UNIVERSITÉ Grenoble Alpes

INSTITUT FOURIER

Nonlinear Quantum Adiabatic Approximation*

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* joint with Clotilde Fermanian

Nonlinearity

CIRM, Sept. 14-18, 2020

Adiabatic Quantum Approximation

- Time dependent Hamiltonian $[0,1] \ni t \mapsto H(t) \in \mathscr{L}(\mathscr{H})$

- Slow time scale $1/\varepsilon$ $i\varepsilon\partial_t\psi^\varepsilon(t) = H(t)\psi^\varepsilon(t), \quad \psi^\varepsilon(0) = \psi_0$

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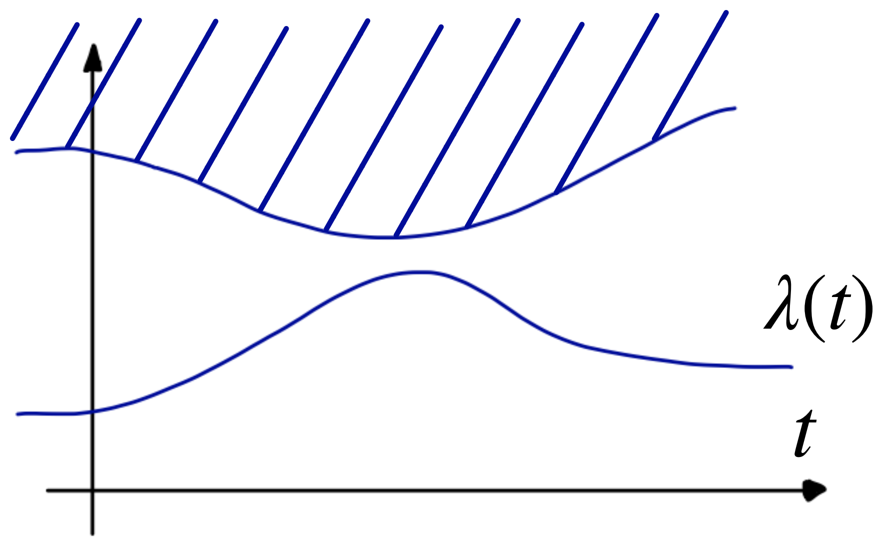

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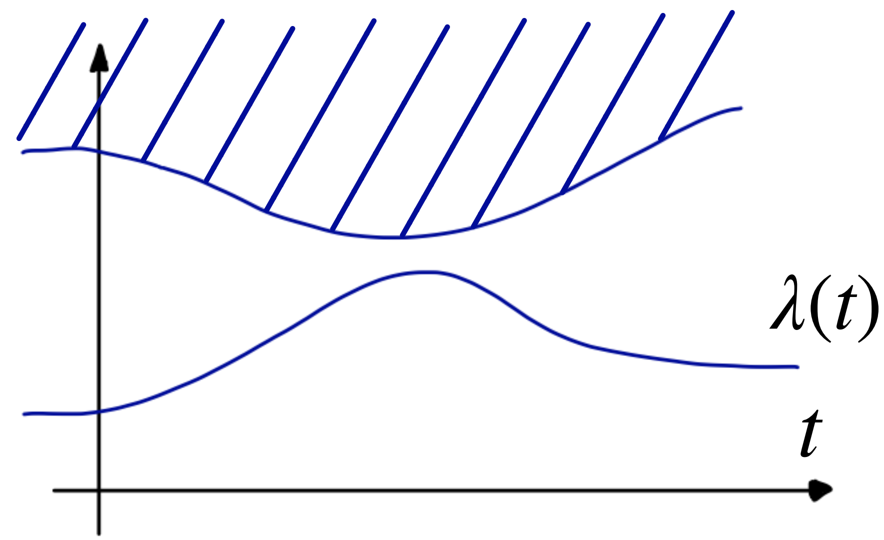

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$$\langle \varphi(t) \,|\, \partial_t \varphi(t) \rangle \equiv 0,$$

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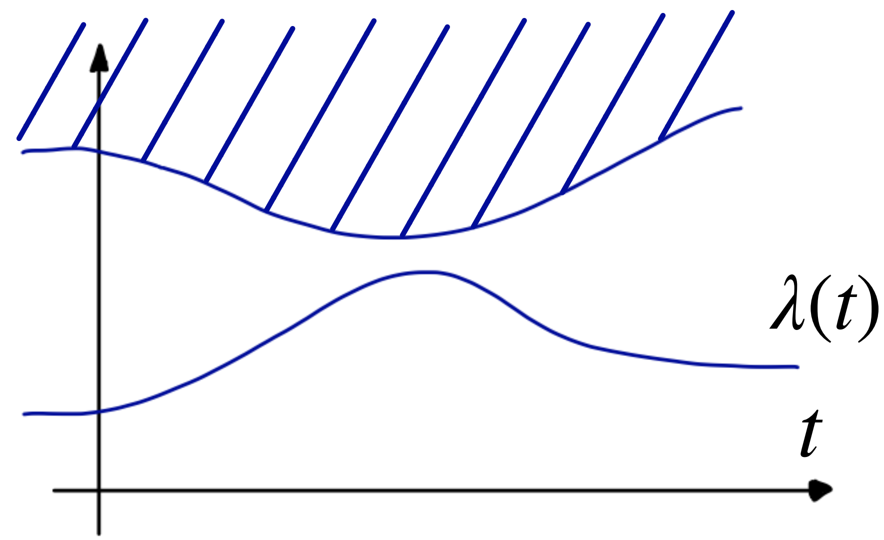

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Born & Fock '28, Kato '50, ...

