Continued Fractions and Orthogonal Polynomials

Mourad E. H. Ismail
University of Central Florida, Orlando.

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Continued fractions:

$$\cfrac{A_1}{B_1 + \cfrac{A_2}{B_2 + \cfrac{A_3}{B_3 + \cdots}}}$$

R. William Gosper, More Mathematical People, M. A. A.

Continued Fractions and Chopsticks.

Generating functions are important in combinatorics.

Orthogonal polynomials: Positive measure μ and consider the function

$$F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t} \tag{1}$$

Formally $1/(t-z) = z^{-1} \sum_{n=0}^{\infty} t^n/z^n$. Thus

$$F(z) = \sum_{n=0}^{\infty} z^{-n-1} \int_{\mathbb{R}} t^n d\mu(t). \tag{2}$$

Essentially a generating function for the moments $\int_{\mathbb{R}} t^n d\mu(t)$. I could have started with

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1-zt}. \tag{3}$$

Asymptotic series versus formal power series.

Important: No need to assume μ is positive, just think of a linear functional L, with moments replaced by $L(x^n)$.

If $\{P_n(x)\}$ are monic orthogonal polynomials, $P_0(x) = 1, P_1(x) = x - \alpha_0$ then

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x),$$

for $n > 0$, with $\alpha_n \in \mathbb{R}, n \geq 0, \beta_n > 0, n > 0$. Converse is also true and is the spectral theorem for orthogonal polynomials.

The recurrence relation or the orthogonality measure generate the polynomials uniquely.

The transform in (1) is called the Stieltjes transform.

Problem: Given the moments find the measure.

Inversion: $F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}$ means the poles of F are the mass points and we can find μ' by using the behavior of F above and below the x-axis.

Given the orthogonal polynomials identifies the moments. Just write

$$x^n = \sum_{k=0}^{n} c_{n,k} P_k(x), \quad \int_{\mathbb{R}} x^n d\mu(x) = c_{n,o}.$$

Thus the recurrence relation gives the moments. The measure may not be unique.

Continued Fractions: Define P_n^* by the recurrence relation, namely

(4) $y_{n+1}(x) = (x - \alpha_n) y_n(x) - \beta_n y_{n-1}(x)$

but $P_0^*(x) = 0, P_1^*(x) = 1$. So P_n^* has degree $n - 1$.

Motivation:

$$\frac{y_{n+1}(x)}{y_n(x)} = (x - \alpha_n) - \beta_n \frac{y_{n-1}(x)}{y_n(x)}$$

Or

$$\frac{y_n(x)}{y_{n+1}(x)} = \frac{1}{(x - \alpha_n) - \beta_n \frac{y_{n-1}(x)}{y_n(x)}}$$

Theorem: If μ is unique then

$$\lim_{n\to\infty} \frac{P_n^*(z)}{P_n(z)} = F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t} \tag{5}$$

Moreover $F(z)$ is

$$\cfrac{1}{z-\alpha_0-\cfrac{\beta_1}{z-\alpha_1-\cfrac{\beta_2}{z-\alpha_2-\cdots}}} \tag{6}$$

The moment generating function $\sum_{n=0}^{\infty} z^n \int_{\mathbb{R}} t^n d\mu(t)$ encodes the measure.

Flajolet (1980) Combinatorial aspects of continued fractions.

Change of notation: $Q_n(z) = z^n P_n(1/z)$. Then

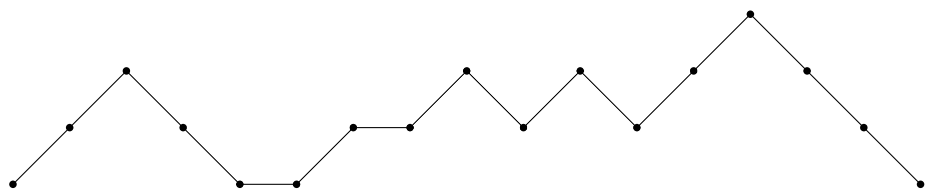
$$Q_{n+1}(z) = (1 - z\alpha_n)Q_n(z) - \beta_n z^2 Q_{n-1}(z)$$

and the continued fraction becomes

$$\cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{1 - z\alpha_2 - \cdots}}}$$

$$= \frac{Q_n^*(z)}{Q_n(z)} + z^n[\text{Power series}]$$

J-fractions.



$a = (1,1), b = (1,-1), c = (1,0).$

The segments are:

$a_0, a_1, b_2, b_1, c_0, a_0, c_1, a_1, b_2, a_1, a_2, b_3, b_2, b_1.$

Continued Fractions:

$$J = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{1 - c_2 z - \cdots}}}$$

Fact (Flajolet)

$$J = \sum_{n=0}^{\infty} R_n z^n,$$

where R_n is a polynomial. The sum of the coefficients of R_n is the nth Motzkin number.

$$S = \cfrac{1}{1 - \cfrac{a_0 b_1 z^2}{1 - \cfrac{a_1 b_2 z^2}{1 - \cdots}}}$$

Flajolet:

$$S = \sum_{n=0}^{\infty} S_n z^n,$$

where S_n is a polynomial. The sum of the coefficients of S_{2n} is the nth Catalan number.

