The Causal Structure of Quantum Information

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Inria Grenoble – Rhône-Alpes

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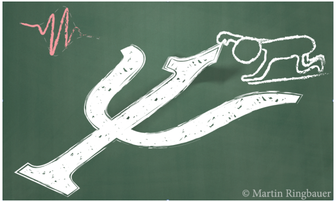
Quantum Foundations

Quantum foundations: the study of the conceptual and mathematical underpinnings of quantum theory



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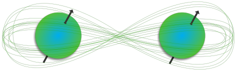
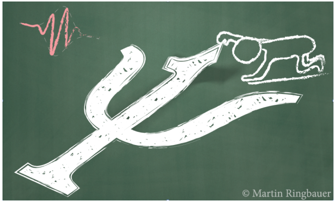
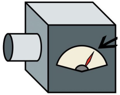
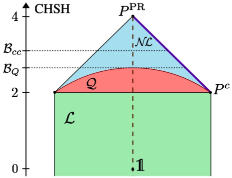


Quantum entanglement

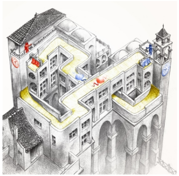




Quantum measurement



Quantum correlations



Quantum causality

Quantum Foundations and Quantum Information

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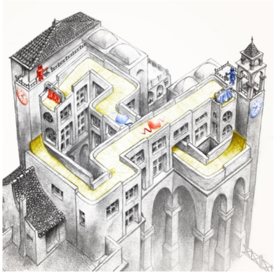
This lecture: Quantum causality



- Understand causal structure of quantum theory
- Fundamentally quantum causal structures?
- Exploit this for quantum information processing?

Outline

Cause and Effect

What does it mean to say that X causes Y?

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Correlation $\neq$ causation

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- Causal (Bayesian) models: Framework to describe causal relations from observed correlations
  - J. Pearl, *Causality* (2000).

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How to define causal relations between quantum events?

- What do we take to be quantum events?

  Classically: **Events:** random variables; **Interventions:** stochastic maps

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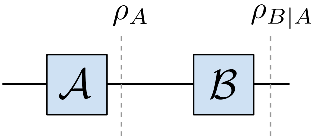

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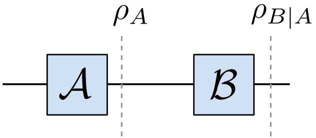

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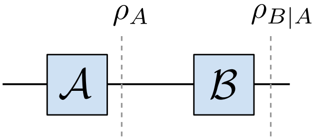

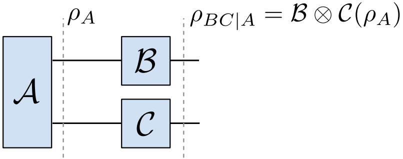

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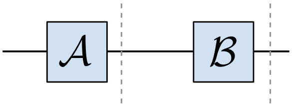

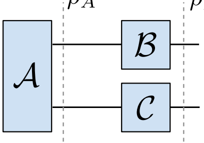

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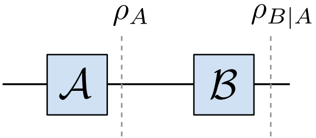

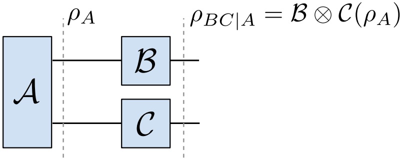

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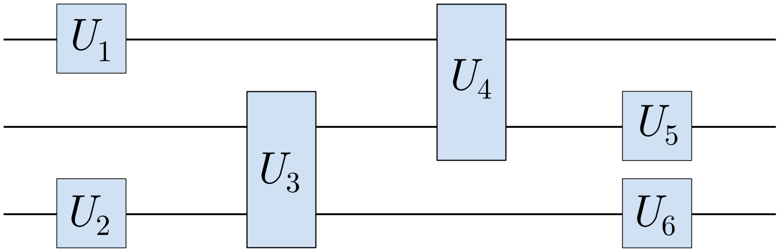
Causal structure defined by ability to influence or "signal" from one operation to another

- Quantum causal models [Barrett, Lorenz, Oreshkov, 2019]

Causal Structure of Quantum Circuits

Quantum circuits have a causal structure



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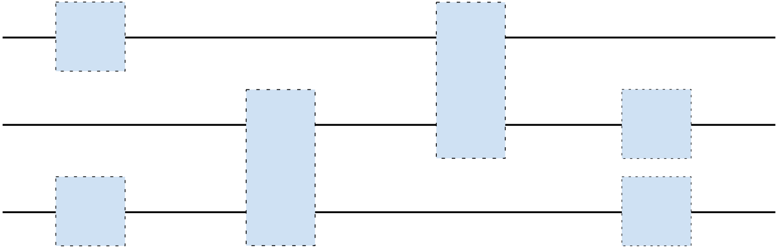

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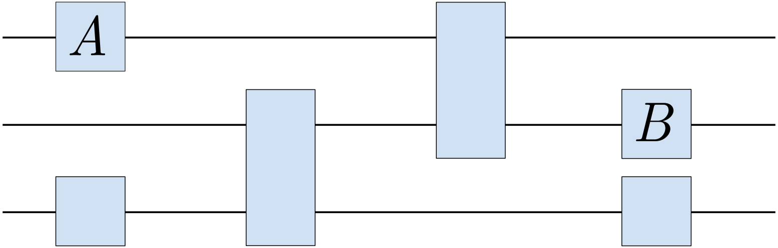