Thm:

$$\psi^\varepsilon(0) = \varphi(0) \;\Rightarrow\; \psi^\varepsilon(t) = e^{-i\int_0^t \lambda(s)ds/\varepsilon}\varphi(t) + \mathcal{O}(\varepsilon), \;\; \forall\, t \in [0,1]$$

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- Tunnelling of BEC in double well potentials

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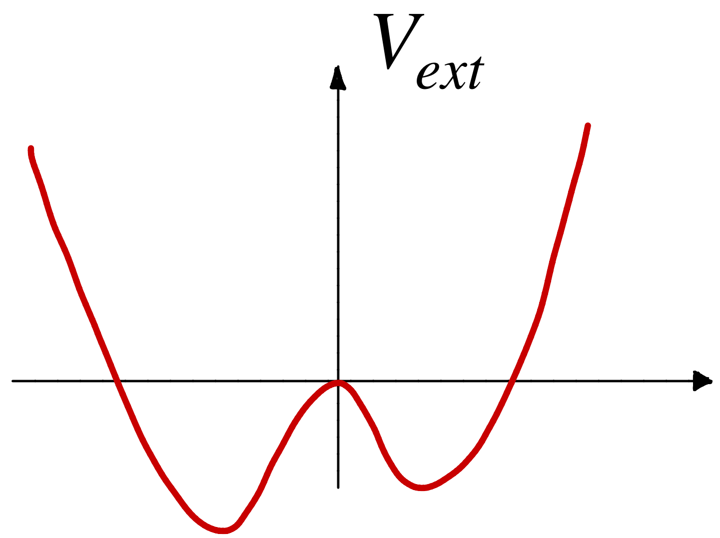

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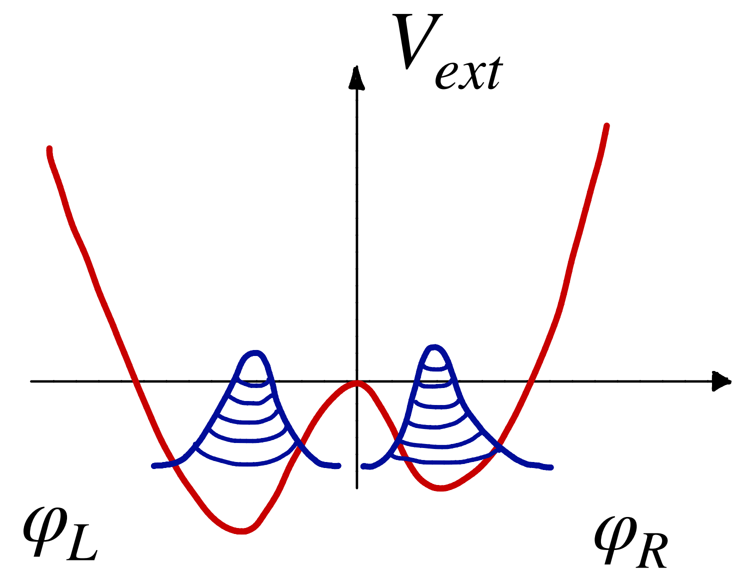

Ansatz

$$\Psi(r,t) \simeq v_1(t)\varphi_L(r,t) + v_2(t)\varphi_R(r,t)$$

φ_L, φ_R gnd. states in each well of $H = -\frac{\Delta_r}{2} + V_{L/R}(r,t)$

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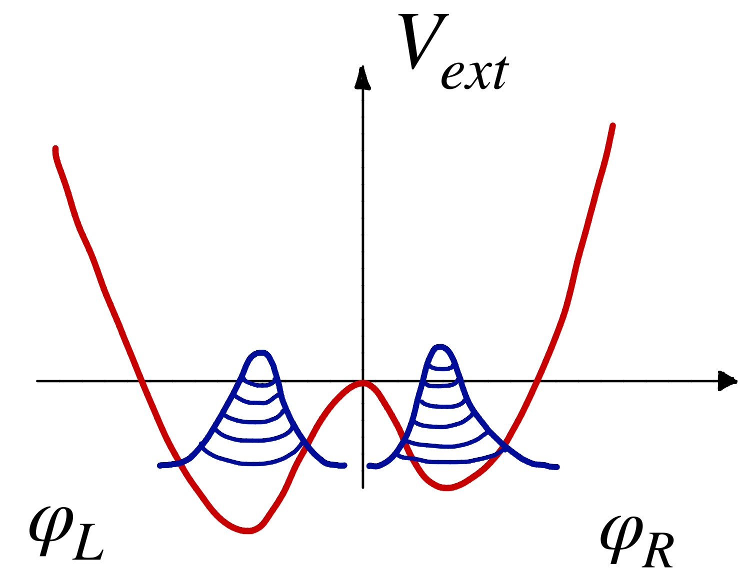

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$\rightsquigarrow$ Milburn et al '97

$$\begin{cases} i\partial_t v_1 & = (R + C(|v_2|^2 - |v_1|^2))v_1 + Wv_2 \\ i\partial_t v_2 & = Wv_1 - (R + C(|v_2|^2 - |v_1|^2)v_2 \end{cases}$$

where R, C, W are (time-dep.) parameters

Nonlinear Setup

- H depends on t and p parameters $x = (x_1, x_2, \ldots, x_p)$

$[0,1] \times [0,1]^p \ni (t, x) \mapsto H(t, x) \in \mathscr{L}(\mathscr{H})$ smooth

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- Example in $\mathbb{C}^2$

$$i\varepsilon\partial_t \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix} = \begin{pmatrix} R + C(|v_2^\varepsilon|^2 - |v_1^\varepsilon|^2) & W \\ W & -(R + C(|v_2^\varepsilon|^2 - |v_1^\varepsilon|^2)) \end{pmatrix} \begin{pmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{pmatrix}$$

