Generic aspects of entanglement in high-dimensional quantum systems

Some applications of asymptotic geometric analysis in quantum information theory

Cécilia Lancien

Institut de Mathématiques de Toulouse & CNRS

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Outline

Overview of the lecture's topic

Goal of this lecture: Understand how *asymptotic geometric analysis* can be useful to tackle problems arising in *quantum information theory*.
[The reference: *Alice and Bob meet Banach*, by G. Aubrun and S. Szarek.]

Question: What is asymptotic geometric analysis?
Use of probabilistic techniques in the study of Banach spaces of high but finite dimension.

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Why "high dimension"?
Quantum system composed of 1 particle: described by a complex Hilbert space H.
Quantum system composed of M such particles: the associated space is $\mathrm{H}^{\otimes M}$.
$\longrightarrow$ When studying multipartite quantum systems, one has to deal with high dimensions as soon as more than a few particles are involved.

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Why "probabilistic techniques"?

1. Identify typical properties of high dimensional quantum systems.
2. Prove the existence of quantum systems having certain properties, using random constructions.

In this talk: We will focus on point (1), with a particular interest for multipartite quantum systems and properties related to entanglement.

Background

Two seminal works from 2006:

1. Aspects of generic entanglement, by P. Hayden, D. Leung and A. Winter.
   Goal: Quantify the typical amount of entanglement in bipartite quantum states.
2. Tensor products of convex sets and the volume of separable states on N qudits, by G. Aubrun and S. Szarek.
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Note: “typical” $=$ “with probability going to 1 (exponentially) as the underlying dimension grows”.
So there are usually two steps in the argument:

1. Identify the average behavior of the property under consideration.
2. Show that this average behavior is generic in high dimension.
   Concentration of measure phenomenon: a sufficiently ‘well-behaved’ function has an exponentially small probability of deviating from its average as the dimension grows.

Reminder: separability vs entanglement in multipartite quantum systems

$\mathrm{H}_1, \ldots, \mathrm{H}_M$ complex Hilbert spaces. In this talk: of finite, but usually large, dimension.
$\longrightarrow \mathrm{H}_i \equiv \mathbf{C}^{d_i}$ with $d_i \gg 1$, for $1 \leqslant i \leqslant M$.

Definition [Separability and entanglement]

A state ρ on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ is called *separable* if it is a convex combination of product states, i.e.
↳ positive semidefinite operator with trace 1 on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$

$$\rho = \sum_{k=1}^{r} \lambda_k \rho_1^k \otimes \cdots \otimes \rho_M^k, \text{ with } \begin{cases} \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant r,\ \sum_{k=1}^{r} \lambda_k = 1 \\ \rho_i^k \text{ state on } \mathrm{H}_i,\ 1 \leqslant k \leqslant r,\ 1 \leqslant i \leqslant M \end{cases} .$$

Otherwise it is called *entangled*.

[Note: If ρ is a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$ for some unit vector $\psi \in \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$, then ρ is separable iff $\psi = \psi_1 \otimes \cdots \otimes \psi_M$ for some unit vectors $\psi_i \in \mathrm{H}_i$, $1 \leqslant i \leqslant M$.]

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Fact: If a multipartite quantum system is in a separable state, there is no intrinsically quantum correlation between its subsystems. It thus does not provide any advantage over a classical system in information processing tasks.
$\longrightarrow$ Characterizing and quantifying the entanglement of multipartite quantum states is an important issue in practice.

Outline

1. Introduction
2. Quantifying the typical amount of entanglement/correlations in multipartite pure states
3. Typical strength of entanglement criteria for bipartite mixed states

Outline

Let $\psi \in \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ be a pure state, i.e. $\|\psi\|_2 = 1$.
↳ Euclidean norm

Its *injective norm* is: $\|\psi\|_\varepsilon := \sup\left\{ |\langle \varphi_1 \otimes \cdots \otimes \varphi_M | \psi \rangle| \, : \, \varphi_i \in \mathrm{H}_i,\ \|\varphi_i\|_2 = 1 \right\}$.

Its *projective norm* is: $\|\psi\|_\pi := \inf\left\{ \sum_{k=1}^{r} |\alpha_k| \, : \, \chi_i^k \in \mathrm{H}_i,\ \|\chi_i^k\|_2 = 1,\ \psi = \sum_{k=1}^{r} \alpha_k \chi_1^k \otimes \cdots \otimes \chi_M^k \right\}$.

