Adjoining roots to ring spectra and algebraic K-theory

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January 24, 2023

Overview

- **Main goal:** Algebraic K-theory of ring spectra.
- Motivation for algebraic K-theory of ring spectra (Morava E-theories E_n) through manifold topology.
- A new construction for adjoining roots to ring spectra.
- A new definition of logarithmic THH: root adjunction is logarithmic THH étale.
- Root adjunction results in a splitting at the level of K-theory
- We compute $V(1)_* \mathrm{K}(ko_p)$
- A simplified computation of $T(2)_* \mathrm{K}(ku_p)$
- We obtain $T(2)_* \mathrm{K}(ku/p)$

Motivation for K-theory of ring spectra

- For a topological space U,

$$A(U) := K(\Sigma^\infty_+ \Omega U)$$

- Let M be a closed differentiable manifold.

Theorem (Waldhausen)

There is an equivalence

$$A(M) \simeq \Sigma^\infty_+ M \oplus Wh^{Diff}(M)$$

of spectra.

- $Wh^{diff}(M)$ is the smooth Whitehead spectrum of M. Contains information on pseudo-isotopy space, h-cobordism space and diffeomorphism space of M.
- $A(M)$ is a module over $A(*) = K(\mathbb{S})$.
- **Goal:** Understand $K(\mathbb{S})$.

Ausoni-Rognes, Waldhausen program for $K(\mathbb{S})$

- Let L_n denote localization with the Johnson-Wilson spectrum $E(n)$.

Conjecture

$$K(\mathbb{S}_{(p)}) \xrightarrow{\simeq} \operatorname{holim} K(L_n \mathbb{S})$$

Conjecture

$$K(L_{K(n)}\mathbb{S}) \to K(E_n)^{hG_n}$$

is an isomorphism in sufficiently large dimensions (after smashing with a chromatic type $n+1$ finite spectrum).

Computations of Ausoni and Rognes

- $E_1 = KU_p$, $\pi_* ku_p \cong \mathbb{Z}_p[u_2]$, $|u_2| = 2$.

$$ku_p \simeq \bigoplus_{0 \le i < p-1} \Sigma^{2i} \ell_p$$

- ℓ is the Adams summand, $\pi_* \ell_p \cong \mathbb{Z}_p[v_1]$, $|v_1| = 2p-2$. Indeed,

$$\pi_* ku_p \cong \pi_*(\ell_p)(\sqrt[p-1]{v_1}) := \pi_*(\ell_p)[z]/(z^{p-1} - v_1).$$

- Ausoni and Rognes compute $V(1)_* K(\ell_p)$ for $p \ge 5$.

Theorem (Ausoni)

$$T(2)_* K(ku_p) \cong \big(T(2)_* K(\ell_p)\big)(\sqrt[p-1]{-v_2}).$$

- KU - Classifies virtual complex vector bundles.
- *Motivation for Ausoni's computation of* $V(1)_*\,\mathrm{K}(ku_p)$:

(Baas-Dundas-Richter-Rognes) $\mathrm{K}(ku)$ classifies virtual complex 2-vector bundles.

- KO classifies virtual real vector bundles and $\mathrm{K}(ko)$ classifies virtual real 2-vector bundles.

We compute $T(2)_*\,\mathrm{K}(ko)$

Adjoining a root to an E_2-algebra

- $m > 0$
- $k > 0$ be even
- X be an E_2-ring spectrum with even homotopy, let $x \in \pi_{mk} X$.
- Let $\mathbb{S}[\sigma_k]$ denote the free E_1-algebra over $\mathbb{S}^k$.
  - **Lurie:** $\mathbb{S}[\sigma_k]$ is an E_2-algebra, $\mathbb{S}[\sigma_2] \simeq \Sigma^\infty_+ \Omega S^3$.
  - **Lurie, Hahn-Wilson:** There is an E_2-map $\mathbb{S}[\sigma_{mk}] \to X$ carrying σ_{mk} to x in homotopy.
  - There is an E_2-map $\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$ carrying σ_{mk} to σ_k^m in homotopy.