Types of Continued Fractions:

A J-fraction is

$$\cfrac{1}{z-\alpha_0-\cfrac{\beta_1}{z-\alpha_1-\cfrac{\beta_2}{z-\alpha_2-\cdots}}}$$

or

$$J=\cfrac{1}{1-\alpha_0 z-\cfrac{\beta_1 z^2}{1-\alpha_1 z-\cfrac{\beta_2 z^2}{1-\alpha_2 z-\cdots}}}$$

Orthogonal polynomials on the unit circle.

$$\int_0^{2\pi} \phi_m(e^{i\theta})\overline{\phi_n(e^{i\theta})}d\mu(\theta) = \delta_{m,n}.$$

$\phi_n(z) = \kappa_n z^n + \ell_n z^{n-1} +$ lower order terms,

the polynomials $\{\phi_n(z)\}$ satisfy the recurrence relations

$$\kappa_n z\phi_n(z) = \kappa_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)\phi^*_{n+1}(z),$$
$$\kappa_n\phi_{n+1}(z) = \kappa_{n+1}z\phi_n(z) + \phi_{n+1}(0)\phi^*_n(z).$$

Here

$$\left[\sum_{k=0}^{n} a_k z^k\right]^* = \sum_{k=0}^{n} \overline{a_k} z^{n-k}.$$

It is convenient to write the recurrence relations in terms of the monic polynomials

$$\Phi_n(z) = \phi_n(z)/\kappa_n.$$

Indeed we have

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\,\Phi_n^*(z),$$
$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z),$$

where

$$\alpha_n = -\overline{\Phi_{n+1}(0)} = -\overline{\phi_{n+1}(0)}/\kappa_{n+1}.$$

This can be written as a system

$$\begin{pmatrix}\Phi_{n+1}(z)\\ \Phi_{n+1}^*(z)\end{pmatrix} = \begin{pmatrix} z & -\overline{\alpha}_n \\ -\alpha_n z & 1\end{pmatrix}\begin{pmatrix}\Phi_n(z)\\ \Phi_n^*(z)\end{pmatrix}.$$

Analogue of the J-fraction: Let

$$f(z) = a_0 + \frac{1-|a_0|^2\,z}{\overline{a_0}\,z+}\,\frac{1}{a_1+}\,\frac{1-|a_1|^2\,z}{\overline{a_1}\,z+}\cdots,$$

with $\alpha_n = -\overline{\phi_{n+1}(0)}/\kappa_{n+1}$ then

$$\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta) = \frac{1+z\,f(z)}{1-z\,f(z)}.$$

The combinatorics is not known.

A Special Feature: Chebyshev polynomials: $T_n(\cos\theta) = \cos(n\theta)$, $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$ are orthogonal on $[-1,1]$ with respect to the weights $(1-x^2)^{\mp 1/2}$. On the other hand z^n is orthogonal on the unit circle with respect to $w = 1$. The Chebyshev polynomials are essentially the real and imaginary parts of $z^n = e^{in\theta}$. This is a general phenomenon every orthogonal polynomial on the unite circle generates two orthogonal polynomials on $[-1,1]$.

Can this be explaend combinatorially?

I want to rewrite the monic form

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

in the orthonormal form. With $a_n^2 = \beta_n$, it is

$$\begin{pmatrix} \alpha_0 & a_1 & 0 & 0 & \cdots \\ a_1 & \alpha_1 & a_2 & 0 & \cdots \\ 0 & a_2 & \alpha_2 & a_3 & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}$$

Formal eigenvalue problem.

Generalized eigenvalue problem $AX = \lambda BX$, where A, B are tridiagonal and B is positive definite.

Why not $B^{-1}AX = \lambda X$.

Ismail-Masson 1995 JAT.

R_I Fractions.

$$P_{-1}(x) = 0, P_0(x) = 1,$$
$$P_n = (x - c_n)P_{n-1}(x) - \lambda_n(x - a_n)P_{n-2}(x).$$

R_{II} Fractions

$$P_{-1}(x) = 0, P_0(x) = 1,$$
$$P_n = (x - c_n)P_{n-1}(x) - \lambda_n(x - a_n)(x - b_n)P_{n-2}(x).$$

Let

$$S_n(x) = P_n(z) / \prod_{k=1}^{n} [(x - a_{k+1})(x - b_{k+1})]$$

Theorem: Given N_0 and N_1, there exits a linear functional $\mathcal{L}$ such that $\mathcal{L}(1) = N_0, \mathcal{L}(xS_1(x)) = N_1$, $\mathcal{L}(x^k S_n(x)) = 0$, for $0 \leq k < n$ holds.

The continued fraction:

$$\frac{1}{z - c_1 -} \; \frac{\lambda_2(z - a_2)(z - b_2)}{z - c_2 -} \; \frac{\lambda_3(z - a_3)}{z - c_3 - \cdots}$$

equals $\mathcal{L}[1/(z - t)]$.

Zhedanov: Generalized eigenvalue problem JAT 2000.

Combinatorics: Jang Soo Kim and Dennis Stanton.

Conjecture: Joseph Brennan

$$\frac{\begin{bmatrix}\alpha+r\\ r\end{bmatrix}_q \cdots \begin{bmatrix}\alpha+\beta\\ r\end{bmatrix}_q}{\begin{bmatrix}\rho\\ r\end{bmatrix}_q \cdots \begin{bmatrix}\beta\\ r\end{bmatrix}_q}$$

is a polynomial in q. The case $q = 1$ is known. Indeed

$$\frac{\binom{\alpha+r}{r} \cdots \binom{\alpha+\beta}{r}}{\binom{r}{r} \cdots \binom{\beta}{r}}$$

is an integer.