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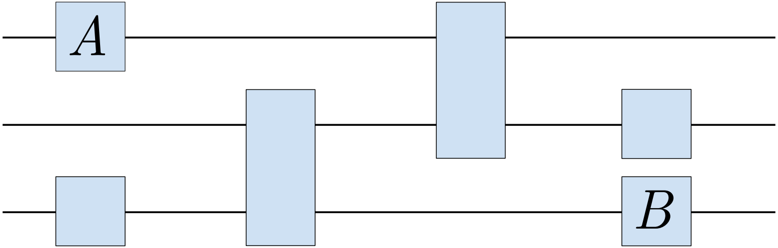
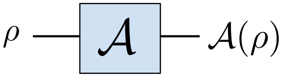


- Need to consider circuits with modifiable operations
- A circuit location A can influence another B if there is there is a path from A to B

Defining the Scenario

Consider causal structure in a computational scenario

- Black-box "operations" (quantum channels)
  - $\mathcal{A} : \mathcal{L}(\mathcal{H}^{A^I}) \to \mathcal{L}(\mathcal{H}^{A^O})$



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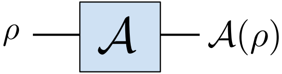
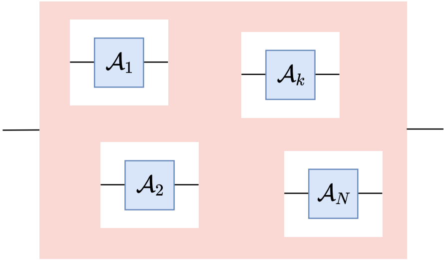
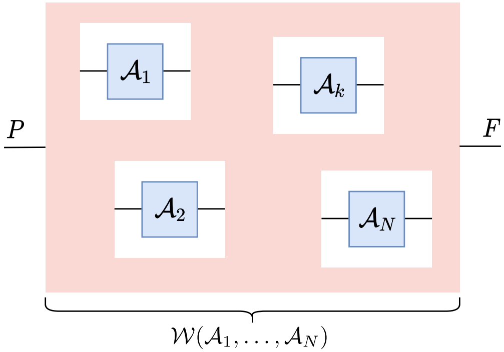


- Consume N queries $\mathcal{A}_1, \ldots, \mathcal{A}_N$ in some "computation"
  - May have all $\mathcal{A}_i \equiv \mathcal{A}$ (N queries to $\mathcal{A}$), but they could also be different operations



Higher Order Operations (Quantum Supermaps)



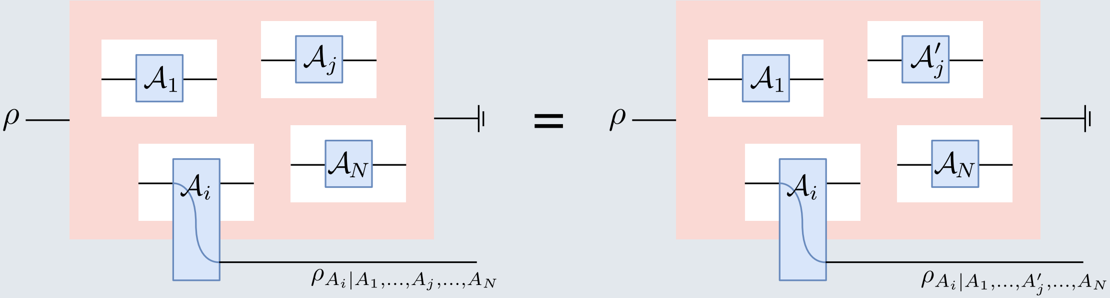
Higher order operation: $(\mathcal{A}_1, \dots, \mathcal{A}_N) \mapsto \mathcal{W}(\mathcal{A}_1, \dots, \mathcal{A}_N) : \mathcal{L}(\mathcal{H}^P) \to \mathcal{L}(\mathcal{H}^F)$

- Multilinear in its arguments
- $\mathcal{W}(\mathcal{A}_1, \dots, \mathcal{A}_N)$ a quantum channel whenever $\mathcal{A}_1, \dots, \mathcal{A}_N$ are quantum channels

Formalising Causal Order

Compatibility with causal order

$\mathcal{W}$ is compatible with $A_1 \prec A_2 \prec \cdots \prec A_N$ if, for all $i < j$, A_j cannot signal to A_i:



for all $\rho, \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \dots, \mathcal{A}_j, \mathcal{A}_j', \dots, \mathcal{A}_N$.

Note: $\mathcal{W}$ can be consistent with several causal orders

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What is the most general way to combine $\mathcal{A}_1, \ldots, \mathcal{A}_N$ compatible with $A_1 \prec A_2 \prec \cdots \prec A_N$?

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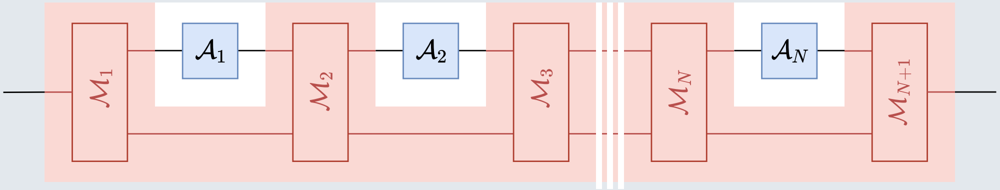
Theorem (Chiribella, D'Ariano, Perinotti (2009))

$\mathcal{W}$ is compatible with $A_1 \prec A_2 \prec \cdots \prec A_N$ if and only if it has the form:



This quantum circuit with N open slots is called a quantum comb

- *Alternatively: process operator, channel with memory, process tensor, ...*

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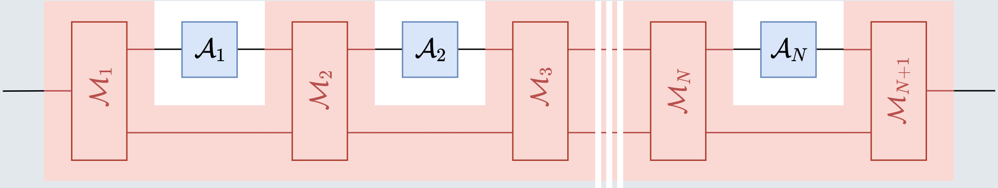

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Quantum circuits are the most general computation with a *fixed causal structure*.