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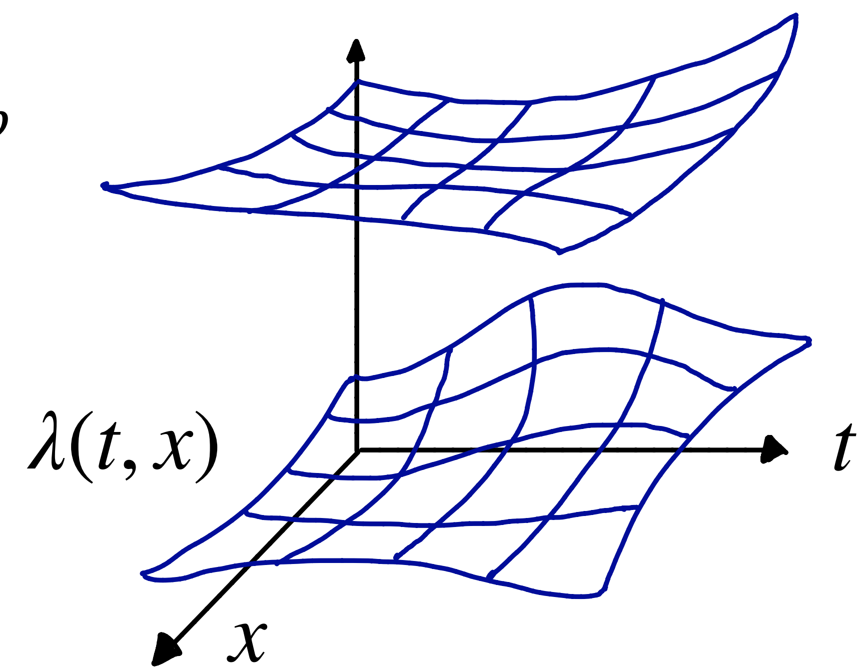
- Gauge invariance $\quad H(t,[e^{i\theta}v]) \equiv H(t,[v])$

Nonlinear Eigenvectors

- Gap assumption $\forall (t,x) \in [0,1] \times [0,1]^p$



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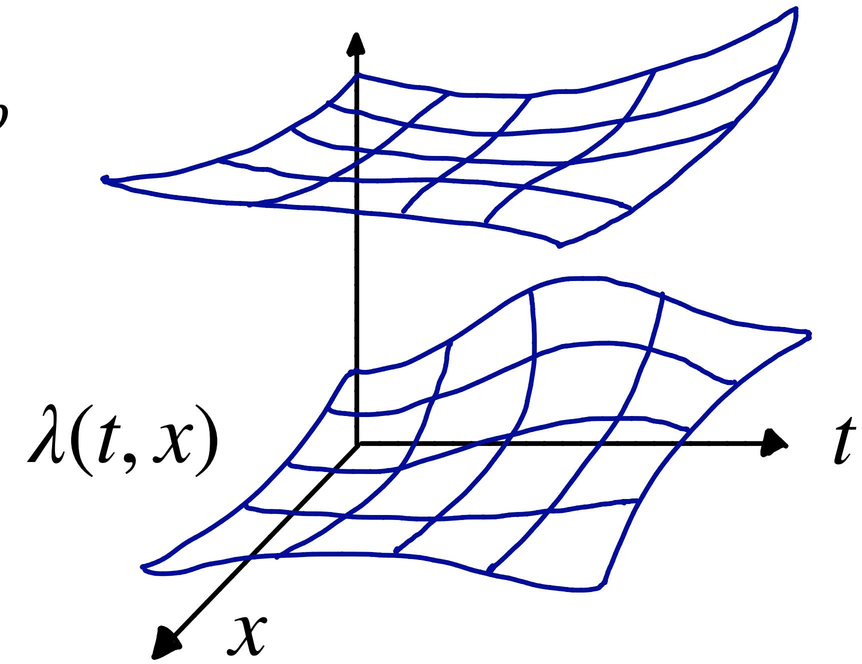

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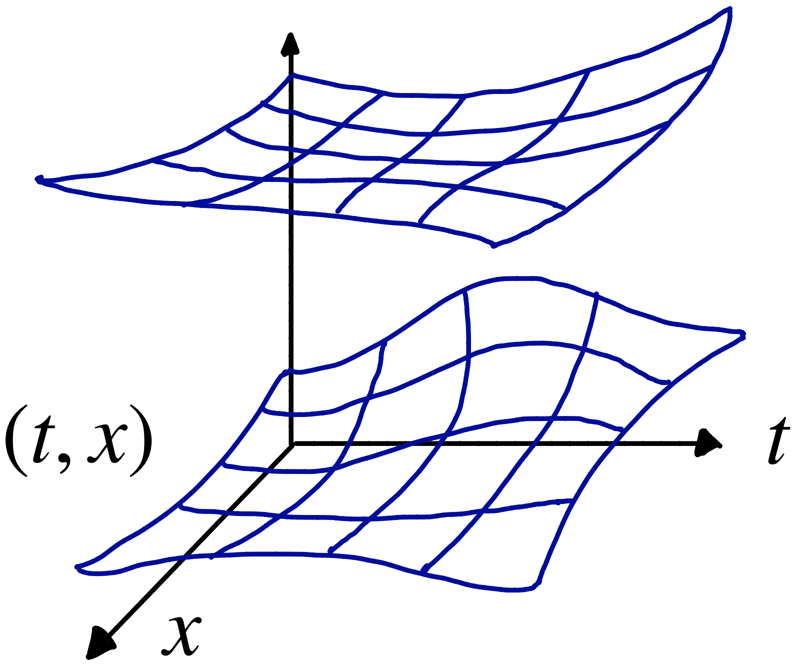

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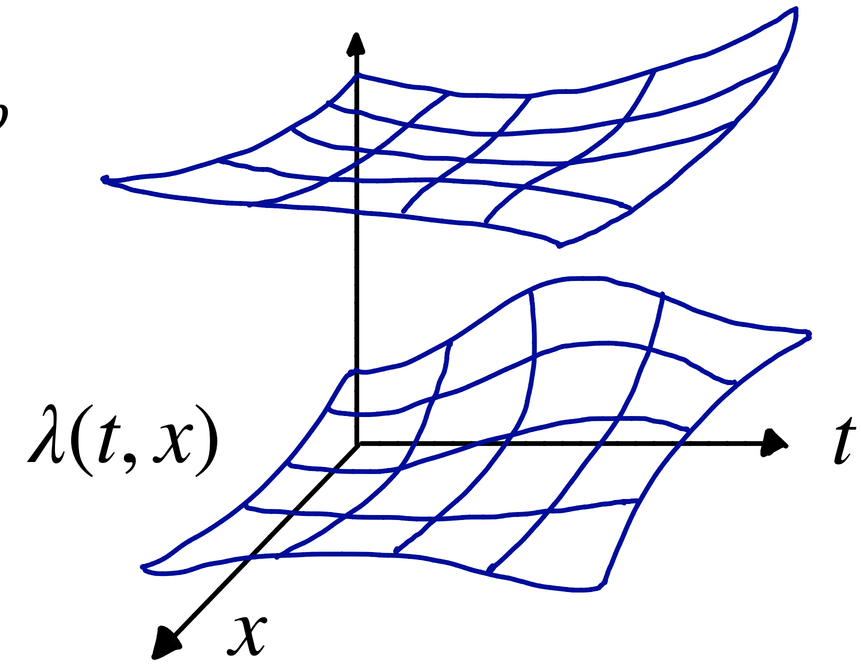