Definition (Ausoni-B.-Moulinos)

$$X(\sqrt[m]{x}) := X \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k]$$

- $X(\sqrt[m]{x})$ is an X-algebra
- **Künneth spectral sequence:** $\pi_* X(\sqrt[m]{x}) \cong \pi_*(X)(\sqrt[m]{x})$

Examples

Theorem (Ausoni-B.-Moulinos)

There is an equivalence of E_1-ring spectra:

$$E_n \simeq \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} L_{K(n)} BP\langle n\rangle(\sqrt[p^n-1]{v_n}),$$

For $n = 1$,

$$ku_p \simeq \ell_p(\sqrt[p-1]{v_1}).$$

Graded ring structures

- The ring:

$$R(\sqrt[m]{r}) := R[z]/(z^m - r)$$

comes equipped with an R-module basis $\{1, z, ..., z^{m-1}\}$ compatible with the multiplicative structure.
- We say $R(\sqrt[m]{r})$ is a $\mathbb{Z}/m$-graded ring.
- Similarly, $X(\sqrt[m]{x})$ is a $\mathbb{Z}/m$-graded ring spectrum.

$\mathbb{Z}/m$-Graded ring spectra

- Y is a $\mathbb{Z}/m$-graded ring spectrum if

$$Y = \bigoplus_{[i]\in\mathbb{Z}/m} Y_i$$

and the multiplication map on Y

$$Y \wedge Y \to Y$$

splits into

$$Y_{[i]} \wedge Y_{[j]} \to Y_{[i+j]}$$

and the unit factors as $\mathbb{S} \to Y_0 \to Y$.

  - The symmetric monoidal ∞-category of $\mathbb{Z}/m$-graded spectra is given by

$$[(\mathbb{Z}/m)^{ds}, \mathcal{S}p]_{Day}.$$

A $\mathbb{Z}/m$-graded E_n-ring spectrum is an E_n-algebra in $[(\mathbb{Z}/m)^{ds}, \mathcal{S}p]_{Day}$.

Morava E-theories are $p^n - 1$-graded ring spectra

- **Lurie:** $\mathbb{S}[\sigma_k] \simeq \bigoplus_{i \geq 0} \Sigma^{ik}\mathbb{S}$ is a graded E_2-algebra.
- $X(\sqrt[m]{x}) := X \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k]$ is a $\mathbb{Z}/m$-graded E_1 X-algebra with

$$X(\sqrt[m]{x})_i \simeq \Sigma^{ik} X.$$

Theorem

Morava E-theory spectrum E_n is a $\mathbb{Z}/(p^n - 1)$-graded E_1 $E(n)$-algebra with

$$(E_n)_i \simeq \Sigma^{2i}\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} L_{K(n)}(BP\langle n\rangle);$$

ku_p is a $\mathbb{Z}/(p-1)$-graded ℓ-algebra with

$$ku_i \simeq \Sigma^{2i}\ell_p.$$

- Improves the classical result:

$$ku_p \simeq \bigoplus_{0 \leq i < p-1} \Sigma^{2i}\ell_p.$$

Graded ring spectra and THH

- $Y = \bigoplus_{i \in \mathbb{Z}/m} Y_i$ be a $\mathbb{Z}/m$-graded ring spectrum
- (Antieau-Mathew-Morrow-Nikolaus) $\mathrm{THH}(Y)$ is a $\mathbb{Z}/m$-graded S^1-spectrum in a canonical way, i.e. there is a canonical S^1-splitting

$$\mathrm{THH}(Y) = \bigoplus_{i \in \mathbb{Z}/m} \mathrm{THH}(Y)_i$$

THH and adjoining roots

- X a p-local E_2-ring spectrum with even homotopy
- $x \in \pi_{mk} X$.
- $\mathrm{THH}(X(\sqrt[m]{x}))$ is a $\mathbb{Z}/m$-graded S^1-spectrum:

$$\mathrm{THH}(X(\sqrt[m]{x})) \simeq \bigoplus_{i \in \mathbb{Z}/m} \mathrm{THH}(X(\sqrt[m]{x}))_i$$

Theorem (Ausoni-B.-Moulinos)

If $p \nmid m$, then the map

$$\mathrm{THH}(X) \xrightarrow{\simeq} \mathrm{THH}(X(\sqrt[m]{x}))_0$$

is an equivalence. In particular,

$$\mathrm{THH}(X) \to \mathrm{THH}(X(\sqrt[m]{x}))$$

is the inclusion of a wedge summand.