Formalising Quantum Combs

Higher order operations can be nicely formulated in the Choi picture

Choi-Jamiołkowski isomorphism

CP maps $\mathcal{C} : \mathcal{L}(\mathcal{H}^X) \to \mathcal{L}(\mathcal{H}^Y)$ are in a bijection with PSD operators $C \in \mathcal{L}(\mathcal{H}^X \otimes \mathcal{H}^Y)$

$$C = \mathcal{I} \otimes \mathcal{C}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|), \quad \text{where } |\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$$

- TP condition: $\mathrm{Tr}_Y[C] = \mathbb{1}^X$
- Inverse: $\mathcal{C}(\rho) = \mathrm{Tr}_X[(\rho^T \otimes \mathbb{1})C] = \rho * C$

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Higher order maps:

- $\mathcal{W} \leftrightarrow W \in \mathcal{L}(\otimes_i(\mathcal{H}^{A_i^I} \otimes \mathcal{H}^{A_i^O}))$

Quantum combs: Choi operator W has nice additional structure

- Can be characterised with semidefinite programming

Outline

Beyond Fixed Causal Structures

Fundamental questions:

- What are the physical limits of information processing?
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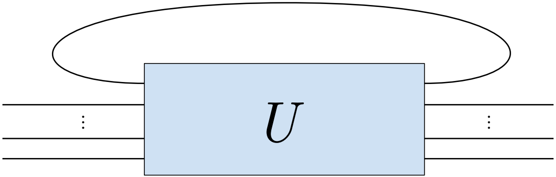
One idea: Quantum circuits with closed-timelike curves (CTCs) [Deutsch, 1991]:



- Nature "magically" finds a consistent fixed point solution
- Compatible with general relativity
- Computationally (too?) powerful ($\mathsf{P}_{\mathsf{CTC}} = \mathsf{BQP}_{\mathsf{CTC}} = \mathsf{PSPACE}$) [Aaronson & Watrous, 2008]
- Nonlinear, ...

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Quantum Causal Structure: Can we have intrinsically quantum causal relations?

- For example, superposition of cause and effect relations?
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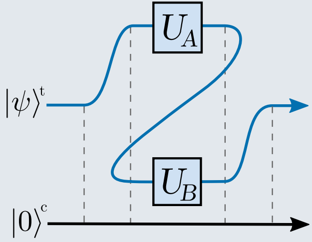
Use a quantum system to coherently control order that two quantum operations (channels/unitaries) are applied.

- U_A, U_B: two unknown unitaries
- $|\psi\rangle^t$: target system; $|\phi\rangle^c$: control system



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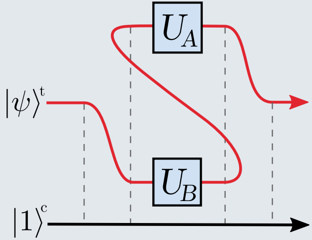

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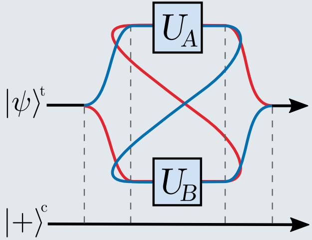
$$U_B U_A \otimes |0\rangle\langle 0|^c + U_A U_B \otimes |1\rangle\langle 1|^c$$



The Quantum Switch

The Quantum Switch is a quantum supermap: $(\mathcal{A},\mathcal{B}) \mapsto \mathcal{W}_{\text{switch}}(\mathcal{A},\mathcal{B})$

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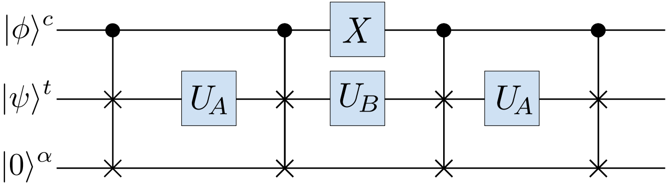

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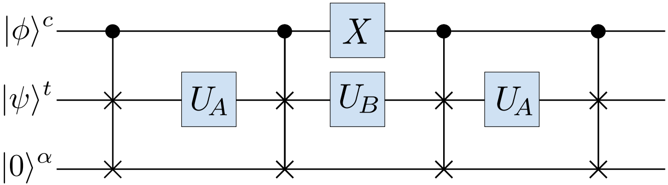

The Quantum Switch is physically meaningful

- Several experimental realisations with quantum optics
  - Vienna, Brisbane, Shanghai, Concepción, ...