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Moreover, $\langle \omega(t) \,|\, \partial_t \omega(t) \rangle \equiv 0$ & $\omega(t) \propto \varphi(t,[\omega(t)])$

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  Let $x \mapsto \theta(x) \in \mathbb{R}$ and

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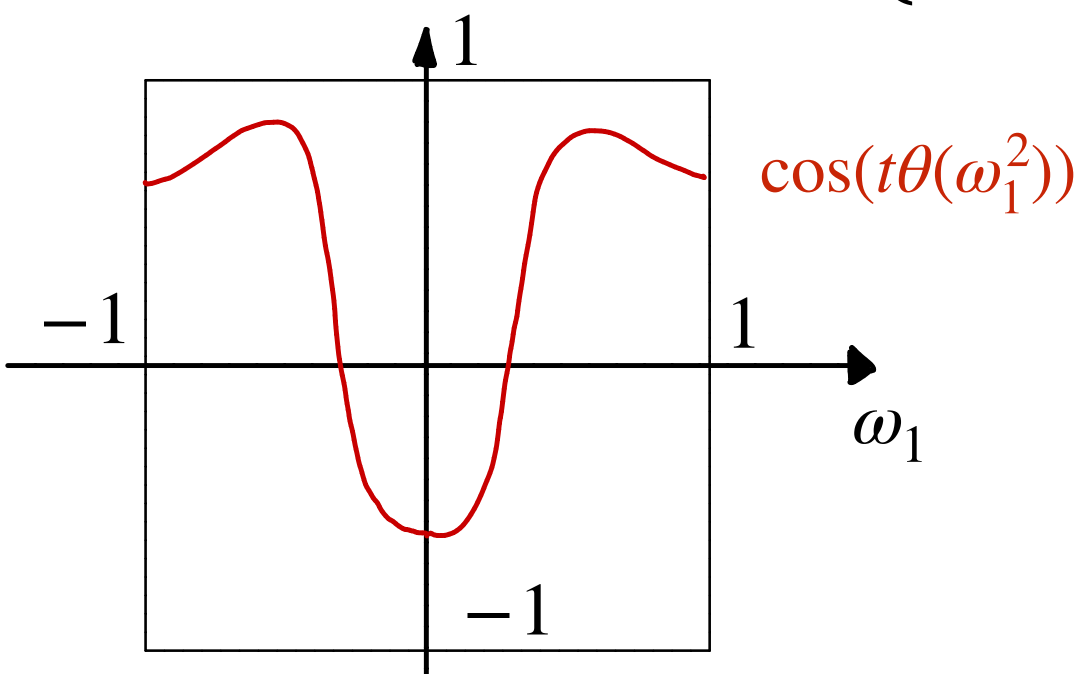

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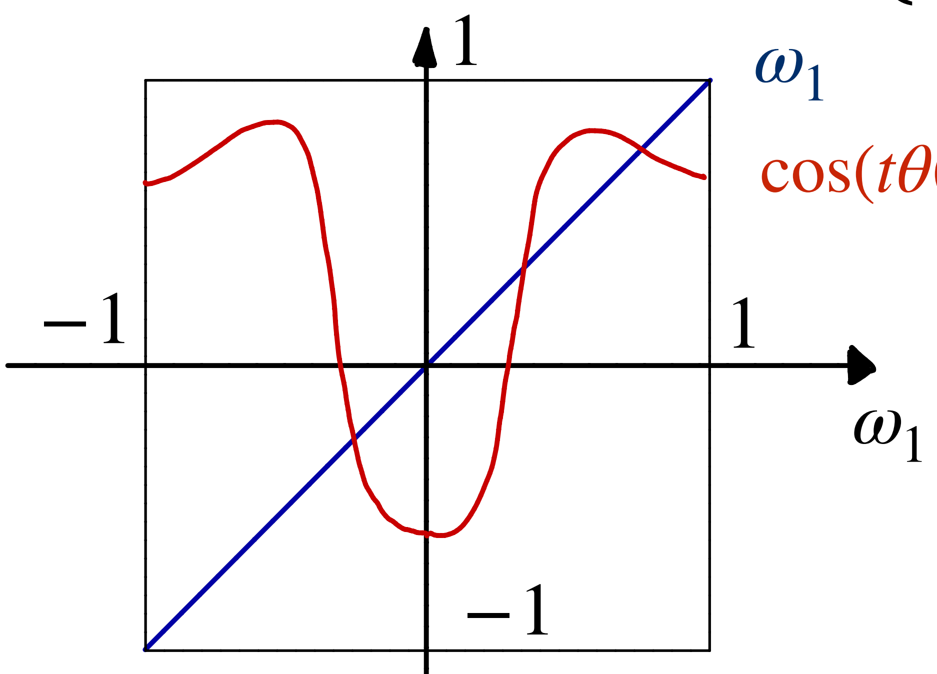