- We identify the other summands using log THH.

logarithmic THH and adjoining roots

- $m > 0$, $k > 0$ even, X- p-local E_2-ring spectrum with even homotopy, $x \in \pi_{km} X$.
  - A new definition of logarithmic THH (denoted by $\mathrm{THH}(- \mid -)$). Idea, include $d\log$ x as $\frac{dx}{x}$ in differential forms.

Definition (Ausoni-B.-Moulinos)

Logarithmic THH of X relative to x is

$$\mathrm{THH}(X \mid x) := \mathrm{THH}(X) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_{mk}])} \mathrm{THH}(\mathbb{S}[\sigma_{mk}^{\pm}])_{\geq 0}$$

where $(-)_{\geq 0}$ is the connective cover in the grading direction.

- (**Devalapurkar-Moulinos**): This is equivalent to the Rognes' definition of log THH.

Theorem (Ausoni-B.-Moulinos)

If $p \nmid m$, adjoining a root is logarithmic THH étale, i.e.

$$\mathrm{THH}(X(\sqrt[m]{x}) \mid \sqrt[m]{x}) \simeq X(\sqrt[m]{x}) \wedge_X \mathrm{THH}(X \mid x) \simeq \bigoplus_{0 \leq i < m} \Sigma^{ik} \mathrm{THH}(X \mid x)$$

logarithmic THH and adjoining roots

- $m > 0$, $k > 0$ even, X- p-local E_2-ring spectrum with even homotopy, $x \in \pi_{km}X$.

Theorem (Ausoni-B.-Moulinos)

If $p \nmid m$, then

$$\mathrm{THH}(X(\sqrt[m]{x})) \simeq \mathrm{THH}(X) \oplus \left(\bigoplus_{0<i<m} \Sigma^{ik}\, \mathrm{THH}(X \mid x) \right).$$

Algebraic K-theory

- $m > 0$, $|x| > 0$ even

Theorem (Ausoni, B., Moulinos)

If X is p-local and $p \nmid m$, then

$$K(X(\sqrt[m]{x})) \simeq K(X) \oplus M$$

for some spectrum M.

Theorem (Ausoni, B., Moulinos)

There is a $T(n+1)$-equivalence of spectra

$$K(E_n) \simeq K(\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} BP\langle n\rangle_p) \oplus N$$

for some spectrum N.

2-Vector bundles

- *Motivation for Ausoni's computation of $V(1)_* \mathrm{K}(ku_p)$:* (Baas-Dundas-Richter-Rognes) $\mathrm{K}(ku)$ classifies virtual complex 2-vector bundles.
- KO classifies virtual real vector bundles and $\mathrm{K}(ko)$ classifies virtual real 2-vector bundles.
- (Ausoni, B., Moulinos) For $p > 3$, there is an equivalence of E_1-ring spectra:

$$ku_p \simeq ko_p(\sqrt[2]{\alpha}),$$

and therefore, $\mathrm{K}(ku_p) \simeq \mathrm{K}(ko_p) \oplus M$.
- We obtain a complete description of $V(1)_* \mathrm{K}(ko_p)$ and:

Theorem (Ausoni, B., Moulinos)

There is an isomorphism

$$T(2)_* \mathrm{K}(ko) \cong \Big(T(2)_* \mathrm{K}(\ell)\Big)\big(\sqrt[\frac{p-1}{2}]{-v_2}\big)$$