Application: Commuting/Anticommuting Unitaries

Commuting vs. Anticommuting Unitary Problem (Chiribella [2012])

Input: Unitaries U_A, U_B (oracle access)

Promise: U_A and U_B either:

- Commute: $[U_A, U_B] = U_A U_B - U_B U_A = 0$,
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- Recall: $\mathcal{W}_{\text{switch}} : (U_A, U_B) \mapsto U_B U_A \otimes |0\rangle\langle 0|^c + U_A U_B \otimes |1\rangle\langle 1|^c$

$$|\psi\rangle^t \otimes \tfrac{1}{\sqrt{2}}(|0\rangle^c + |1\rangle^c) \xrightarrow{\mathcal{W}_{\text{switch}}} \tfrac{1}{\sqrt{2}}(U_B U_A |\psi\rangle^t \otimes |0\rangle^c + U_A U_B |\psi\rangle^t \otimes |1\rangle^c)$$
$$= \tfrac{1}{2}\{U_A, U_B\} |\psi\rangle^t \otimes |+\rangle^c - \tfrac{1}{2}[U_A, U_B] |\psi\rangle^t \otimes |-\rangle^c$$

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$$|\psi\rangle^t \otimes \tfrac{1}{\sqrt{2}}(|0\rangle^c + |1\rangle^c) \xrightarrow{\mathcal{W}_{\text{switch}}} \tfrac{1}{\sqrt{2}}(U_B U_A |\psi\rangle^t \otimes |0\rangle^c + U_A U_B |\psi\rangle^t \otimes |1\rangle^c)$$
$$= \tfrac{1}{2}\{U_A, U_B\} |\psi\rangle^t \otimes |+\rangle^c - \tfrac{1}{2}[U_A, U_B] |\psi\rangle^t \otimes |-\rangle^c$$

- Impossible with a quantum comb

Quantum N-Switch

How powerful is quantum control?

Quantum N-Switch

How powerful is quantum control?

- First step: Generalise the quantum switch

Quantum N-Switch

The quantum N-switch is a supermap $\mathcal{W}_N$:

$$(U_1, \ldots, U_N) \mapsto \sum_\pi U_{\pi(N)} \cdots U_{\pi(1)} \otimes |\pi\rangle\langle\pi|^c,$$

where π is a permutation of $(1, \ldots, N)$.

- Coherent control of all $N!$ orders of gates

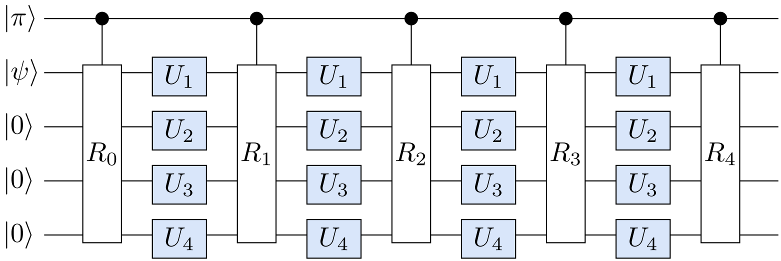
If we initialise the control to $|\phi\rangle^c = \frac{1}{\sqrt{N!}} \sum_\pi |\pi\rangle_c$, we apply the gates in a superposition of all possible orders:

$$|\psi\rangle^t \otimes |\phi\rangle^c \to \frac{1}{\sqrt{N!}} \sum_\pi U_{\pi(N)} \cdots U_{\pi(1)} |\psi\rangle^t \otimes |\pi\rangle^c.$$

Simulating the N-Switch

How much overhead would simulating this with a quantum comb require?

- Naive simulation: N^2 queries (N of each U_i)



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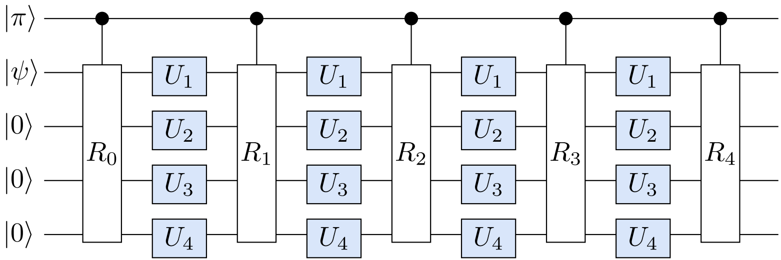

Theorem (Facchini and Perdrix (2014))

Any circuit simulating $\mathcal{W}_N$ *requires at least* $N^2 - o(n^{7/4+\epsilon})$ *queries to* $\{U_1, \ldots, U_N\}$.

Fourier Promise Problem

Fourier Promise Problem (Araújo, Costa, Brukner [2014])

Input: Unitaries $U_1, \ldots, U_N$ (oracle access).

Promise: Let $x = 0, \ldots, N! - 1$ be a labelling of permutations π_x and

$$\Pi_x = U_{\pi_x(N-1)} \cdots U_{\pi_x(1)} U_{\pi_x(0)}.$$

Then the unitaries satisfy

$$\forall x, \quad \Pi_x = \omega^{xy} \Pi_0, \quad (\text{where } \omega = e^{i\frac{2\pi}{N!}})$$

for some $y \in \{0, \ldots, N! - 1\}$.

Problem: Find y.

- Quantum N-Switch: Solves perfectly (N total queries)

Quantum N-Switch: Discussion

What do we learn from the N-Switch?

- There are physically meaningful computations beyond the circuit model
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  - Other variants: Hadamard promise problem, ...
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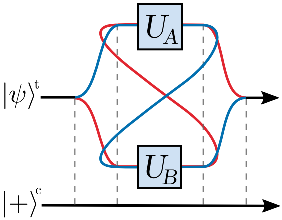
Not a general model of computation with quantum control!