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Facts:

- The ε and π norms are *dual norms*: $\|\psi\|_\varepsilon = \sup_{\|\varphi\|_\pi \leqslant 1} |\langle \varphi|\psi\rangle|$ and $\|\psi\|_\pi = \sup_{\|\varphi\|_\varepsilon \leqslant 1} |\langle \varphi|\psi\rangle|$.
- The ε and π norms are, respectively, minimal and maximal norms among *tensor norms* (i.e. norms which factorize on product vectors: $\|\psi_1 \otimes \cdots \otimes \psi_M\| = \|\psi_1\| \cdots \|\psi_M\|$).
  In particular: $\|\psi\|_\varepsilon \leqslant \|\psi\|_2 \leqslant \|\psi\|_\pi$.
- A pure state ψ is separable iff $\|\psi\|_\varepsilon = \|\psi\|_\pi = 1$.
  $\longrightarrow$ If $\|\psi\|_\varepsilon \ll 1$ or $\|\psi\|_\pi \gg 1$, then ψ is 'very' entangled.

Particular case of bipartite pure states

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Then clearly, $\|\psi\|_2 = \|M_\psi\|_2$.

And the *Schmidt decomposition* of ψ corresponds to the *singular value decomposition* of M_ψ:

$$|\psi\rangle = \sum_{k=1}^{r} \sqrt{\lambda_k} |e_k f_k\rangle \longleftrightarrow M_\psi = \sum_{k=1}^{r} \sqrt{\lambda_k} |e_k\rangle\langle f_k|,$$

with $r \leqslant d$ the *Schmidt rank* of ψ, $\sum_{k=1}^{r} \lambda_k = 1$, $\{e_k\}_{k=1}^{r}, \{f_k\}_{k=1}^{r}$ orthonormal sets in $\mathbf{C}^d$.

So $\|\psi\|_\varepsilon = \max_{1\leqslant k\leqslant r} \sqrt{\lambda_k} = \|M_\psi\|_\infty$ and $\|\psi\|_\pi = \sum_{k=1}^{r} \sqrt{\lambda_k} = \|M_\psi\|_1$.

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$\longrightarrow$ Checking bipartite pure state separability is easy.
Quantitatively, for all $\psi \in \mathbf{C}^d \otimes \mathbf{C}^d$ s.t. $\|\psi\|_2 = 1$, $\frac{1}{\sqrt{d}} \leqslant \|\psi\|_\varepsilon \leqslant 1$ and $1 \leqslant \|\psi\|_\pi \leqslant \sqrt{d}$.

But no such simple characterization for $M > 2$ (no equivalent of the Schmidt decomposition).

Geometric measure of entanglement

Definition [Geometric measure of entanglement (Wei/Goldbart)]

Let $\psi \in \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ be a pure state. Its *geometric measure of entanglement (GME)* is

$$E(\psi) := -\log \sup \left\{ \left| \langle \varphi_1 \otimes \cdots \otimes \varphi_M | \psi \rangle \right|^2 : \varphi_i \in \mathrm{H}_i ,\ \|\varphi_i\|_2 = 1 \right\} .$$

By definition, $E(\psi) = -2\log \|\psi\|_\varepsilon$. So $E(\psi) = 0$ iff ψ is separable.
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Remark: The definition of the GME can be extended to mixed states on $\mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M$ (but it is not an entanglement measure anymore), as

$$\begin{aligned} E(\rho) &:= -\log \sup \{ \langle \varphi_1 \otimes \cdots \otimes \varphi_M | \rho | \varphi_1 \otimes \cdots \otimes \varphi_M \rangle \, : \, \varphi_i \in \mathrm{H}_i, \ \|\varphi_i\|_2 = 1 \} \\ &= -\log \sup \{ \mathrm{Tr}(\rho\sigma) \, : \, \sigma \text{ separable state on } \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_M \}. \end{aligned}$$

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Fact: For any unit vector $\psi \in (\mathbf{C}^d)^{\otimes M}$, $\|\psi\|_\varepsilon \geqslant \frac{1}{\sqrt{d^{M-1}}}$, i.e. $E(\psi) \leqslant (M-1)\log d$.
This can be checked recursively, starting from the bipartite case:
For any unit vector $\psi \in \mathbf{C}^{d_1} \otimes \mathbf{C}^{d_2}$, $\|\psi\|_\varepsilon \geqslant \frac{1}{\sqrt{d}}$, where $d := \min(d_1, d_2)$.
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Question: Are multipartite pure states generically 'very' or 'little' entangled?
$\longrightarrow$ What is the typical value of the GME for a unit vector $\psi \in (\mathbf{C}^d)^{\otimes M}$ sampled at random?

