Bijections between walks inside a triangular domain and Motzkin paths of bounded amplitude

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Lattice Paths, Combinatorics and Interactions
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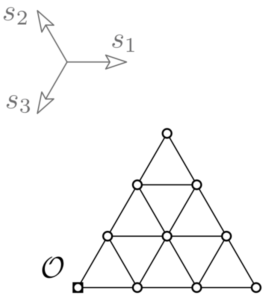
Forward triangular path: path confined within a triangular lattice $\mathcal{T}_L$ of size L, where the only allowed steps are in direction s_1, s_2, s_3.



Triangular lattice of size $L = 3$

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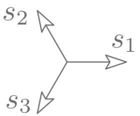

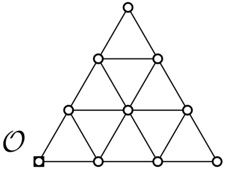

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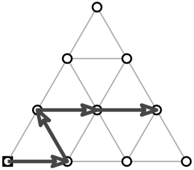
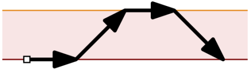


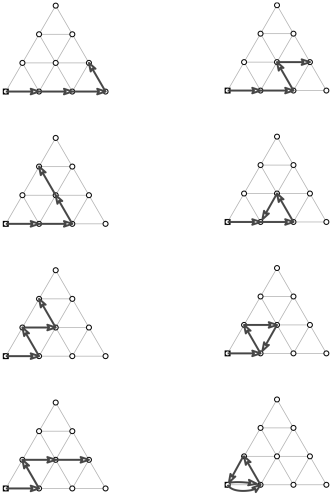
Example of forward path of length 4 starting from $\mathcal{O}$

Motzkin path: path made of steps $\nearrow$, $\rightarrow$ and $\searrow$, starting from height 0, ending at height 0, and never going below the horiz. axis.



Example of Motzkin path of length 4,
with maximal height equal to 1

Equinumeracy



Forward triangular paths of length 4,
on $\mathcal{T}_3$, starting from $\mathcal{O}$

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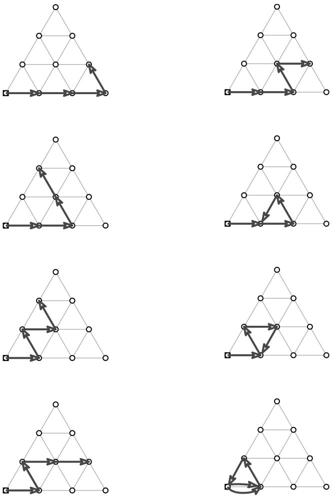
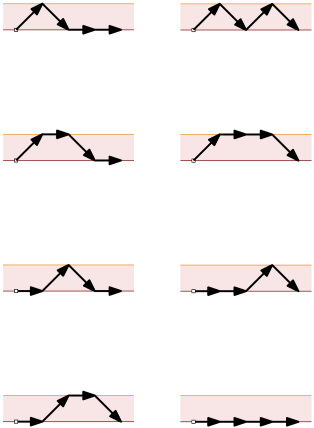


Forward triangular paths of length 4, on $\mathcal{T}_3$, starting from $\mathcal{O}$



Motzkin paths of length 4, with maximal height ≤ 1

Mortimer and Prellberg - EJC 2015

Observation: there are as many:

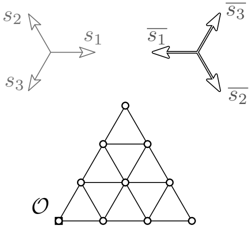
- forward triangular paths on $\mathcal{T}_{2L+1}$ of length n starting from $\mathcal{O}$
- Motzkin paths of length n with height bounded by L

Proof: genetaring functions, kernel method

Open problem: find a bijective proof!

Triangular paths

- A **forward** step is a step in direction s_1, s_2, or s_3.
- A **backward** step is a step in direction $\overline{s_1}, \overline{s_2}$, or $\overline{s_3}$.