Adams' splitting result for $K(ku)$

- $Y = ku_p \simeq \ell(\sqrt[p-1]{v_1})$.
- Fiber sequence:

$$\mathrm{TC}(ku_p) \to \prod_{i\in\mathbb{Z}/(p-1)} \mathrm{THH}(ku_p)_i^{hS^1} \xrightarrow{\varphi_p^{hS^1} - can} \prod_{i\in\mathbb{Z}/(p-1)} (\mathrm{THH}(ku_p)_i^{tC_p})^{hS^1}$$

- $\varphi_p\colon\ \mathrm{THH}(ku_p)_i \to \mathrm{THH}(ku_p)_{pi}^{tC_p}$
- $p = 1$ in $\mathbb{Z}/(p-1)$ so we have

$$\varphi_p\colon\ \mathrm{THH}(ku_p)_i \to \mathrm{THH}(ku_p)_i^{tC_p}$$

- The fiber sequence above splits providing a splitting

$$\mathrm{TC}(ku_p) \simeq \prod_{i\in\mathbb{Z}/(p-1)} \mathrm{TC}(ku_p)_i$$

Simplified computation of the algebraic K-theory of ku

- Simplifying the proof of Ausoni's theorem:

$$T(2)_* \mathrm{K}(ku_p) \cong T(2)_* \mathrm{K}(\ell_p)[b]/(b^{p-1} + v_2).$$

  - **Ausoni**: There is a map $K(\mathbb{Z}, 2) \simeq BU(1) \to GL_1(ku)$ providing a class $b \in T(2)_{2p+2} \mathrm{TC}(ku_p)$ satisfying $b(b^{p-1} + v_2) = 0$. Indeed, $b \in T(2)_{2p+2} \mathrm{TC}(ku_p)_1$.
  - $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$ is a $\mathbb{Z}/(p-1)$-graded E_∞-ring spectrum.
  - $\mathrm{TC}(ku_p)$ is a $\mathbb{Z}/(p-1)$-graded E_∞-ring spectrum:

$$\mathrm{TC}(ku_p) \simeq \bigoplus_{i \in \mathbb{Z}/(p-1)} \mathrm{TC}(ku_p)_i$$

  - Using logarithmic THH computations of Rognes, Sagave and Schlichtkrull: $b \in T(2)_* \mathrm{TC}(ku_p)$ is a unit, therefore, $b^{p-1} = -v_2$.
  - The following facts complete the proof:
    1)$\mathrm{TC}(ku_p)_0 \simeq \mathrm{TC}(\ell_p)$,
    2)$b \in T(2)_* \mathrm{TC}(ku_p)_1$ satisfy $b^{p-1} = -v_2$ and
    3)$T(2)_* \mathrm{TC}(ku_p)$ is a $\mathbb{Z}/(p-1)$-graded commutative ring.

Algebraic K-theory of two-periodic Morava K-theory

- $V(1)_* \mathrm{K}(k(1))$ is known due to Ausoni-Rognes.
- We compute $T(2)_* \mathrm{K}(ku/p)$, ku/p is the two-periodic Morava K-theory of height 1.

Theorem (B.)

Let $b^{p-1} = -v_2$, there is an isomorphism:

$$T(2)_* \mathrm{K}(ku/p) \cong T(2)_* \mathrm{K}(k(1)) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b].$$

- We construct ku/p as an E_1 ku_p-algebra in $\mathbb{Z}/(p-1)$-graded spectra. Therefore, $\mathrm{TC}(ku/p)$ is a module over $\mathrm{TC}(ku_p)$ in $\mathbb{Z}/(p-1)$-graded spectra.
- As $\mathbb{Z}/(p-1)$-graded E_1-rings: $ku/p \simeq k(1)(\sqrt[p-1]{v_1})$.
- Result follows by the following facts:
  1) $\mathrm{TC}(ku/p)_0 \simeq \mathrm{TC}(k(1))$
  2) There is a unit $b \in T(2)_* \mathrm{TC}(ku_p)_1$ with $b^{p-1} = -v_2$