GME of uniformly distributed multipartite pure states

Theorem [Typical ε norm of a random unit vector (Aubrun/Szarek)]

There exist constants $c, C, c_0 > 0$ s.t., for $\psi \in (\mathbf{C}^d)^{\otimes M}$ a uniformly distributed unit vector,

$$\mathbf{P}\left(c\sqrt{\frac{M\log M}{d^{M-1}}} \leqslant \|\psi\|_\varepsilon \leqslant C\sqrt{\frac{M\log M}{d^{M-1}}} \right) \geqslant 1 - e^{-c_0 dM\log M}.$$

Consequence: For $\psi \in (\mathbf{C}^d)^{\otimes M}$ a uniformly distributed unit vector, when d or M is large, $E(\psi) = (M-1)\log d - \log(M\log M) + O(1)$ with high probability.
$\longrightarrow$ A random multipartite pure state is typically close to maximally entangled.

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Proof idea: Observe that $\psi \sim g/\|g\|_2$, where $g \in (\mathbf{C}^d)^{\otimes M}$ has independent complex Gaussian entries with mean 0 and variance 1.
• By the standard Gaussian concentration inequality: $\mathbf{P}\left(\|g\|_2 \gtrless \sqrt{d^M}(1\pm\varepsilon)\right) \leqslant e^{-d^M\varepsilon^2}$.
• Set $\mathcal{V} := \left\{\varphi_1\otimes\cdots\otimes\varphi_M \,:\, \varphi_i \in \mathbf{C}^d,\ \|\varphi_i\|_2 = 1\right\}$, so that $\mathbf{E}\|g\|_\varepsilon = \mathbf{E}\sup_{\varphi\in\mathcal{V}} |\langle\varphi|g\rangle|$.
To estimate the latter quantity, use results about suprema of Gaussian processes.
Upper bound: 'small' covering subset of $\mathcal{V}$. Lower bound: 'large' separated subset of $\mathcal{V}$.
Conclusion: $\mathbf{E}\|g\|_\varepsilon$ is of order $\sqrt{dM\log M}$.
Then show that $g \mapsto \|g\|_\varepsilon$ also concentrates around its average.

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Question: Are 'interesting' multipartite pure states really captured by the uniform distribution?

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‘Physical’ states of many-body quantum systems and tensor network states

Curse of dimensionality in many-body quantum systems: A system composed of M d-dimensional subsystems has dimension d^M, which is exponential in M.

However, ‘physically relevant’ states of many-body quantum systems, such as *ground states of gapped local Hamiltonians*, are (conjectured to be) well approximated by so-called *tensor network states (TNS)*, which form a small subset of the global state space (Hastings, Landau/Vazirani/Vidick).

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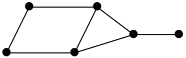
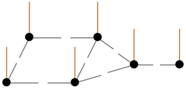
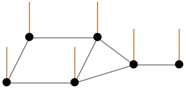
Tensor network state on $(\mathbf{C}^d)^{\otimes M}$: Take a graph G with M vertices and L edges.
Put at each vertex v a tensor $\chi_v \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes \delta(v)}$ to get a tensor $\hat{\chi}_G \in (\mathbf{C}^d)^{\otimes M} \otimes (\mathbf{C}^q)^{\otimes 2L}$.
Contract together the indices of $\hat{\chi}_G$ associated to a same edge to get a tensor $\chi_G \in (\mathbf{C}^d)^{\otimes M}$.
$\longrightarrow$ If $\delta(v) \leqslant \delta$ for all v, then χ_G is described by at most $Mq^\delta d$ parameters, which is linear in M.



G with 6 vertices and 7 edges



$\hat{\chi}_G \in (\mathbf{C}^d)^{\otimes 6} \otimes (\mathbf{C}^q)^{\otimes 14}$



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d-dimensional indices: *physical* indices. q-dimensional indices: *bond* indices.