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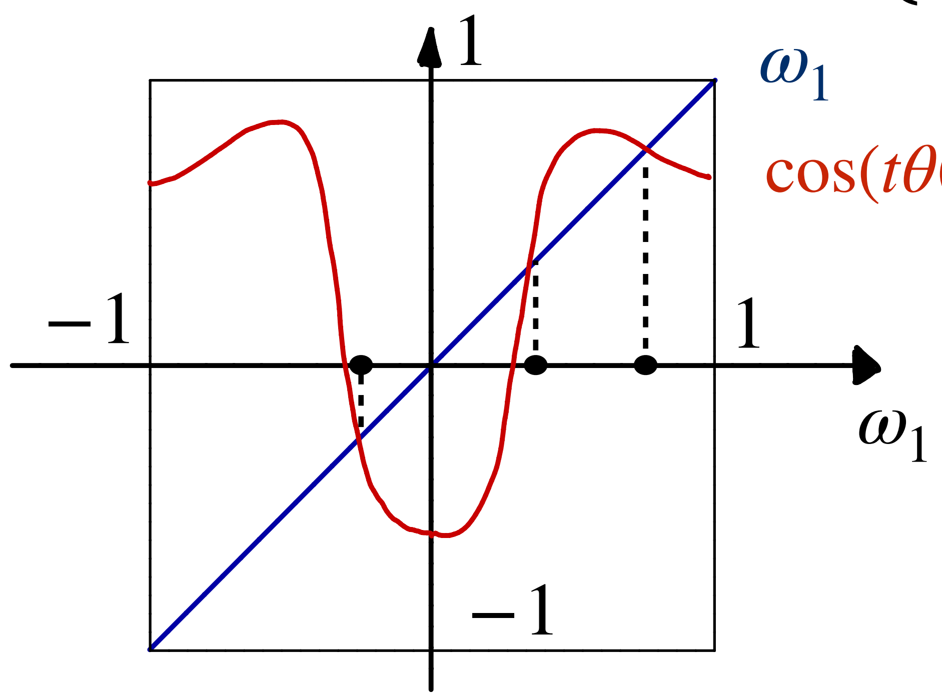

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$$\boxed{v^\varepsilon(t) = e^{-i\int_0^t \lambda(u,[\omega(u)])du/\varepsilon}\omega(t) + \mathcal{O}_t(\varepsilon)} \qquad \varepsilon \to 0, \quad t \in [0,\tau_0]$$

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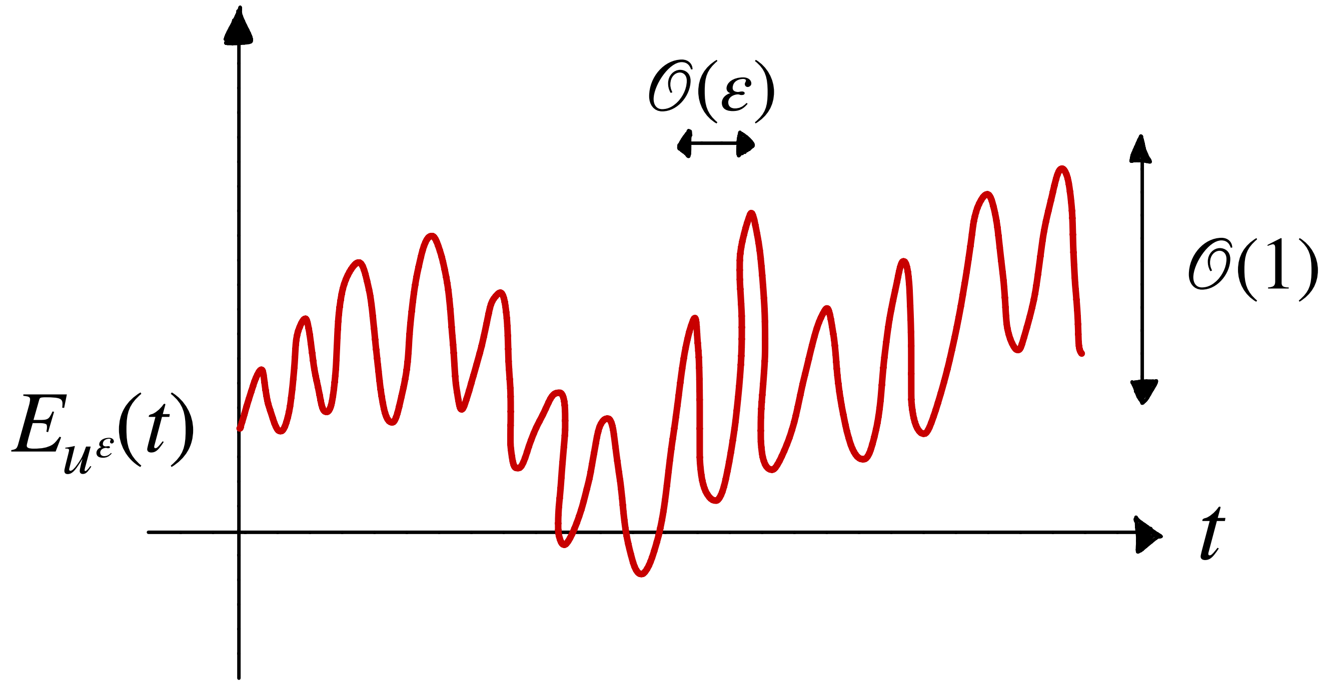

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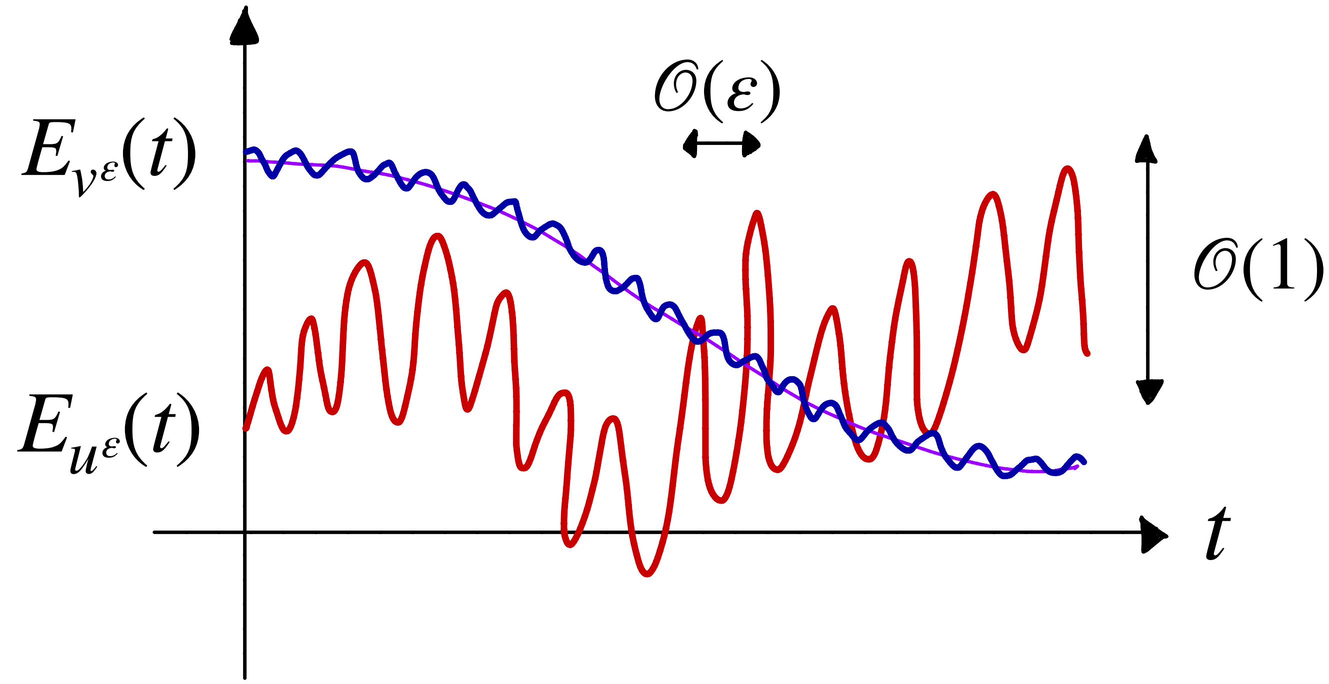

