Random triangulations and bijective paths to Liouville quantum gravity

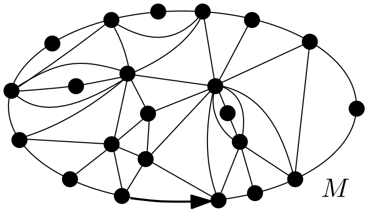
Nina Holden

ETH Zürich

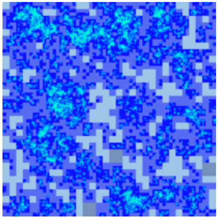
Based on works with Marie Albenque, Olivier Bernardi, and Xin Sun.

June 22, 2021

Two random surfaces



random planar map (RPM)



Liouville quantum gravity (LQG)

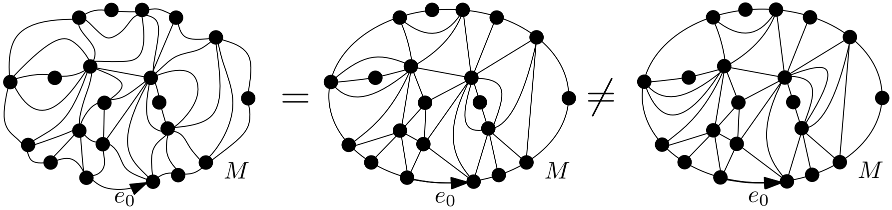
Main result (informal): RPM converge to LQG in the scaling limit

Bijections between planar maps and lattice paths essential in proofs

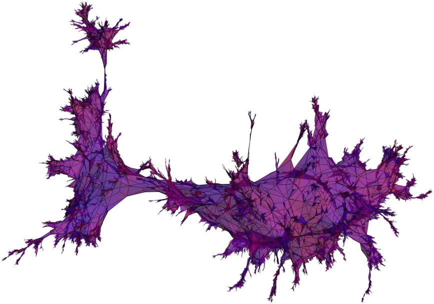
Right figure by Miller and Sheffield

Random planar maps (RPM)

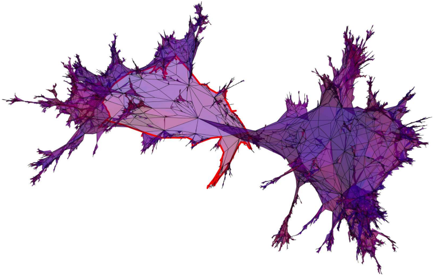
- A **planar map** M is a finite connected graph drawn in the sphere, viewed up to continuous deformations.
- A **triangulation of a disk** is a planar map where all the faces have three edges, except one distinguished face (the exterior face) with arbitrary degree and simple boundary.
- Given $n, m \in \mathbb{N}$ let M be a **uniformly** chosen triangulation of a disk with n interior vertices and m boundary vertices.



Uniformly sampled triangulations



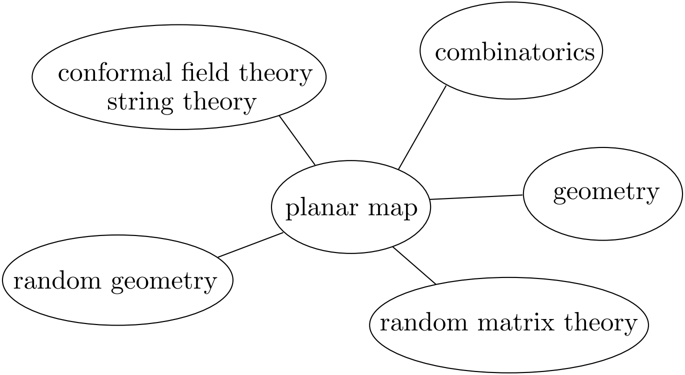
Triangulation



Triangulation of disk

Simulations by Bettinelli

Planar maps


conformal field theory
string theory
combinatorics
planar map
geometry
random geometry
random matrix theory

Gaussian free field (GFF)

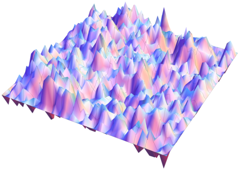
- The free boundary Gaussian free field h in $\mathbb{D}$ is the Gaussian random field with mean zero and covariance

$$\mathrm{Cov}(h(z), h(w)) = G(z, w),$$

where $G : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ is the Neumann Green's function

$$G(z, w) = \log |z - w|^{-1} + \log |1 - z\overline{w}|^{-1}.$$

- h **not** well defined as a function since $G(z, z) = \infty$.
- h well-defined as a **random generalized function (distribution)**.
  - $\int_{\mathbb{D}} hf \, d^2 z$ is well-defined for f a smooth test function.