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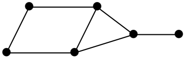

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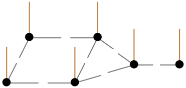

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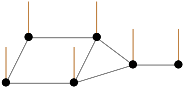

$\chi_G \in (\mathbf{C}^d)^{\otimes 6}$

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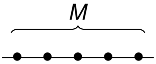
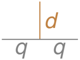
If the underlying graph G is 1-dimensional (line or circle), χ_G is a *matrix product state (MPS)*.

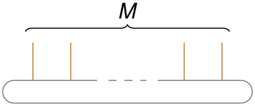
A simple model of random translation-invariant MPS

M particles on a circle



Pick a tensor $\chi \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes 2}$ whose entries are independent complex Gaussians with mean 0 and variance $1/dq$.
Repeat it on all sites and contract neighboring q-dimensional indices.
$\longrightarrow$ Obtained tensor $\chi_M \in (\mathbf{C}^d)^{\otimes M}$: *random translation-invariant MPS with periodic boundary conditions*.





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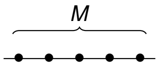
A simple model of random translation-invariant MPS

M particles on a circle



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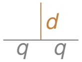

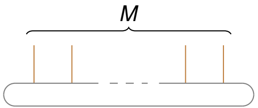

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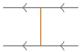
$T : \mathbf{C}^q \otimes \mathbf{C}^q \to \mathbf{C}^q \otimes \mathbf{C}^q$,
obtained by contracting the d-dimensional indices of χ and $\bar{\chi}$.



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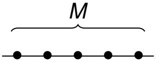

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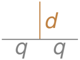

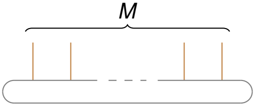

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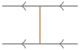

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Remark: The parameter q quantifies the amount of bipartite entanglement: Across any bipartite cut preserving the ordering of subsystems, χ_M has Schmidt rank at most $q^2 \ll d^{M/2}$.
Now what about genuinely multipartite entanglement? $\hookrightarrow$ *area vs volume law*
$\longrightarrow$ If $q = 1$, $\chi_M = \chi^{\otimes M}$ is separable. But what can we say for $q \gg 1$?

Correlations in an MPS

Let A, B be 1-site observables, i.e. observables on $\mathbf{C}^d$.
Goal: Quantify the correlations between the outcomes of A and B, when performed on 'distant' sites.

$A : \mathbf{C}^d \to \mathbf{C}^d$

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Compute the value on the MPS χ_M of the observable $A_1 \otimes I_k \otimes B_1 \otimes I_{M-k-2}$, i.e.

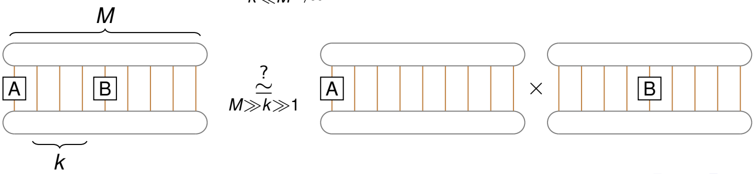
$$v_\chi(A,B,k) := \frac{\langle \chi_M | A_1 \otimes I_k \otimes B_1 \otimes I_{M-k-2} | \chi_M \rangle}{\langle \chi_M | \chi_M \rangle}.$$

Compare it to the product of the values on χ_M of $A_1 \otimes I_{M-1}$ and $I_{k+1} \otimes B_1 \otimes I_{M-k-2}$, i.e.

$$v_\chi(A)v_\chi(B) := \frac{\langle \chi_M | A_1 \otimes I_{M-1} | \chi_M \rangle \langle \chi_M | I_{k+1} \otimes B_1 \otimes I_{M-k-2} | \chi_M \rangle}{\langle \chi_M | \chi_M \rangle^2}.$$

Correlations in the MPS χ_M: $\gamma_\chi(A,B,k) := \left| v_\chi(A,B,k) - v_\chi(A)v_\chi(B) \right|$.

Question: Do we have $\gamma_\chi(A,B,k) \underset{k \ll M \to \infty}{\longrightarrow} 0$? And if so, at which speed?



Exponential decay of correlations in random translation-invariant MPS

Clearly, separability implies no correlation between 1-site observables:
If $\chi_M = \chi^{\otimes M}$, then $\gamma_\chi(A, B, k) = 0$ for any $k \leqslant M$ and any observables A, B on $\mathbf{C}^d$.

