraani-Hmidi (rewriting Merle's proof),

$$\rho_n e^{i\theta_n}\psi(\tau_n, \rho_n x + x_n) \underset{n\to\infty}{\longrightarrow} Q(x) \text{ in } H^1(\mathbb{R}^2).$$

We then show:

- $(x_n)_n$ is bounded, because $\int |x|^2|\psi(t,x)|^2dx \lesssim 1$, hence $x_{n'} \to \underline{x}$,
- A priori estimate (based on virial and V. Banica's trick): $\int |x-\underline{x}|^2|\psi(t,x)|^2dx \lesssim t^2$,
- Hence (uncertainty principle) $\|\nabla\psi(t)\|_{L^2} \gtrsim \frac{1}{t}$.

Therefore, $|\rho_n| \lesssim |\tau_n|$, and we obtain a contradiction with the a priori property $\|u(t)\|_{L^6}^6 \lesssim \frac{1}{1+t^2}$.

Stability of solitary waves: two notions

$$-\frac{1}{2}\Delta\phi + \omega\phi - |\phi|^2\phi + |\phi|^4\phi = 0, \quad \phi \in H^1(\mathbb{R}^d) \setminus \{0\}.$$

Definition (First notion)

Action: $S(\phi) = \frac{1}{2}\|\nabla\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - \frac{1}{2}\|\phi\|_{L^4}^4 + \frac{1}{3}\|\phi\|_{L^6}^6 = E + \omega M.$

Ground state: $S(\phi) \leqslant S(\varphi)$ for any solution φ.

The standing wave $e^{i\omega t}\phi(x)$ is orbitally stable in $H^1(\mathbb{R}^d)$, if $\forall \varepsilon > 0$, $\exists \delta > 0$,

$$\|u_0 - \phi\|_{H^1} \leqslant \delta \Longrightarrow \sup_{t\in\mathbb{R}} \inf_{\substack{\theta\in\mathbb{R} \\ y\in\mathbb{R}^d}} \left\| u(t,\cdot) - e^{i\theta}\phi(\cdot - y) \right\|_{H^1(\mathbb{R}^d)} \leqslant \varepsilon.$$

Otherwise, the standing wave is said to be unstable.

Galilean invariance: $\tilde{u}(t,x) = e^{iv\cdot x - i|v|^2 t/2} e^{i\omega t}\phi(x - vt)$ solution, $\forall v \in \mathbb{R}^d$.
$\|u(0) - \tilde{u}(0)\|_{H^1} \lesssim |v|$, but $\|u(t) - \tilde{u}(t)\|_{H^1} \geqslant \|\phi\|_{H^1}$ for $t \gg 1$.

Stability of solitary waves: two notions

Definition (Second notion)

For $\rho > 0$, denote $\Gamma(\rho) = \left\{u \in H^1(\mathbb{R}^d),\ M(u) = \rho\right\}$, and assume that the minimization problem

$$(1) \qquad u \in \Gamma(\rho), \quad E(u) = \inf\{E(v)\ ;\ v \in \Gamma(\rho)\}$$

has a solution. Denote by $\mathcal{E}(\rho)$ the set of such solutions. We say that solitary waves are $\mathcal{E}(\rho)$-orbitally stable, if $\forall \varepsilon > 0$, $\exists \delta > 0$,

$$\inf_{\phi \in \mathcal{E}(\rho)} \|u_0 - \phi\|_{H^1} \leqslant \delta \Longrightarrow \sup_{t \in \mathbb{R}} \inf_{\phi \in \mathcal{E}(\rho)} \|u(t) - \phi\|_{H^1(\mathbb{R}^d)} \leqslant \varepsilon.$$

Lagrange: an element of $\mathcal{E}(\rho)$ solves

$$-\frac{1}{2}\Delta\phi + \omega\phi - |\phi|^2\phi + |\phi|^4\phi = 0, \quad \phi \in H^1(\mathbb{R}^d) \setminus \{0\}$$

for some $\omega \in \mathbb{R}$.

Stability of solitary waves

For a given ω, uniqueness results are available for a large class of nonlinearities (positive, radial solutions).

However, it is not known in general:

- Does a ground state belong to $\mathcal{E}(\rho)$, where ρ denotes its mass? In particular, it is not even clear that the first notion is stronger than the second.
- If solitary waves are $\mathcal{E}(\rho)$-orbitally stable, and if the ground state belongs to $\mathcal{E}(\rho)$ but is unstable, what is the nature of the instability?

When the nonlinearity is homogeneous, the two notions are known to be equivalent, and instability well understood (blow-up or dispersion).

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Orbital stability

Two methods of proof:

- Cazenave-Lions 1982: based on concentration-compactness property. Second notion.
- Grillakis-Shatah-Strauss 1987, after M. Weinstein : coercivity of the action. First notion. Typically, up to spectral assumptions (of the linearized operator about the ground state),
  - If $\frac{\partial}{\partial\omega}\|\phi\|_{L^2}^2 > 0$, then orbital stability holds.
  - If $\frac{\partial}{\partial\omega}\|\phi\|_{L^2}^2 < 0$, then instability holds.