Discrete GFF

Liouville quantum gravity (LQG)

- If $h : \mathbb{D} \to \mathbb{R}$ is smooth and $\gamma \in (0,2)$, then the following defines a **measure** μ and a **distance function (metric)** D on $\mathbb{D}$:
$$\mu(U) = \int_U e^{\gamma h(z)} d^2 z, \qquad D(z_1, z_2) = \inf_{P: z_1 \to z_2} \int_P e^{\frac{\gamma h(z)}{2}} dz.$$
where $U \subset \mathbb{D}$ and $z_1, z_2 \in \mathbb{D}$.
- γ-Liouville quantum gravity (LQG): h is the **Gaussian free field**.
- The definition of an LQG surface does not make literal sense since h is a distribution and not a function.
- **Measure** μ and **distance function (metric)** D defined by considering regularization h_ϵ of h.[1]

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \qquad D(z_1, z_2) = \lim_{\epsilon \to 0} c_\epsilon \inf_{P: z_1 \to z_2} \int_P e^{\frac{\gamma h_\epsilon(z)}{d(\gamma)}} dz.$$

- LQG for $\gamma = \sqrt{8/3}$: Brownian map; scaling limit of **uniform** maps.
- Key takeaway: LQG defines random **measure & distance function.**

[1] Metric construction: Gwynne-Miller'19, Ding-Dubedat-Dunlap-Falconet'19, Dubedat-Falconet-Gwynne-Pfeffer-Sun'19. Hausdorff dim. $(\mathbb{D}, D)$ denoted by $d(\gamma)$.

Random planar maps converge to LQG

Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

What does it mean for a RPM to converge?

Random planar maps converge to LQG

Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

What does it mean for a RPM to converge?

(i) Metric space structure (Gromov-Hausdorff topology)
  - Le Gall'11, Miermont'11, several others

(ii) Statistical physics decorations (variants of mating-of-trees topology)
  - Duplantier-Miller-Sheffield'14, Sheffield'11, several others

(iii) Conformal structure (weak topology for measures on $\mathbb{C}$)
  - H.-Sun'19

Random planar maps converge to LQG

Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

What does it mean for a RPM to converge?

(i) Metric space structure (Gromov-Hausdorff topology)
  - Le Gall'11, Miermont'11, several others

(ii) Statistical physics decorations (variants of mating-of-trees topology)
  - Duplantier-Miller-Sheffield'14, Sheffield'11, several others

(iii) Conformal structure (weak topology for measures on $\mathbb{C}$)
  - H.-Sun'19

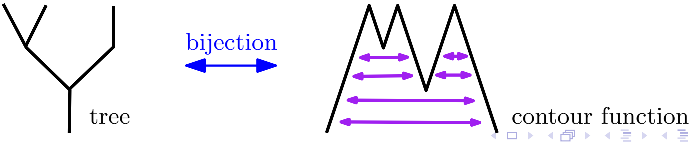
Convergence in (i) and (ii) established via **bijections to lattice paths**:

(i) metric bijections

(ii) mating-of-tree bijections

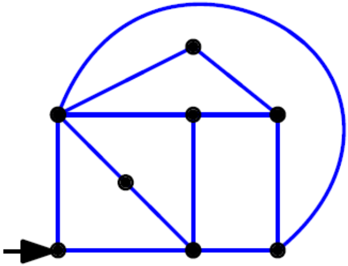
Bijections between **planar maps** and **lattice paths**

- Two families: (i) metric bijections and (ii) mating-of-trees bijections
- Examples (i): Cori–Vauquelin'81, Schaeffer'98, Bouttier-Di Francesco-Guitter'04, Poulalhon-Schaeffer'06, etc.
- Examples (ii): Mullin'67, Bernardi'08, Li-Sun-Watson'17, Bernardi'07/Ber.-H.-Sun'19, Kenyon-Miller-Sheffield-Wilson'15, etc.
- Both families of bij. involve lattice paths encoding **pair of trees**.
- **Continuum analogue** of bijections:
  - (i) Brownian map (Marckert–Mokkadem'06, Le Gall'07'11, Miermont'11)
  - (ii) LQG as mating of trees (Duplantier-Miller-Sheffield'14).
- (ii) for **decorated** planar maps, i.e., with statistical physics model. Therefore also for **non-uniform** planar maps.
- The lattice path encodes important information: (i) metric properties; (ii) observables of the map with a statistical physics model.