Intuition: In an MPS χ_M, the correlations between 1-site observables decay exponentially with the distance separating the sites, i.e. there exist $C(\chi), \tau(\chi) > 0$ s.t., for any $k \ll M$ and any observables A, B on $\mathbf{C}^d$,

$$\gamma_\chi(A, B, k) \leqslant C(\chi) e^{-\tau(\chi) k} \|A\|_\infty \|B\|_\infty.$$

Correlation length in the MPS χ_M: $\xi(\chi) := 1/\tau(\chi)$.

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There exist constants $C, c_0 > 0$ s.t., for $\chi_M \in (\mathbf{C}^d)^{\otimes M}$ a random translation-invariant MPS,

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Proof idea: Let $\lambda_1(T), \lambda_2(T)$ be the two largest eigenvalues of the transfer operator T and set $\varepsilon(T) := |\lambda_2(T)|/|\lambda_1(T)|$. Then, $\gamma_\chi(A, B, k) \leqslant C(T)\varepsilon(T)^k \|A\|_\infty \|B\|_\infty$. So $\xi(\chi) = 1/|\log \varepsilon(T)|$.
We can then prove that $\mathbf{P}\left(|\lambda_1(T)| \geqslant 1 - \frac{C}{\sqrt{d}} \text{ and } |\lambda_2(T)| \leqslant \frac{C}{\sqrt{d}} \right) \geqslant 1 - e^{-c_0 q}$.
↳ spectral analysis for a non-normal random matrix with tensor product structure

Perspectives

- The amount of correlations in a random MPS is generically small. Is it also the case for the amount of multipartite entanglement?
  $\longrightarrow$ Can we estimate the GME of a random MPS? Work in progress with I. Nechita...
- What about more complicated models of random MPS, where the random 1-site tensor has some symmetries?
- Can the generic amount of correlations and multipartite entanglement be computed in TNS with a more complicated geometry?
  Typically small correlation length can be proven for random TNS on a 2-dimensional regular lattice (Lancien/Pérez-García), but everything else remains essentially open.

Outline

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Entanglement criteria

Known: The problem of deciding whether a given multipartite quantum state is entangled or separable (and even just approximate versions of it) is in general computationally hard (Gharibian).
$\longrightarrow$ Solution in practice: Look for necessary conditions to separability, which are easier to check than separability itself, aka *entanglement criteria*.

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Here, we focus on the bipartite case. And we look at two such necessary conditions to separability, which can be efficiently checked and are thus widely used in practice:

- being *positive under partial transposition (PPT)*,
- being *k-extendible*.

The PPT criterion

Definition [Partial transposition]

The *partial transposition* (on B) of a state ρ_{AB} on $\mathrm{A}\otimes\mathrm{B}$ is defined as

$$\Gamma_{AB}(\rho_{AB}) := I_A \otimes T_B(\rho_{AB}),$$

where I denotes the identity map and T denotes the transposition map.

Theorem [Necessary condition to separability (Peres)]

On a bipartite Hilbert space $\mathrm{A}\otimes\mathrm{B}$, if a state is separable, then it is positive under partial transposition (PPT).

Remarks:

- This is obvious since $\Gamma_{AB}(\sigma_A\otimes\tau_B) = \sigma_A\otimes T_B(\tau_B)$.
- NSC for separability on $\mathbf{C}^2\otimes\mathbf{C}^2$ or $\mathbf{C}^2\otimes\mathbf{C}^3$ (Horodecki's). In higher dimensions, there exist PPT entangled states.
- Special instance in the class of separability relaxations built on: ρ_{AB} is separable iff for any positive map Λ_B, $Id_A\otimes\Lambda_B(\rho_{AB})$ is positive (Horodecki's).

The k-extendibility criterion

Definition [k-extendibility]

Let $k \geqslant 2$. A state ρ_{AB} on $\mathrm{A}\otimes\mathrm{B}$ is *k-extendible* (w.r.t. B) if there exists a state ρ_{AB^k} on $\mathrm{A}\otimes\mathrm{B}^{\otimes k}$ which is invariant under any permutation of the B subsystems and s.t. $\rho_{AB} = \mathrm{Tr}_{\mathrm{B}^{k-1}}[\rho_{AB^k}]$.

Theorem [Necessary and sufficient condition to separability (Doherty/Parrilo/Spedalieri)]

On a bipartite Hilbert space $\mathrm{A}\otimes\mathrm{B}$, a state is separable iff it is k-extendible for all $k \geqslant 2$.