Towards a General Model

We can have more general "switch-like" computations

- Transformations on target between unitaries
- Allow a quantum memory between queries



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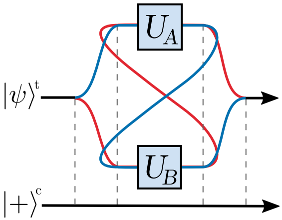

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- In quantum switch, control is fixed initially: static control
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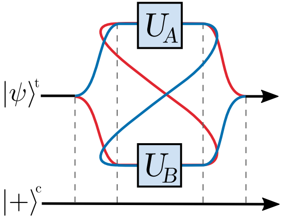

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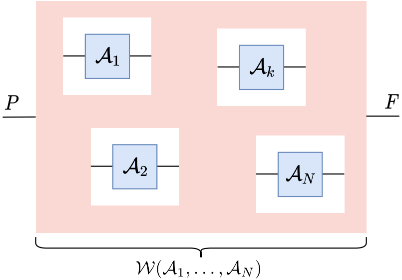
Goal: A generalised model of computation incorporating quantum control

Reminder of Scenario

Goal

A generalised model of computation incorporating quantum control:

- A physically well-defined (linear) quantum supermap $\mathcal{W}$
- Composition of $\mathcal{A}_1, \ldots, \mathcal{A}_N$ not necessarily in a well-defined, classical order



A subtle requirement:

- At the end, every operation should have been applied exactly once
- Necessary for $\mathcal{W}$ to be linear and well-defined

Circuits with Classical Control of Causal Order

Simpler situation: quantum circuits with classical control of causal order

- Dynamical control structure: determine at each step which operation to apply

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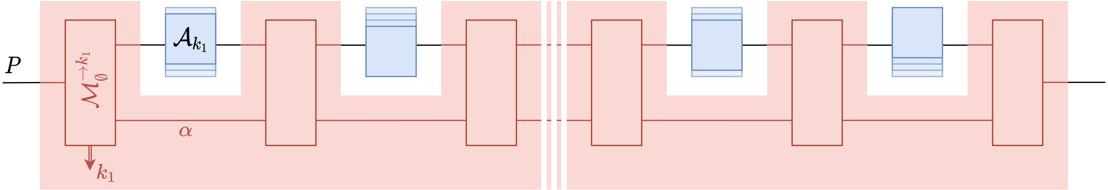

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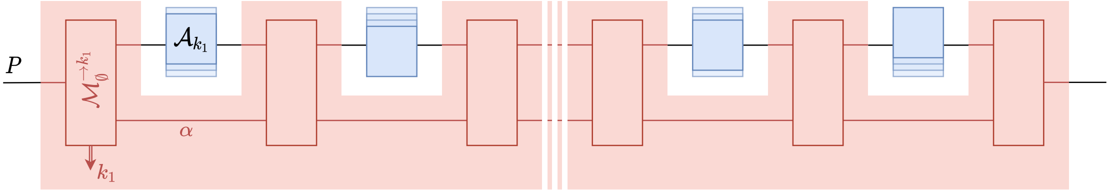

1) Input to circuit: a state $\rho \in \mathcal{L}(\mathcal{H}_P)$
2) Perform a quantum instrument $\{\mathcal{M}_\emptyset^{\to k_1}\}_{k_1}$. Apply $\mathcal{A}_{k_1}$ to the target subsystem of $\mathcal{M}_\emptyset^{\to k_1}(\rho) \in \mathcal{L}(\mathcal{H}^t \otimes \mathcal{H}^\alpha)$

Quantum instruments (generalised quantum measurements)

A quantum instrument is a set $\{\mathcal{M}_a\}_a$ of CP maps such that $\mathcal{M} = \sum_a \mathcal{M}_a$ is CPTP.
Obtain outcome a with probability $\mathrm{Tr}[\mathcal{M}_a(\rho)]$ and state becomes $\mathcal{M}_a(\rho)/\,\mathrm{Tr}[\mathcal{M}_a(\rho)]$.

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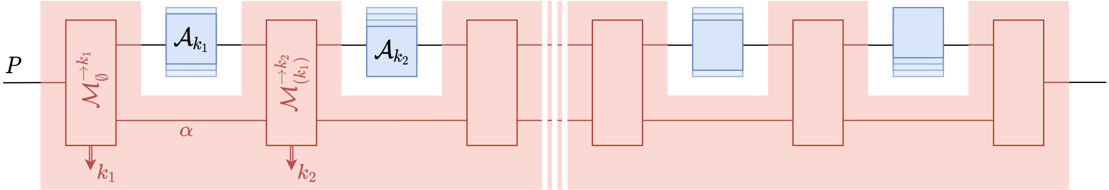

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4) Etc.

[Wechs, Dourdent, Abbott, Branciard (2021)]

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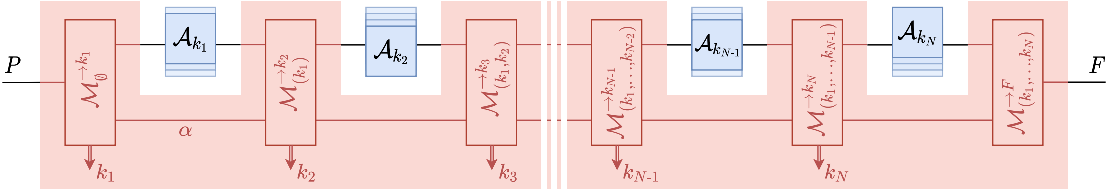

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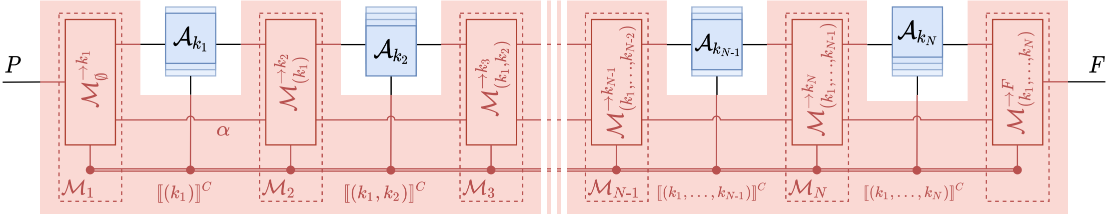