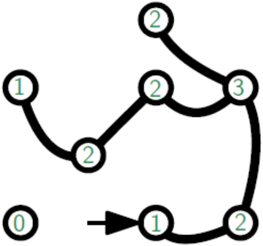


Scaling limits of planar maps via metric bijections

Cori-Vauquelin-Schaeffer (CVS) bijection



Quadrangulation



Well-labeled tree

Well-labeled tree: tree with positive labels such that root has label 1; adjacent labels differ by $0, \pm 1$.

Key property: Graph distance to root = label

Figure due to Olivier Bernardi

Natural topology on compact **metric spaces**.

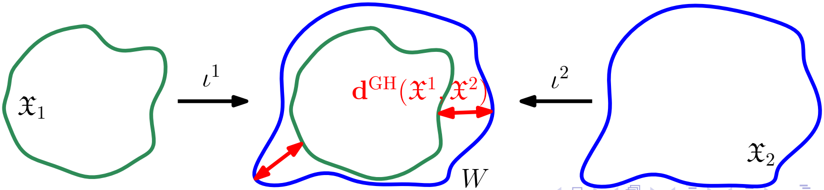
Hausdorff distance for $E_1, E_2 \subset W$ and (W, D) a metric space

$$\mathbf{d}_D^{\mathrm{H}}(E_1, E_2) := \max\{\sup_{x \in E_1} \inf_{y \in E_2} D(x,y), \sup_{y \in E_2} \inf_{x \in E_1} D(x,y)\}.$$

Gromov-Hausdorff distance between $\mathfrak{X}^1 = (X^1, d^1)$ and $\mathfrak{X}^2 = (X^2, d^2)$

$$\mathbf{d}^{\mathrm{GH}}\left(\mathfrak{X}^1, \mathfrak{X}^2\right) = \inf_{(W,D),\iota^1,\iota^2} \mathbf{d}_D^{\mathrm{H}}\left(\iota^1(X^1), \iota^2(X^2)\right),$$

where the infimum is over all compact metric spaces (W, D) and isometric embeddings $\iota^1 : X^1 \to W$ and $\iota^2 : X^2 \to W$.



Gromov-Hausdorff-Prokhorov topology

Natural topology on compact **metric measure spaces**.

Prokhorov distance for Borel measures μ^1, μ^2 on (W, D):

$$\mathbf{d}_D^{\mathrm{P}}(\mu^1, \mu^2) = \inf\{\epsilon > 0 \,:\, \mu^1(A) \leq \mu^2(A^\epsilon) + \epsilon \text{ and } \mu^2(A) \leq \mu^1(A^\epsilon) + \epsilon \text{ for all closed sets } A \subset W\},$$

where A^ϵ is the set of elements of W at distance less than ϵ from A, i.e. $A^\epsilon = \{x \in W \text{ such that } \exists a \in A,\ D(a, x) < \epsilon\}$.

Gromov-Hausdorff-Prokhorov distance between $\mathfrak{X}^1 = (X^1, d^1, \mu^1)$ and $\mathfrak{X}^2 = (X^2, d^2, \mu^2)$:

$$\mathbf{d}^{\mathrm{GHP}}\left(\mathfrak{X}^1, \mathfrak{X}^2\right) = \inf_{(W,D),\iota^1,\iota^2} \mathbf{d}_D^{\mathrm{H}}\left(\iota^1(X^1), \iota^2(X^2)\right) + \mathbf{d}_D^{\mathrm{P}}((\iota^1)_*\mu^1, (\iota^2)_*\mu^2),$$

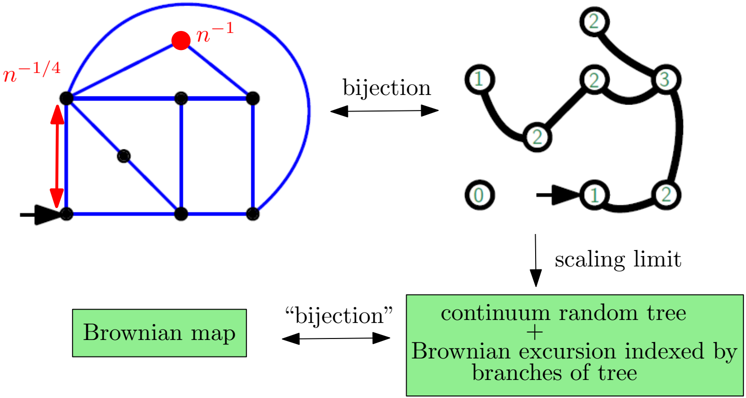
where the infimum is over all compact metric spaces (W, D) and isometric embeddings $\iota^1 : X^1 \to W$ and $\iota^2 : X^2 \to W$.