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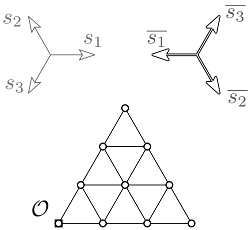

For $W \in \{F, B\}^n$, a path of direction vector W is a path of length n such that step i is forward if $W_i = F$ and backward if $W_i = B$.

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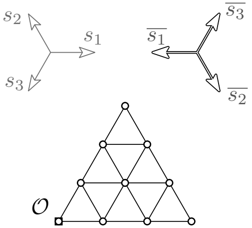

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Proposition

For any $W, W' \in \{F, B\}^n$, and any $z \in \mathcal{T}_L$, there are as many paths of direction vector W starting from z and confined in $\mathcal{T}_L$ as paths of direction vector W' starting from z and confined in $\mathcal{T}_L$.

Bijective proof

We define reversible operations that can be applied to any triangular path $(\omega_1, \dots, \omega_n)$ of $\mathcal{T}_L$ and s.t. the image is still in $\mathcal{T}_L$.

- A **swap flip** modifies two consecutive steps (ω_i, ω_{i+1}) according to:
  - $(s_j, \overline{s_k}) \longleftrightarrow (\overline{s_k}, s_j)$
    if $(\omega_i, \omega_{i+1}) = (s_j, \overline{s_k})$ or $(\omega_i, \omega_{i+1}) = (\overline{s_k}, s_j)$, with $j \neq k$
  - $(s_k, \overline{s_k}) \longleftrightarrow (\overline{s_{k-1}}, s_{k-1})$
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  This induces a flip $(F, B) \longleftrightarrow (B, F)$ in the direction vector.

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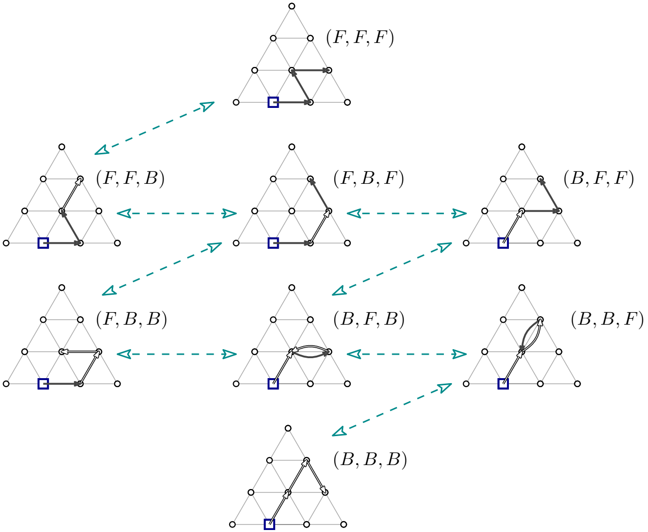
  This induces a flip $(F, B) \longleftrightarrow (B, F)$ in the direction vector.
- A **last-step flip** changes the direction of the last step ω_n according to:

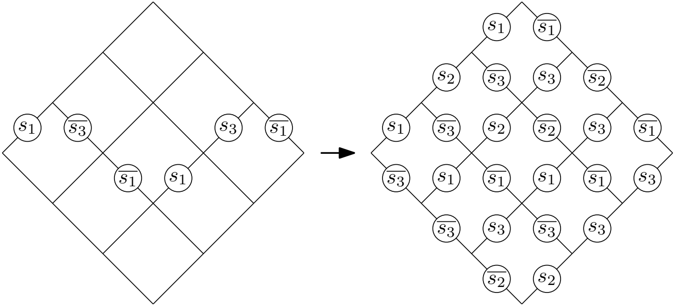
$$s_i \longleftrightarrow \overline{s_{i-1}}$$

Bijective proof



Another more symetric way to see the bijection, using only swap flips: we represent the "folding of the path" on a diamond lattice (with step $\nearrow$ when direction F and $\searrow$ for B).