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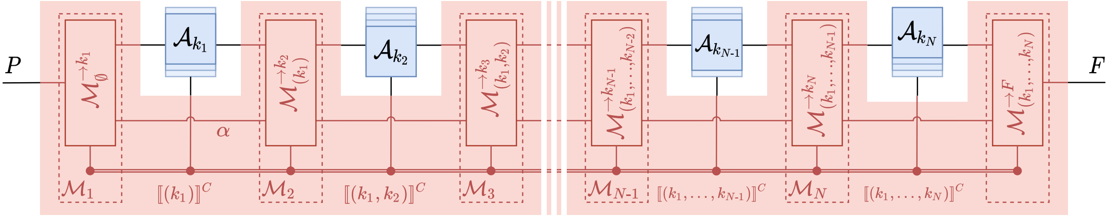

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- Classically controlled operations:

$$\mathcal{M}_1 = \textstyle\sum_{k_1} \mathcal{M}_\emptyset^{\to k_1} \otimes [\![(k_1)]\!]$$

$$\mathcal{M}_2 = \textstyle\sum_{k_1,k_2} \mathcal{M}_{\{k_1\}}^{\to k_2} \otimes \Pi_{(k_1),k_2}$$

$$\vdots$$

From Classical to Quantum Control

Turning the classical into quantum control requires a few tweaks:

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- Quantum control system: $|\{k_1, \dots, k_{n-1}\}, k_n\rangle$
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  - $\{k_1, \dots, k_{n-1}\}$: History recording which operations have already been used
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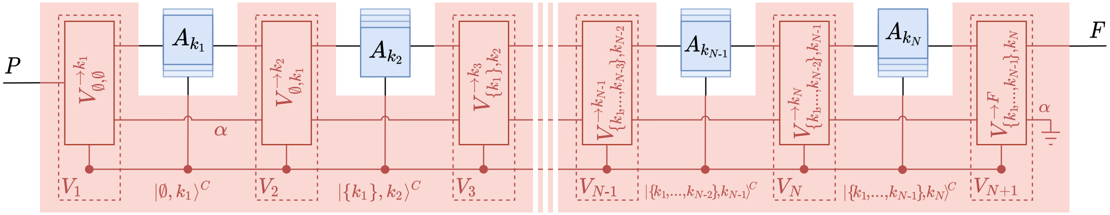
- Circuit evolves coherently, exploring causal structures in a quantum superposition



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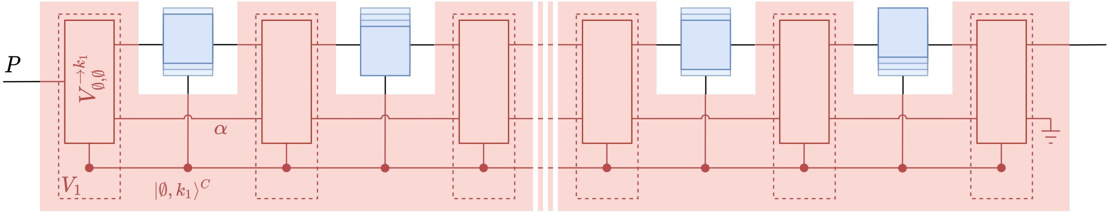
Circuits with Quantum Control of Causal Order



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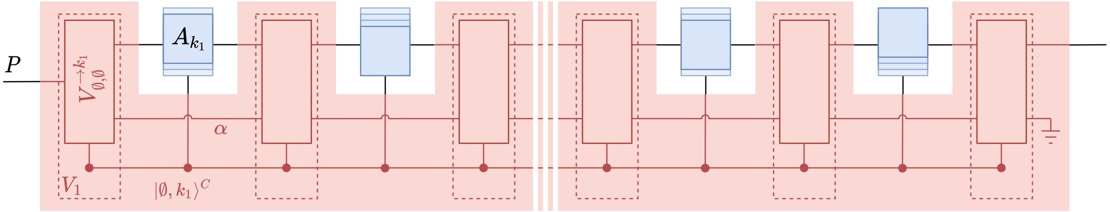

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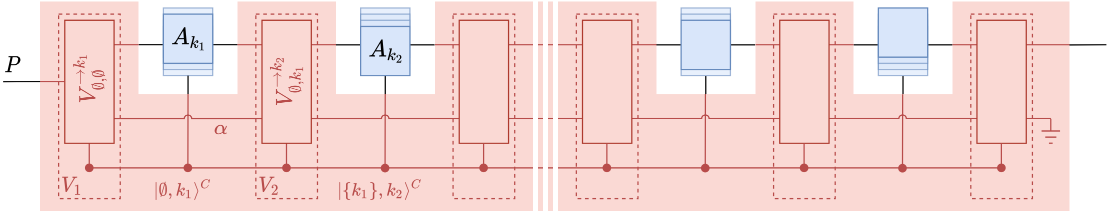

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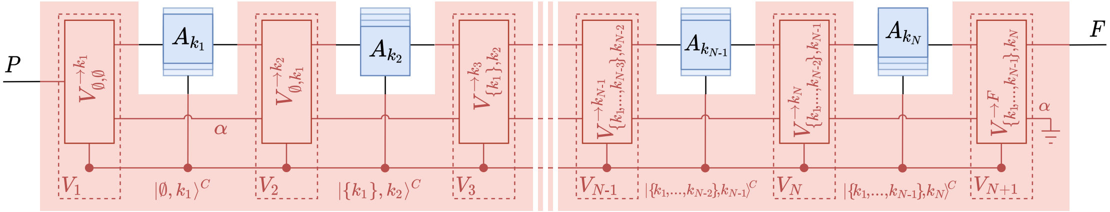