Random quadrangulation $\Rightarrow$ Brownian map

M_n is a quadrangulation, M is the Brownian map ($\sqrt{8/3}$-LQG)

Theorem 1 (Le Gall'11, Miermont'11)

$M_n \Rightarrow M$ for the Gromov-Hausdorff-Prokhorov topology.

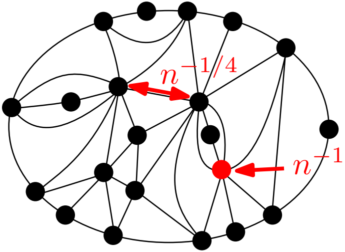


Random triangulation of a disk ⇒ Brownian disk

M_n is a triangulated disk, M is the Brownian disk ($\sqrt{8/3}$-LQG)

Theorem 2 (Albenque-H.-Sun'19)

$M_n \Rightarrow M$ for the Gromov-Hausdorff-Prokhorov topology.



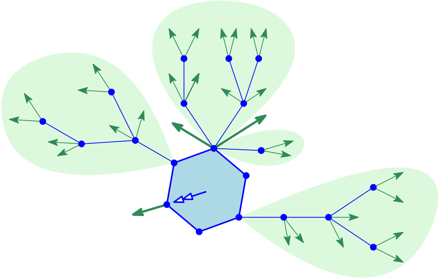
Related works: Le Gall'11, Miermont'11, Bettinelli–Miermont'15, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'13, Addario-B.–Wen'15

Random triangulation of a disk ⇒ Brownian disk

M_n is a triangulated disk, M is the Brownian disk ($\sqrt{8/3}$-LQG)

Theorem 2 (Albenque-H.-Sun'19)

$M_n \Rightarrow M$ for the Gromov-Hausdorff-Prokhorov topology.



Blossoming forest

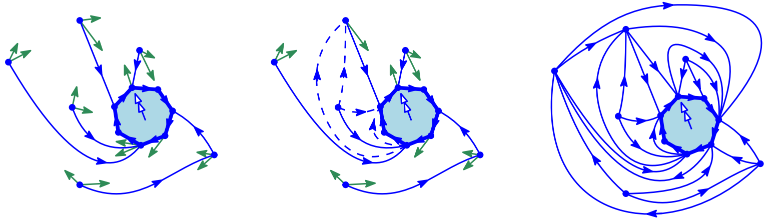
Related works: Le Gall'11, Miermont'11, Bettinelli–Miermont'15, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'13, Addario-B.–Wen'15

Random triangulation of a disk $\Rightarrow$ Brownian disk

M_n is a triangulated disk, M is the Brownian disk ($\sqrt{8/3}$-LQG)

Theorem 2 (Albenque-H.-Sun'19)

$M_n \Rightarrow M$ for the Gromov-Hausdorff-Prokhorov topology.



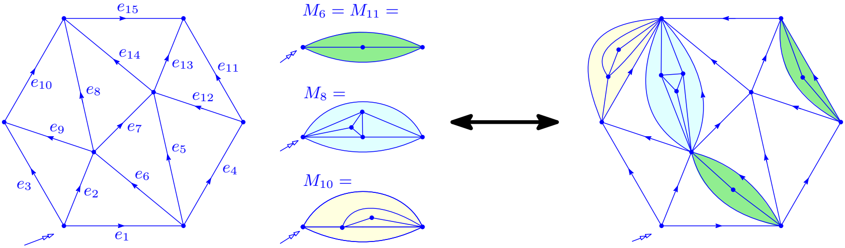
Related works: Le Gall'11, Miermont'11, Bettinelli–Miermont'15, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'13, Addario-B.–Wen'15

Random triangulation of a disk ⇒ Brownian disk

M_n is a triangulated disk, M is the Brownian disk ($\sqrt{8/3}$-LQG)

Theorem 2 (Albenque-H.-Sun'19)

$M_n \Rightarrow M$ for the Gromov-Hausdorff-Prokhorov topology.