Cor: For $v^\varepsilon(t) = e^{-i\int_0^t \lambda(u,[\omega(u)])du/\varepsilon}\omega(t) + \mathcal{O}_t(\varepsilon)$ an adiabatic sol.

$$E_{v^\varepsilon}(t) = \lambda(t, [\omega(t)]) + \mathcal{O}_t(\varepsilon)$$

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- Method of proof

  Schauder fixed point theorem for $\omega(t)$

  Linear approx. $\rightsquigarrow$ non self-adjoint generator

  "Integration by parts" & control of remainders

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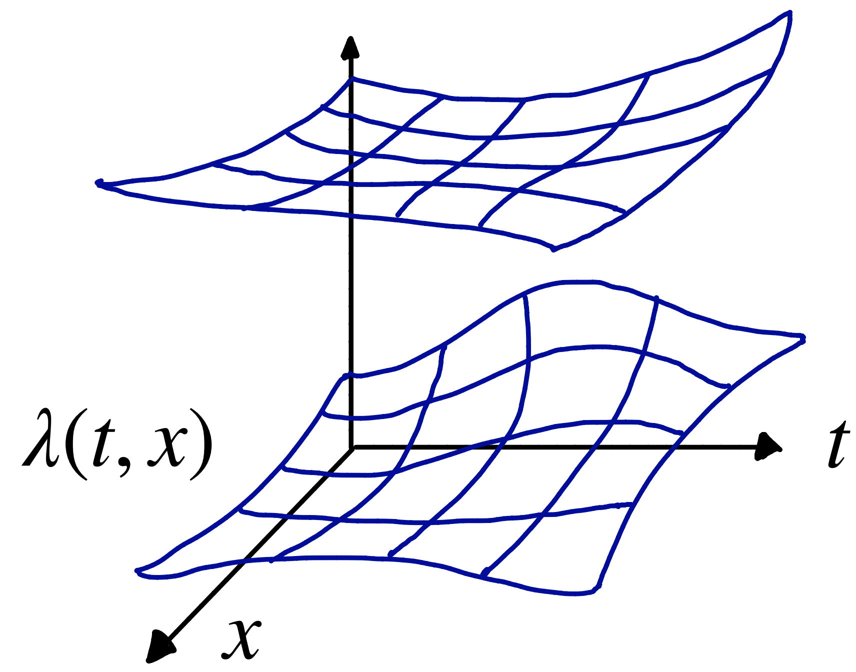
$H(t,x)\varphi(t,x) = \lambda(t,x)\varphi(t,x), \quad \varphi(t,x) \in \mathscr{H}$



Nonlinear eigenvector

- **Setup** $\dim \mathscr{H} = \infty$ & $H(t,x) \in \mathscr{L}(\mathscr{H})$ smooth

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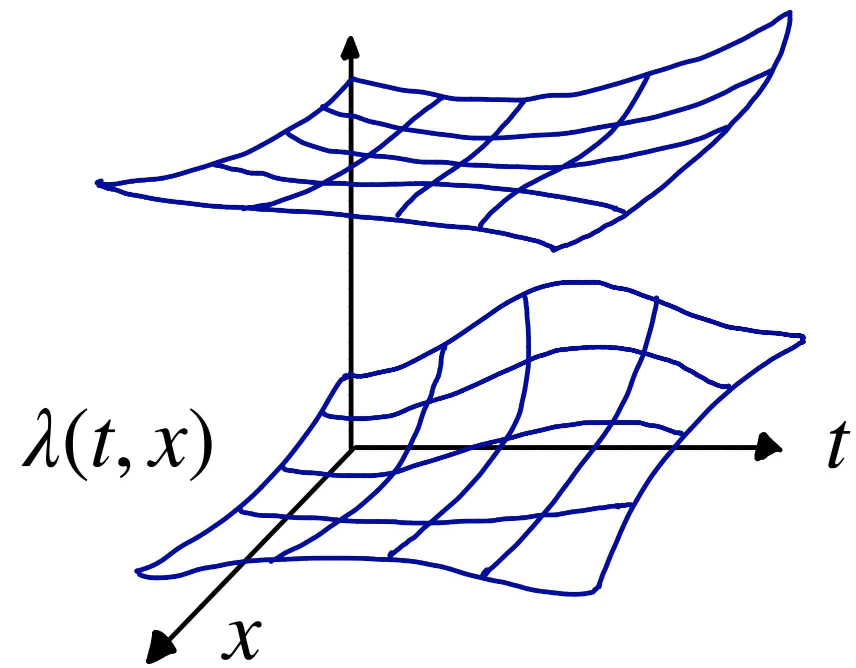

Fix t, let $\mathscr{B}_1(\mathscr{H}) = \{v \in \mathscr{H} \mid \|v\| \leq 1\}$ convex