Example of the path $s_1\ \overline{s_3}\ \overline{s_1}$.



A **Motzkin meander** is a suffix of a Motzkin path: it can start at any height and ends at height 0.

Notations:

- $f_n(z)$ =nb. of **forward paths** of length n starting from z and confined in $\mathcal{T}_{2L+1}$
- $m_n(\ell)$ =nb. of **Motzkin meanders** of length n starting from height ℓ and of maximal height $\leq L$

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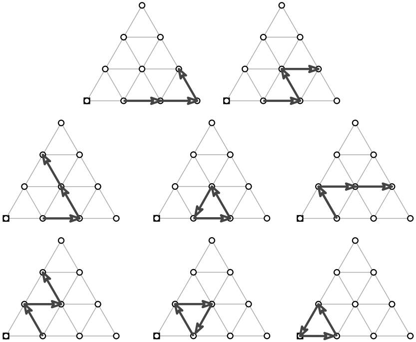
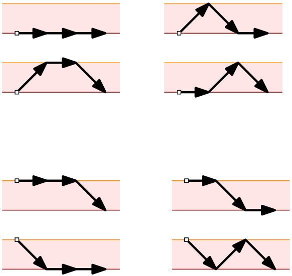
Proposition

For any $\ell \in \{0, \ldots, L\}$,

$$f_n(\mathcal{O} + \ell s_1) = \sum_{i=0}^{\ell} m_n(i).$$

For $\ell = 0$, this gives: $f_n(\mathcal{O}) = m_n(0)$.





forward paths of length 3
starting from $\mathcal{O} + s_1$
and confined in $\mathcal{T}_3$

Motzkin meanders of length n
starting from height 0 or 1
and of max. height ≤ 1

- Motzkin meanders:

$$m_n(\ell) = m_{n-1}(\ell - 1) + m_{n-1}(\ell) + m_{n-1}(\ell + 1), \quad \ell \in \{1, \ldots, L-1\}$$
$$m_n(0) = m_{n-1}(0) + m_{n-1}(1)$$
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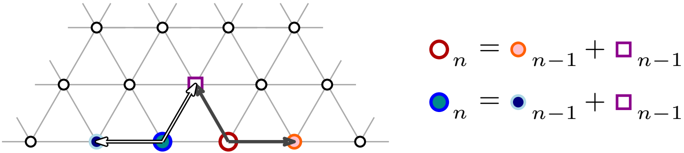
- Forward paths:



$$f_n(\mathcal{O}+\ell s_1) - f_n(\mathcal{O}+(\ell-1)s_1) = f_{n-1}(\mathcal{O}+(\ell+1)s_1) - f_{n-1}(\mathcal{O}+(\ell-2)s_1)$$

Extension to inner points of the triangle: to each point $z \in \mathcal{T}_{2L+1}$, we can associate coeff. $p_1(z), \dots, p_L(z)$ s.t.:

$$\forall n \geq 0, \quad f_n(z) = \sum_{i=0}^{L} p_i(z) m_n(i).$$

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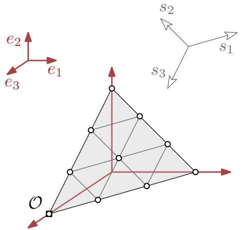
For $z = ie_1 + je_2 + ke_3$,

$$\frac{(1-x^{i+1})(1-x^{j+1})(1-x^{k+1})}{(1-x)^2}$$

$$= p_0(z) + p_1(z)x + \cdots + p_L(z)x^H + \cdots + p_{2L+1}(z)x^{2L+1}$$



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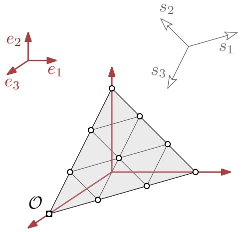
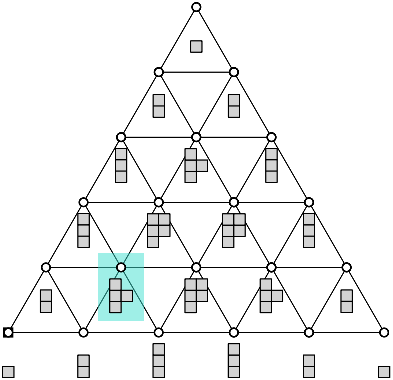