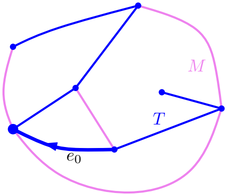


simple (type III) loopless (type II)

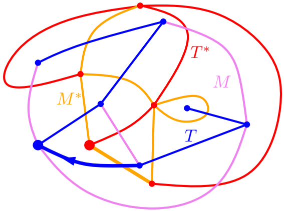
Related works: Le Gall'11, Miermont'11, Bettinelli–Miermont'15, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'13, Addario-B.–Wen'15

Scaling limits of planar maps via mating-of-trees bijections

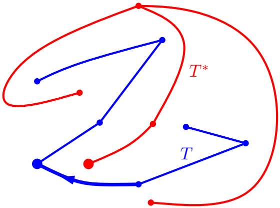
Mullin bijection



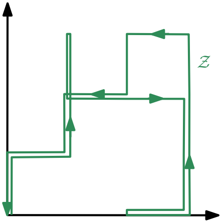
planar map and spanning tree



planar map, spanning tree and their duals



spanning tree and its dual

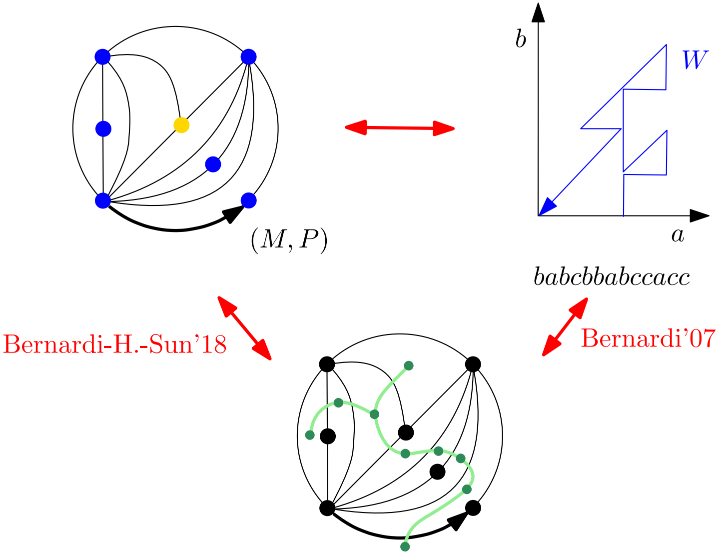


lattice path encoding trees

Lattice path ⇒ 2D Brownian motion, which encodes an LQG surface.

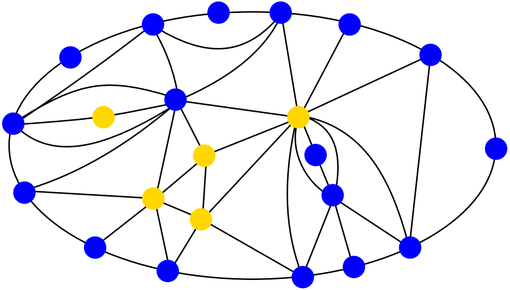
Non-uniform map: Law of map reweighted by number of spanning trees.

Percolated triangulations and Kreweras walks



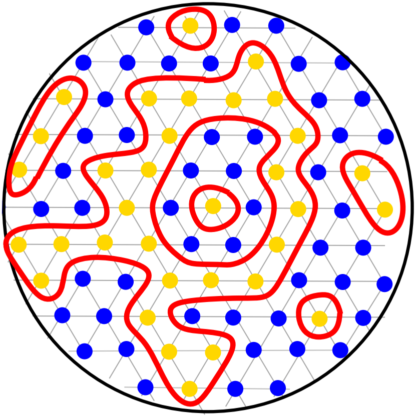
Scaling limit of percolated triangulations

- Let M be uniform triangulation with n interior (resp. $\sqrt{n}$ boundary) vertices; let P be uniform coloring of the interior vertices.



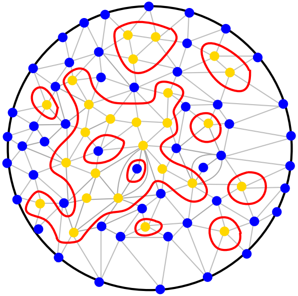
Scaling limit of percolated triangulations

- Let M be uniform triangulation with n interior (resp. $\sqrt{n}$ boundary) vertices; let P be uniform coloring of the interior vertices.
- Conformal loop ensemble (CLE): conformally invariant collection of non-crossing loops in $\mathbb{D}$; loop variant of Schramm-Loewner evolution



Scaling limit of percolated triangulations

- Let M be uniform triangulation with n interior (resp. $\sqrt{n}$ boundary) vertices; let P be uniform coloring of the interior vertices.
- Conformal loop ensemble (CLE): conformally invariant collection of non-crossing loops in $\mathbb{D}$; loop variant of Schramm-Loewner evolution
- Bernardi-H.-Sun'18: There exists an embedding of P into $\mathbb{D}$ such that a number of interesting observables of (M, P) converge jointly in law when $n \to \infty$. The scaling limit can be described in terms of CLE on an independent $\sqrt{8/3}$-LQG surface.