Remarks:

- "ρ_{AB} separable $\Rightarrow$ ρ_{AB} k-extendible for all k" is obvious since $\sigma_A\otimes\tau_B = \mathrm{Tr}_{\mathrm{B}^{k-1}}[\sigma_A\otimes\tau_B^{\otimes k}]$.
- "ρ_{AB} k-extendible for all k $\Rightarrow$ ρ_{AB} separable" relies on the *quantum de Finetti theorem* (Christandl/König/Mitchison/Renner).
- ρ_{AB} k-extendible $\Rightarrow$ ρ_{AB} k'-extendible for all $k' \leqslant k$.
  $\longrightarrow$ Hierarchy of NC for separability, which an entangled state is guaranteed to stop passing at some point (but one cannot tell when a priori).

Problem: When relaxing the separability constraint to one which is easier to check, how ‘rough’ is the approximation?

Quantifying the strength of separability relaxations

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Known: There exist states which are PPT or k-extendible, and nevertheless 'very' entangled (i.e. far away from the set of separable states in some standard or operational distance measure).
$\longrightarrow$ Instead of looking at worst case scenarios, can we say something stronger about average/typical behaviors?
Hope: Implications regarding average case complexity of checking separability...?

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Two possible quantitative strategies:

- Estimate the size of the set of states, either satisfying a given entanglement criterion or being indeed separable.
  $\longrightarrow$ Information on how much bigger than the separable set the relaxed set is.
- Characterize when certain random states are with high probability, either violating a given entanglement criterion or indeed entangled.
  $\longrightarrow$ Information on how powerful the separability test is to detect entanglement.

Interlude: asymptotic spectrum of GUE and Wishart matrices

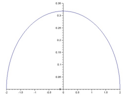
Definitions [Gaussian Unitary Ensemble and Wishart matrices]

- G is an $n \times n$ *GUE matrix* if $G = (H + H^*)/\sqrt{2}$ with H an $n \times n$ matrix whose entries are independent complex Gaussians with mean 0 and variance 1.
- W is an (n,s)-*Wishart matrix* if $W = HH^*$ with H an $n \times s$ matrix whose entries are independent complex Gaussians with mean 0 and variance 1.

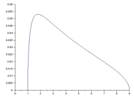
Definitions [Semicircular and Marčenko-Pastur distributions]

- $d\mu_{SC(m,\sigma^2)}(x) := \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}\mathbf{1}_{[m-2\sigma,m+2\sigma]}(x)dx.$
- $d\mu_{MP(\lambda)}(x) := \begin{cases} f_\lambda(x)dx \text{ if } \lambda \geqslant 1 \\ (1-\lambda)\delta_0 + \lambda f_\lambda(x)dx \text{ if } \lambda < 1 \end{cases}$, where

  $f_\lambda(x) := \frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{2\pi\lambda x}\mathbf{1}_{[\lambda_-,\lambda_+]}(x)$, with $\lambda_\pm := (\sqrt{\lambda} \pm 1)^2$.



$\mu_{SC(0,1)}$



$\mu_{MP(4)}$

Fact: For any Hermitian matrix M on $\mathbf{C}^n$, denote by $\mu_M := \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(M)}$ its spectral distribution.

- $(G_n)_{n\in\mathbf{N}}$ sequence of $n \times n$ GUE matrices: $(\mu_{G_n/\sqrt{n}})_{n\in\mathbf{N}}$ converges strongly to $\mu_{SC(0,1)}$.
- $(W_n)_{n\in\mathbf{N}}$ sequence of $(n,\lambda n)$-Wishart matrices: $(\mu_{W_n/\lambda n})_{n\in\mathbf{N}}$ converges strongly to $\mu_{MP(\lambda)}$.

[Convergence in probability of the spectral distribution and of the extreme eigenvalues.]

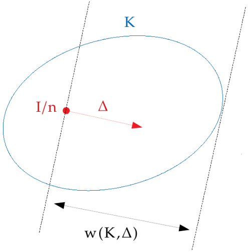
Outline

Mean width of a set of states

Definitions

Let K be a convex set of states on $\mathbf{C}^n$ containing the maximally mixed state I/n.