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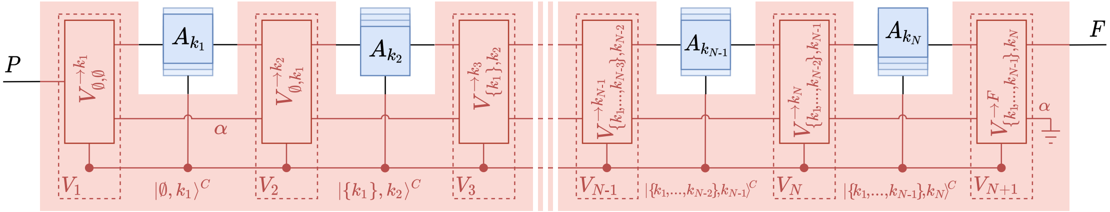

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- ...
- V_{N+1}: $\to \left(\sum_{(k_1,\ldots,k_N)} V_{\{k_1,\ldots,k_{N-1}\},k_N}^{\to F}(A_{k_N} \otimes \mathbb{1}^\alpha) \cdots (V_{\emptyset,\emptyset}^{\to k_1} |\psi\rangle)^{t\alpha}\right) \otimes |\{1,\ldots,N\}, F\rangle$

Beyond the Quantum Switch

The quantum N-switch can be represented as quantum circuit with quantum control of causal order

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Can we do anything else with this model?

- Yes! We can have quantum dynamical causal structure.
- Before slot $n+1$ we apply

$$V_{n+1} = \sum_{\mathcal{K}_{n-1}, k_n, k_{n+1}} V_{\mathcal{K}_{n-1}, k_n}^{\to k_{n+1}} \otimes |\mathcal{K}_{n-1} \cup \{k_n\}, k_{n+1}\rangle\langle \mathcal{K}_{n-1}, k_n|$$

  - k_{n+1} can depend on outcome of previous operations

- Examples that exploit this have very recently been devised
- Computations making use of this are active research topic

[Wechs, Dourdent, Abbott, Branciard (2021)]

Discussion

What can we do with quantum circuits with quantum control of causal order (QC-QCs)?

- Can still simulate with $O(N^2)$ queries
- What types of problems can we solve "efficiently" with QC-QCs?
- Have an SDP characterisation, formulating interesting families of QC-QCs difficult
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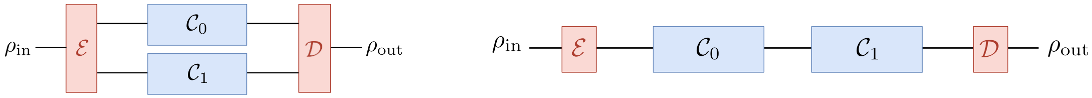
Quantum Shannon Theory without Causal Order

Quantum Shannon theory: quantum states, channels, but classical, causal trajectories for information carriers



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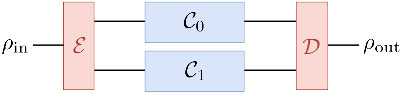
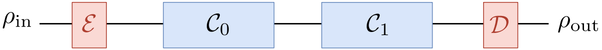
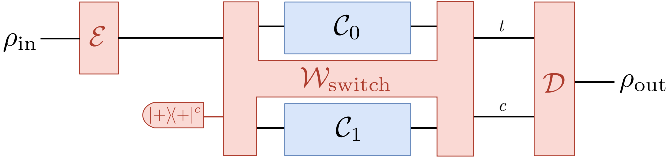




What if we allow for superposition, or quantum control, of trajectories?



[Chiribella & Kristjánsson, *Quantum Shannon theory with superpositions of trajectories* (2019)]

Causal Activation of Capacity

- Recall (fully) depolarising channel:

$$\mathcal{N} : \rho \mapsto \mathrm{Tr}[\rho]\frac{\mathbb{1}}{d}$$

- $\mathcal{N} = \mathcal{N} \circ \mathcal{N}$ has zero capacity (classical and quantum)

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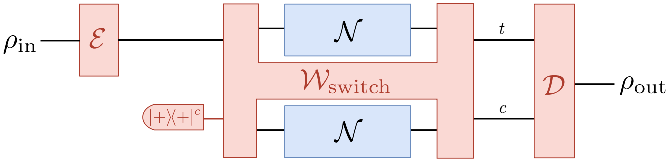
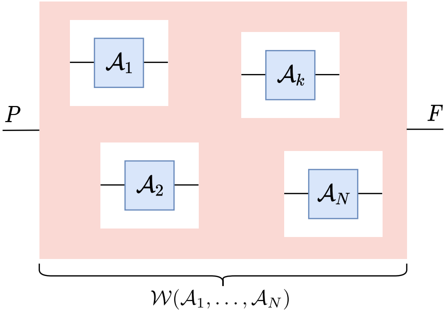
Causal activation: $\mathcal{W}_{|+\rangle\langle+|}(\mathcal{N}, \mathcal{N})$ has nonzero classical capacity



- Similar results for quantum capacity with other channels
- Meaning of these results still debated

[Salek, Ebler, Chiribella (2018)]

Causal Structure of Quantum Supermaps



Higher order operations, a.k.a. quantum supermaps:

- Quantum combs
- Quantum circuits with quantum control (QC-QCs)
- ???