$S : \mathscr{B}_1(\mathscr{H}) \to \mathscr{B}_1(\mathscr{H})$ s.t. $S(v) = \varphi([v])$ continuous

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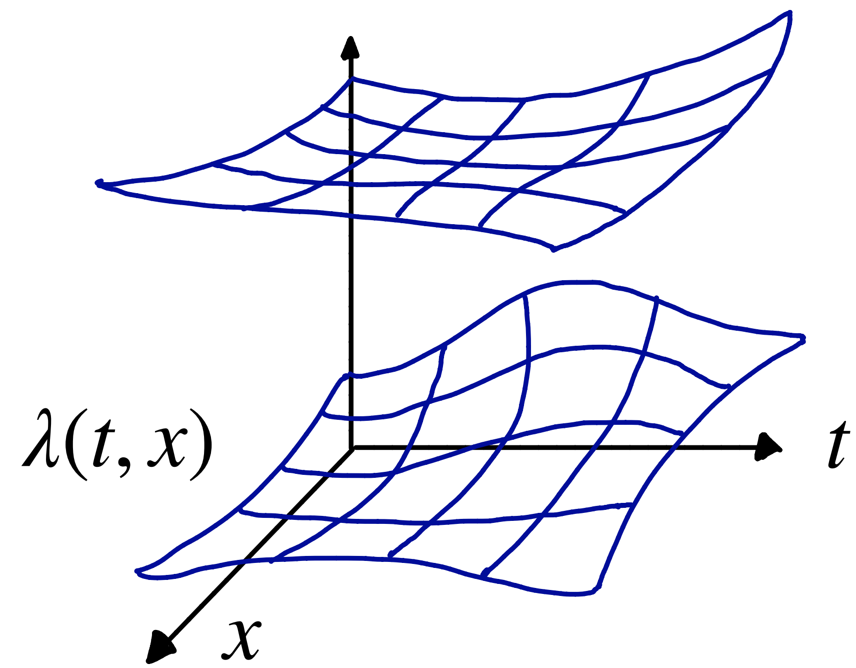

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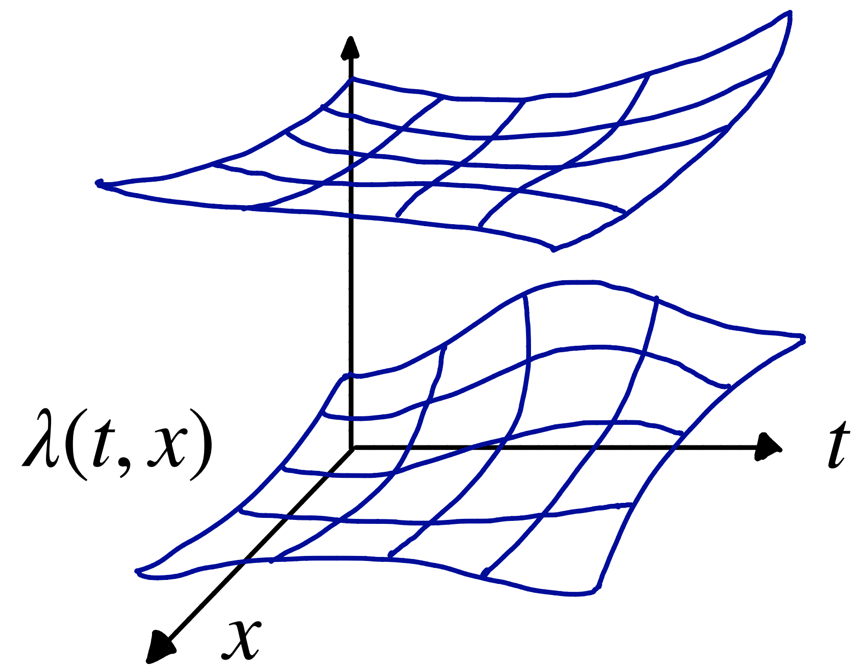

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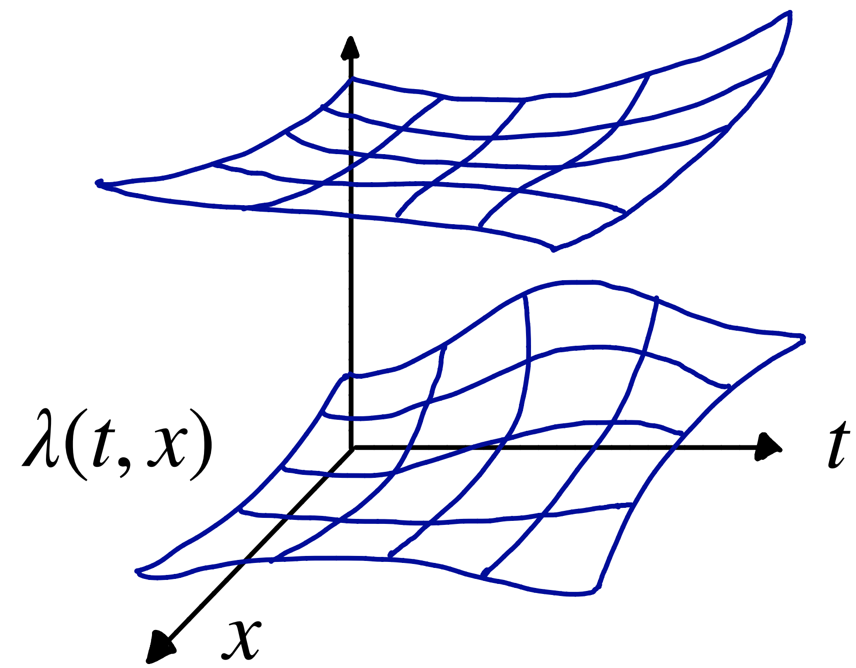

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Then, Implicit Function Thm. for δ small

Linearisation I

- Normalisation

w.l.o.g: $\lambda(t,x) \equiv 0,$ (via $H(t,x) \mapsto H(t,x) - \lambda(t,x)$)

$$\Rightarrow H(t,[\omega(t)])\omega(t) \equiv 0 \;\&\; \langle \omega(t) \,|\, \dot{\omega}(t) \rangle \equiv 0.$$