- For Δ a Hermitian on $\mathbf{C}^n$ s.t. $\|\Delta\|_2 = 1$, the *width of K in the direction Δ* is $w(K,\Delta) := \sup_{\sigma \in K} \mathrm{Tr}\left(\Delta\left(\sigma - \frac{I}{n}\right)\right)$.
- The *mean width of K* is the average of $w(K,\cdot)$ over the Hilbert-Schmidt unit sphere of Hermitians on $\mathbf{C}^n$, equipped with the uniform probability measure. Equivalently: $w(K) = \frac{1}{\gamma_n}\mathbf{E}w(K,G)$, with G a GUE matrix on $\mathbf{C}^n$ and $\gamma_n := \mathbf{E}\|G\|_2 \underset{n\to+\infty}{\sim} n$.



The mean width of a set of states is a measure of its size: for any ‘reasonable’ K, $w(K) \simeq vrad(K)$, where $vrad(K)$ is the *volume-radius* of K (i.e. the radius of the Euclidean ball with same volume as K).
Computing it amounts to estimating the supremum of some Gaussian process.

Mean width of the set of separable states

Observation: Denote by $\mathcal{D}$ the set of all states on $\mathbf{C}^n$. $\mathbf{E}\sup_{\sigma\in\mathcal{D}}\mathrm{Tr}(G(\sigma - I/n)) = \mathbf{E}\|G\|_\infty$. So by the semicircle law, the mean-width of $\mathcal{D}$ is asymptotically $2\sqrt{n}/\gamma_n$, i.e. $2/\sqrt{n}$.

Theorem [Mean width of the set of separable states (Aubrun/Szarek)]

Denote by $\mathcal{S}$ the set of separable states on $\mathbf{C}^d\otimes\mathbf{C}^d$.

There exist universal constants $c, C > 0$ s.t. $\dfrac{c}{d^{3/2}} \leqslant w(\mathcal{S}) \leqslant \dfrac{C}{d^{3/2}}$.

Remark: The mean width of the set of separable states is of order $1/d^{3/2}$, hence much smaller than the mean width of the set of all states (of order $1/d$).
$\longrightarrow$ On high dimensional bipartite systems, most states are entangled.

Proof idea: The extreme points of $\mathcal{S}$ are product pure states, of the form $|\varphi_1\rangle\langle\varphi_1|\otimes|\varphi_2\rangle\langle\varphi_2|$. So one can use the same arguments to estimate $\mathbf{E}\sup_{\sigma\in\mathcal{S}}\mathrm{Tr}(G\sigma)$ for G a GUE matrix on $\mathbf{C}^d\otimes\mathbf{C}^d$, as those used to estimate $\mathbf{E}\|g\|_\varepsilon$ for g a standard Gaussian vector in $\mathbf{C}^d\otimes\mathbf{C}^d$.

Mean width of the set of k-extendible states

Theorem [Mean width of the set of k-extendible states (Lancien)]

Fix $k \geqslant 2$ and denote by $\mathcal{E}_k$ the set of k-extendible states on $\mathbf{C}^d \otimes \mathbf{C}^d$.

Then, $w(\mathcal{E}_k) \underset{d\to\infty}{\sim} \dfrac{2/\sqrt{k}}{d}$.

Remark: The mean width of the set of k-extendible states is of order $1/d$, hence much bigger than the mean width of the set of separable states.
$\longrightarrow$ On high dimensional bipartite systems, the set of k-extendible states is a very rough approximation of the set of separable states.

Proof strategy: $\sup_{\sigma\in\mathcal{E}_k} \mathrm{Tr}(G(\sigma - I/d^2))$ can be expressed as $\|\widetilde{G}\|_\infty$ for some 'modified' GUE matrix $\widetilde{G}$. So one has to estimate $\mathbf{E}\|\widetilde{G}\|_\infty$.
This can be done by computing the p-order moments $\mathbf{E}\,\mathrm{Tr}\,\widetilde{G}^p$, and identifying the limiting spectral distribution (after rescaling by d/k): a centered semicircular distribution $\mu_{SC(0,k)}$.
The latter has $2\sqrt{k}$ as upper-edge.

Note: Similarly, the mean width of the set of PPT states is of order $1/d$ (Aubrun/Szarek).
$\longrightarrow$ On high dimensional bipartite systems, the set of PPT states is also a very rough approximation of the set of separable states.

Outline

Random induced states

Setting: System of interest H coupled with an environment H'.
Given a global pure state $\psi \in \mathrm{H} \otimes \mathrm{H}'$, its *marginal* on H $\rho := \mathrm{Tr}_{\mathrm{H}'} |\psi\rangle\langle\psi|$ is a mixed state.