Process Matrices

Quantum supermaps beyond QC-QCs exist

- Process matrix framework: most general operations compatible with local causality, but without any global causality constraint
- Introduced independently to study quantum gravity
- Beyond QC-QCs, not clear if they have a physical interpretation

[Oreshkov, Costa, Brukner (2012)]

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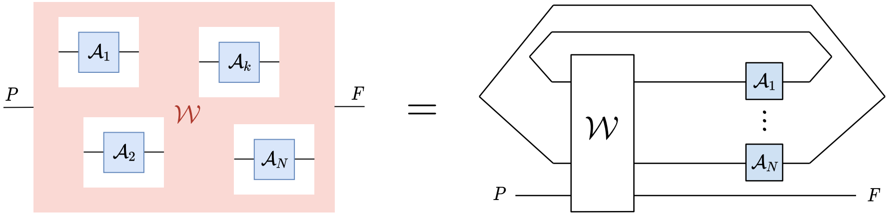

[Araújo, Guérin, Baumeler (2017)]

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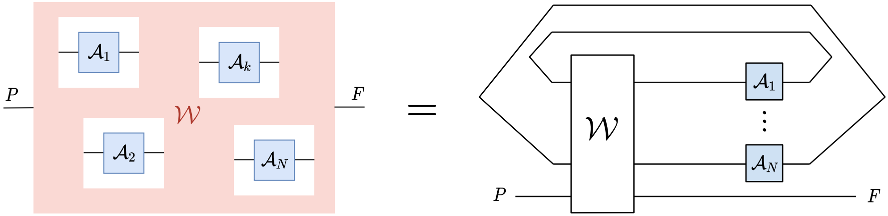


- $\mathsf{BQP}_{\ell\mathsf{CTCs}} \subseteq \mathsf{PP}$

[Araújo, Guérin, Baumeler (2017)]

Summary

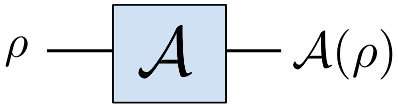
- Need new tools to analyse causal structure of quantum information processing
  - Quantum combs (higher order quantum maps)
  - Quantum causal models
- By exploiting quantum control, can process information in an indefinite causal order
  - Quantum switch
  - Generalised circuit model incorporating quantum control
  - Provides some advantages in quantum information
  - Only beginning to understand its potential and limits
- Fundamental questions
  - What are the limits of physical information processing?

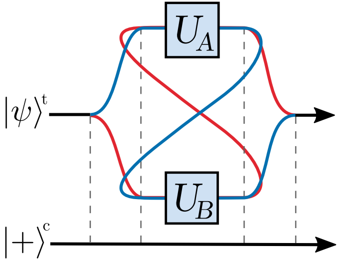
The Quantum Switch as a Supermap

The Quantum Switch is a quantum supermap: $(\mathcal{A},\mathcal{B}) \mapsto \mathcal{W}_{\text{switch}}(\mathcal{A},\mathcal{B})$

- If $\mathcal{A},\mathcal{B}$ are quantum channels with Kraus operators $\{K_i\}_i, \{L_j\}_j$, then $\mathcal{W}_{\text{switch}}(\mathcal{A},\mathcal{B})$ has Kraus operators

$$S_{ij} = L_j K_i \otimes |0\rangle\langle 0|^c + K_i L_j \otimes |1\rangle\langle 1|^c \,.$$





$\mathcal{A}(\rho) = \sum_i K_i \rho K_i^\dagger$

$[\mathcal{W}_{\text{switch}}(\mathcal{A},\mathcal{B})](\rho^{tc}) = \sum_{i,j} S_{ij}\rho^{tc} S_{ij}^\dagger$

Fourier Promise Problem Working

Quantum N-Switch: Solves perfectly (N total queries)

1) Initial state: $\frac{1}{\sqrt{N!}}\sum_{x=0}^{N!-1}|\psi\rangle^t\otimes|x\rangle^c$
2) Apply N-switch: $\to\frac{1}{\sqrt{N!}}\sum_{x=0}^{N!-1}\Pi_x|\psi\rangle^t\otimes|x\rangle^c$
3) Apply QFT to control: $\to\frac{1}{N!}\sum_{x,s=0}^{N!-1}\omega^{-xs}\Pi_x|\psi\rangle^t\otimes|s\rangle^c=\sum_{x,s=0}^{N!-1}\omega^{x(y-s)}\Pi_0|\psi\rangle^t\otimes|s\rangle^c$
4) Measure the control: $p(s)=\frac{1}{(N!)^2}\|\sum_{x=0}^{N!-1}\omega^{x(y-s)}\Pi_0|\psi\rangle^t\otimes|s\rangle^c\|^2=\delta_{s,y}$

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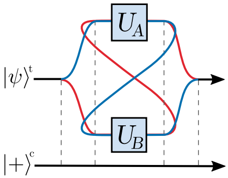
With a quantum comb?

- The simulating the switch requires $\Omega(N^2)$ queries.
- Other approaches: need to determine phase to accuracy $2\pi/N!$
- Rigorous error-robust analysis lacking thus far...

Example: Quantum Switch as a QC-QC



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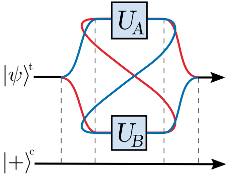

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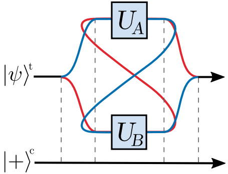

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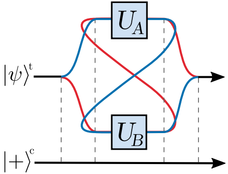

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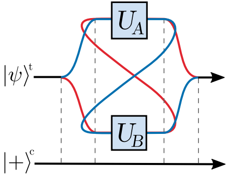

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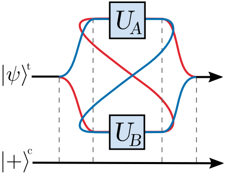

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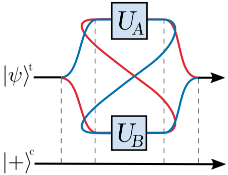

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