2D cubic-quintic case: existence of solitary waves

Theorem

Let $d = 2$. For all $\omega \in]0, \frac{3}{16}[$, $\exists$ solution $u(t,x) = e^{i\omega t}\phi(x)$.

1. *For any $M > \|Q\|_{L^2}^2$, $\exists$ a ground state such that $\|\phi\|_{L^2}^2 = M$.*
2. *The ground state solution is unique, up to translation and multiplication by $e^{i\theta}$, for constant $\theta \in \mathbb{R}$.*

Remark

In 3D, existence for the same range of ω, minimal mass not explicit: Killip, Oh, Pocovnicu, Vişan 2017.

Ground state

$$-\frac{1}{2}\Delta\phi - |\phi|^2\phi + |\phi|^4\phi + \omega\phi = 0.$$

$\rightsquigarrow$ Pohozaev: necessarily, $\omega > 0$.

Berestycki-Gallouët-Kavian 1983: $F(s) = \frac{1}{4}s^4 - \frac{1}{6}s^6$.

Existence+exponential decay for $0 < \omega < \omega_*$,

$$\omega^* = \sup\left\{\omega > 0; \quad \frac{\omega}{2}s^2 - F(s) < 0 \text{ for some } s > 0\right\}.$$

Direct computation: $\omega_* = 3/16$.

Uniqueness of positive radial ground state: J. Jang 2010.

A consequence of Pohozaev:

$$\int_{\mathbb{R}^2} |\phi|^6 = \frac{3(\gamma - 1)}{4}\int_{\mathbb{R}^2} |\nabla\phi|^2, \quad \gamma := \frac{\|\phi\|_{L^4}^4}{\|\nabla\phi\|_{L^2}^2}.$$

$\rightsquigarrow$ $\gamma > 1$, hence (sharp Gagliardo-nirenberg inequality) $\|\phi\|_{L^2} > \|Q\|_{L^2}$.

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Asymptotic $\omega \to 0$

To prove that $\|\phi_\omega\|_{L^2} - \|Q\|_{L^2} > 0$ is arbitrarily small, let

$$\psi_\omega(x) = \frac{1}{\sqrt{\omega}}\phi_\omega\left(\frac{x}{\sqrt{\omega}}\right).$$

Regular limit $\omega \to 0$ in terms of ψ:

$$-\frac{1}{2}\Delta\psi_\omega + \psi_\omega - \psi_\omega^3 + \omega\psi_\omega^5 = 0,$$

One can check:

- $\omega \mapsto \phi_\omega$ is analytic.
- $\psi_\omega \to Q$ in $H^1(\mathbb{R}^2)$ as $\omega \to 0$.
- $\|\phi_\omega\|_{L^2(\mathbb{R}^2)} = \|\psi_\omega\|_{L^2(\mathbb{R}^2)}$.

Remark

As $\omega \to 3/16$, $\|\phi_\omega\|_{L^2} \to \infty$: otherwise, bounded in H^1 (Pohozaev) + radial + definition of $\omega_* \Longrightarrow \|\phi_\omega\|_{L^2} \to 0$.

Orbital stability

In 1D, orbital stability of ground states (first notion): Ohta 1995, thanks to an explicit formula derived by Iliev & Kirchev 1993.

In 2D, for $\rho > \|Q\|_{L^2}^2$, solitary wave are $\mathcal{E}(\rho)$- orbitally stable. First,

$$\inf \left\{ E(u) ;\, u \in H^1(\mathbb{R}^2),\, M(u) = \rho \right\} < 0.$$

$$u_\lambda(x) = \lambda u(\lambda x) : \quad E(u_\lambda) = \frac{\lambda^2}{2}\left(\|\nabla u\|_{L^2}^2 - \|u\|_{L^4}^4 + \frac{2}{3}\lambda^2 \|u\|_{L^6}^6 \right).$$

Pick $u \in H^1$ so that $\|\nabla u\|_{L^2}^2 - \|u\|_{L^4}^4 < 0$, e.g. $u = \left(\frac{\rho}{M(Q)}\right)^{1/2} Q$.
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Extensions and (more) open questions

- 2D case: it is expected that ground states are orbitally stable (GSS+numerical simulation, see e.g. Lewin-Rota Nodari 2020).
- 3D case:
  - Solitary wave are $\mathcal{E}(\rho)$-orbitally stable for ρ sufficiently large.
  - There exists $0 < \omega_0 < \frac{3}{16}$ such that for $0 < \omega < \omega_0$, ϕ_ω is unstable.
  - There exists $\omega_0 \leqslant \omega_1 < \frac{3}{16}$ such that for all $\omega_1 < \omega < \frac{3}{16}$, ϕ_ω is orbitally stable.
  - Conjecture (from numerics, Killip, Oh, Pocovnicu, Vișan 2017, Lewin-Rota Nodari 2020): $\omega_0 = \omega_1$.
  - Nature of the instability?
- What if we add an external potential, e.g. harmonic?

$$-\frac{1}{2}\Delta\phi + \frac{|x|^2}{2}\phi - |\phi|^2\phi + |\phi|^4\phi + \omega\phi = 0.$$

$\mathcal{E}(\rho)$ for ρ sufficiently large: OK. Range for ω? Ground state?