Set $\mathrm{H} \equiv \mathbf{C}^n$ and $\mathrm{H}' \equiv \mathbf{C}^s$.
$\longrightarrow$ Random mixed state model on the system $\mathbf{C}^n$, induced by the environment $\mathbf{C}^s$:
$\rho = \mathrm{Tr}_{\mathbf{C}^s} |\psi\rangle\langle\psi|$ with $\psi \in \mathbf{C}^n \otimes \mathbf{C}^s$ a uniformly distributed unit vector.
Equivalent description: $\rho = \frac{W}{\mathrm{Tr}\, W}$ with W a (n, s)-Wishart matrix.

Question: Given $d \in \mathbf{N}$, consider ρ a random state on $\mathbf{C}^d \otimes \mathbf{C}^d$ induced by an environment $\mathbf{C}^s$, for some $s \in \mathbf{N}$.
For which values of s is ρ typically separable? PPT? k-extendible?
$\hookrightarrow$ with probability going to 1 (exponentially) as d grows

Hence 2 steps:

- Identify the range of s where ρ is, on average, separable / PPT / k-extendible.
- Show that the average behavior is generic in high dimension (concentration of measure phenomenon: a sufficiently 'well-behaved' function has an exponentially small probability of deviating from its average as the dimension grows).

Separability of random induced states

Theorem [Separability threshold for random induced states (Aubrun/Szarek/Ye)]

Let ρ be a random state on $\mathbf{C}^d \otimes \mathbf{C}^d$ induced by $\mathbf{C}^s$. There exists a threshold s_0 satisfying $cd^3 \leqslant s_0 \leqslant Cd^3 \log^2 d$ for some universal constants $c, C > 0$ s.t., if $s < s_0$ then ρ is typically entangled, and if $s > s_0$ then ρ is typically separable.

Intuition: If $s \leqslant d^2$ then ρ is uniformly distributed on the set of states of rank at most s, therefore generically entangled. If $s \gg d^2$ then ρ is expected to be close to Id/d^2, therefore separable.
$\longrightarrow$ Phase transition between these two regimes?

Proof idea: Convex geometry + Comparison of random matrix ensembles (majorization) + Concentration of measure in high dimension.

PPT of random induced states

Theorem [PPT threshold for random induced states (Aubrun)]

Let ρ be a random state on $\mathbf{C}^d \otimes \mathbf{C}^d$ induced by $\mathbf{C}^s$.
If $s < 4d^2$ then ρ is typically not PPT, and if $s > 4d^2$ then ρ is typically PPT.

Remark: The threshold environment dimension at which random induced states are generically either PPT or not is of order d^2, hence much smaller than the one at which they are generically either separable or entangled (of order d^3).
$\longrightarrow$ In the range $d^2 \lesssim s \lesssim d^3$, typical entanglement of random induced states is typically not detected by the PPT test.

Proof strategy: Everything boils down to characterizing when the partial transposition of a (d^2, s)-Wishart matrix W is positive.
This can be done by computing the p-order moments $\mathbf{E}\,\mathrm{Tr}\,\Gamma(W)^p$, and identifying the limiting spectral distribution (after rescaling by s): a non-centered semicircular distribution $\mu_{SC(1,d^2/s)}$.
The latter has positive support iff $1 - 2\sqrt{d^2/s} \geqslant 0$ i.e. iff $s \geqslant 4d^2$.

Note: Similarly, the threshold environment dimension at which random induced states are generically either k-extendible or not is of order d^2 (Lancien).
$\longrightarrow$ In the range $d^2 \lesssim s \lesssim d^3$, typical entanglement of random induced states is typically not detected by the k-extendibility test.

Perspectives

- On high dimensional bipartite systems, the volume of PPT or k-extendible states is more like the one of all states than like the one of separable states.
  For ρ a random state on $\mathbf{C}^d \otimes \mathbf{C}^d$ induced by $\mathbf{C}^s$, if $Cd^2 \leqslant s \leqslant cd^3$, then with high probability ρ is entangled but this entanglement is not detected by the PPT or k-extendibility test.
  $\longrightarrow$ Asymptotic weakness of these necessary conditions for separability.
- Similar features are exhibited by all other known entanglement criteria.
- What about generalizations to the multipartite setting?
  First need: Entanglement criteria which are not designed only for bipartite states.
  Proposal: Construction of such criteria through tensor norms (Jivulescu/Lancien/Nechita).
  $\longrightarrow$ Can their typical performance be quantified?

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