$$p_i(z+s_1) + p_i(z+s_2) + p_i(z+s_3) = p_{i-1}(z) + p_i(z) + p_{i+1}(z)$$
$$p_0(z+s_1) + p_0(z+s_2) + p_0(z+s_3) = p_0(z) + p_1(z)$$
$$p_L(z+s_1) + p_L(z+s_2) + p_L(z+s_3) = p_L(z) + p_{L-1}(z)$$

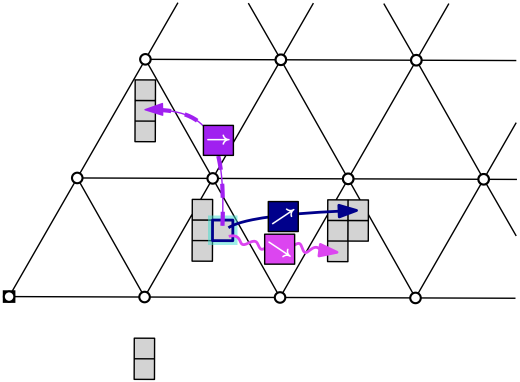
Profile of the triangle



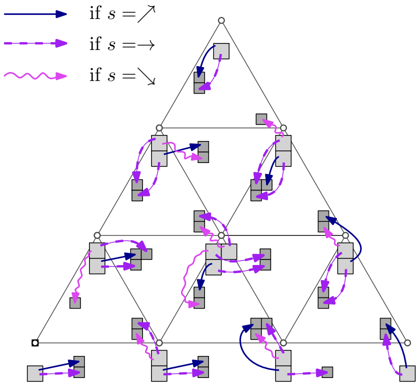
For $z = e_1 + e_2 + 3e_3$, we have $(p_0(z), p_1(z), p_2(z)) = (1, 2, 1)$ and:

$$f_n(z) = m_n(0) + 2m_n(1) + m_n(2)$$

Notion of scaffolding



Complete scaffolding



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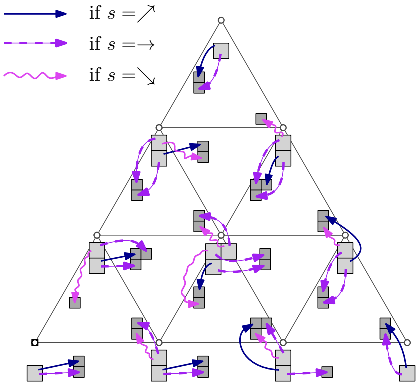

Given a scaffolding, we have a bijection between:

- forward triangular paths on $\mathcal{T}_{2L+1}$ of length n starting from $\mathcal{O}$
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Further results

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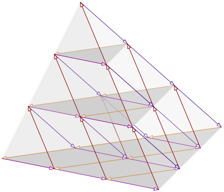
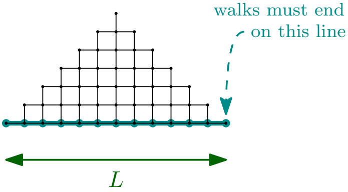
- The same results hold for a triangular lattice of size $2L$, with Motzkin paths of max. height L, where horiz. steps are forbidden at height L.
- Extension to dim. 3, with Motzkin paths replaced by paths on a "waffle".





There are as many:

- pyramid paths starting at $\mathcal{O}$
- waffle walks starting at $(0, 0)$ and ending on the x-axis.