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$$i\epsilon \dot{\Delta}(t) = H(t,[v^{\epsilon}(t)])v^{\epsilon}(t) - i\epsilon \dot{\omega}(t)$$

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$$-i\epsilon\dot{\omega}(t) + O(\|\Delta(t)\|^2)$$

⇝ System for $(\Delta, \overline{\Delta})$ Recall: $H(t,x) = \overline{H(t,x)}$

Linearisation II

- System

Set

$$\mu_j := \begin{pmatrix} \partial_{x_j} H(t,[\omega(t)])\omega(t) \\ -\partial_{x_j} H(t,[\omega(t)])\omega(t) \end{pmatrix} \qquad \nu_j := \begin{pmatrix} \omega_j(t) e_j \\ \omega_j(t) e_j \end{pmatrix}$$

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$$F_0(t) = \begin{pmatrix} H(t,[\omega(t)]) & 0 \\ 0 & -H(t,[\omega(t)]) \end{pmatrix} = F_0(t)^* \qquad \text{s.t.}$$

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- Fact: $\begin{pmatrix} \dot{\omega} \\ \dot{\omega} \end{pmatrix} \in \ker F^{\perp}$ & $\|G(t)\| = O(\delta)$

Integration by parts I

- Spectral prop.

Lemma Assume $\sigma(H(t,x)) \cap \sigma(-H(t,x)) \equiv \{0\}$, $H(t,x)$ real

If $H(t,x)$ has N simple eigenval. matrix case

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Then, for δ small enough,

$$\sigma(F(t)) = \{-l_{N'}(t) < \ldots < -l_1(t) < 0 < l_1(t) < \cdots < l_{N'}(t)\} \subset \mathbb{R}$$

all simple, except 0 of mult. 2

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($F(t)$ semisimple)

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⇝ consequence of Aronszajn-Weinstein theory

Integration by parts II

- Kato's approach of linear approx.

$$i\epsilon \begin{pmatrix} \dot{\Delta} \\ \dot{\overline{\Delta}} \end{pmatrix} \simeq +F(t) \begin{pmatrix} \Delta \\ \overline{\Delta} \end{pmatrix} - i\epsilon \begin{pmatrix} \dot{\omega} \\ \dot{\overline{\omega}} \end{pmatrix} \qquad \text{where} \qquad F(t) = \sum_{j=-N'}^{N'} l_j(t) \mathbb{P}_j(t)$$

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$$i\epsilon\partial_t T^\epsilon(t,s) = F(t)T^\epsilon(t,s), \quad T^\epsilon(s,s) = \mathbb{I}$$

Adiabatic approx.

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Integration by parts II

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Lemma

$$\boxed{T^\epsilon(t,s) = V^\epsilon(t,s) + O_{s,t}(\epsilon)} \quad \& \quad \|T^\epsilon(t,s)\| = O(1)$$

Integration by parts III

- Duhamel's formula

$$\begin{pmatrix} \Delta(t) \\ \overline{\Delta}(t) \end{pmatrix} = -\int_0^t T^\epsilon(t,s) \begin{pmatrix} \dot{\omega}(s) \\ \dot{\omega}(s) \end{pmatrix} ds + \frac{i}{\epsilon} \int_0^t T^\epsilon(t,s) O(\|\Delta(s)\|^2) ds$$

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- Integration by parts

$$\mathbb{P}_0(t)\begin{pmatrix}\dot\omega(t)\\ \dot\omega(t)\end{pmatrix} \equiv 0 \quad \Rightarrow \quad \int_0^t V^\epsilon(t,s)\begin{pmatrix}\dot\omega(s)\\ \dot\omega(s)\end{pmatrix} ds = O_t(\epsilon)$$

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- Control of remainder

$$\sup_{0 \le t \le \tau} \|\Delta(t)\| \le a\epsilon + b\tau(\sup_{0 \le t \le \tau} \|\Delta(t)\|)^2/\epsilon$$ yields

$$4ab\tau \le 1 \quad \Rightarrow \quad \sup_{0 \le t \le \tau} \|\Delta(t)\| \le 2a\epsilon$$

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$$\begin{pmatrix}\Delta(t)\\ \overline{\Delta}(t)\end{pmatrix} = -\int_0^t T^\epsilon(t,s)\begin{pmatrix}\dot\omega(s)\\ \dot\omega(s)\end{pmatrix} ds + \frac{i}{\epsilon}\int_0^t T^\epsilon(t,s) O(\|\Delta(s)\|^2) ds$$

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$$4ab\tau \le 1 \quad \Rightarrow \quad \sup_{0\le t\le\tau} \|\Delta(t)\| \le 2a\epsilon \quad \Rightarrow \quad \boxed{v^\epsilon(t) = \omega(t) + O_t(\epsilon)}$$

Unbounded extension

- Unbounded case $\qquad \dim \mathscr{H} = \infty$

$H(t,x) = H_0 + W(t,x)$ where

$$\left.\begin{array}{l} H_0 = H_0^* \geq 0 \quad \text{on} \quad \mathscr{D} \subset \mathscr{H} \\ W(t,x) = W^*(t,x) \in \mathscr{L}(\mathscr{H}) \ \text{smooth} \end{array}\right\} \text{real}$$

Unbounded extension

- Unbounded case $\qquad \dim \mathscr{H} = \infty$

$H(t,x) = H_0 + W(t,x)$ where

$$\left.\begin{array}{l} H_0 = H_0^* \geq 0 \quad \text{on} \quad \mathscr{D} \subset \mathscr{H} \\ W(t,x) = W^*(t,x) \in \mathscr{L}(\mathscr{H}) \text{ smooth} \end{array}\right\} \text{ real}$$

$\sigma(H_0) = \{\lambda_j\}_{j \in \mathbb{N}}$ simple, and $\lambda_{j+1} - \lambda_j \geq c_0 j^{\alpha}$ for $\alpha > 1/2$, $c_0 > 0$

$\|W(t,x)\| \leq \delta$, $\|\partial_{x_j} W(t,x)\| \leq \delta$

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- Perturbation in δ

Same holds for $\sigma(H(t,x)) = \{\lambda_j(t,x)\}_{j\in\mathbb{N}}$

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- Same statement generated gap hyp.
  weak derivatives