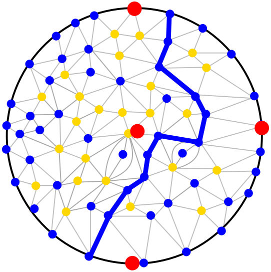
$\Longrightarrow$ LQG and CLE

Scaling limit of percolated triangulations

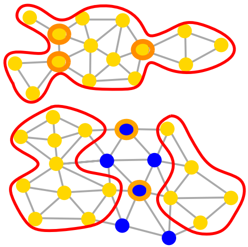
- Let M be uniform triangulation with n interior (resp. $\sqrt{n}$ boundary) vertices; let P be uniform coloring of the interior vertices.
- Conformal loop ensemble (CLE): conformally invariant collection of non-crossing loops in $\mathbb{D}$; loop variant of Schramm-Loewner evolution
- Bernardi-H.-Sun'18: There exists an embedding of P into $\mathbb{D}$ such that a number of interesting observables of (M, P) converge jointly in law when $n \to \infty$. The scaling limit can be described in terms of CLE on an independent $\sqrt{8/3}$-LQG surface.
    - Counting measure on vertices rescaled by $n^{-1} \Rightarrow$ LQG area measure
    - Percolation cycles $\Rightarrow$ CLE loops
    - Crossing events converge
    - Counting measure on pivotals rescaled by $n^{-1/4} \Rightarrow$ CLE touching meas.
    - Percolation exploration tree $\Rightarrow$ CLE exploration tree

Scaling limit of percolated triangulations

- Let M be uniform triangulation with n interior (resp. $\sqrt{n}$ boundary) vertices; let P be uniform coloring of the interior vertices.
- Conformal loop ensemble (CLE): conformally invariant collection of non-crossing loops in $\mathbb{D}$; loop variant of Schramm-Loewner evolution
- Bernardi-H.-Sun'18: There exists an embedding of P into $\mathbb{D}$ such that a number of interesting observables of (M, P) converge jointly in law when $n \to \infty$. The scaling limit can be described in terms of CLE on an independent $\sqrt{8/3}$-LQG surface.

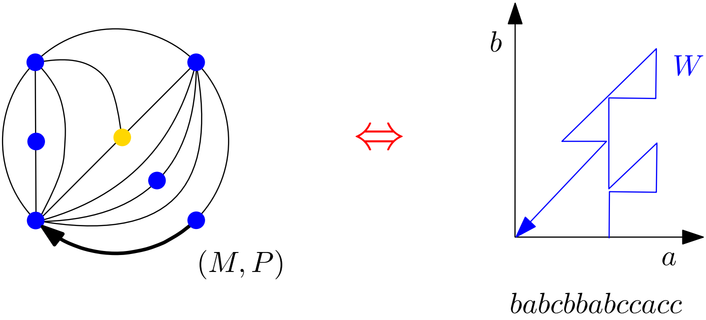


crossing event



pivotal points

Scaling limit via Kreweras walks

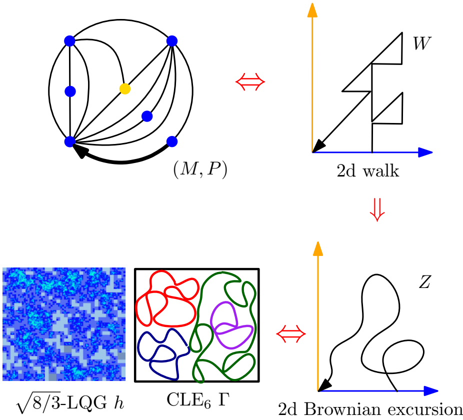


Scaling limit via Kreweras walks

(M,P) $\overset{\text{bijection}}{\Leftrightarrow}$ walk $(W_k)_{k\in[3n]}$

$\Downarrow$ convergence

(h,Γ) $\overset{\text{“bijection”}}{\Leftrightarrow}$ Brownian excursion $(Z_t)_{t\in[0,1]}$

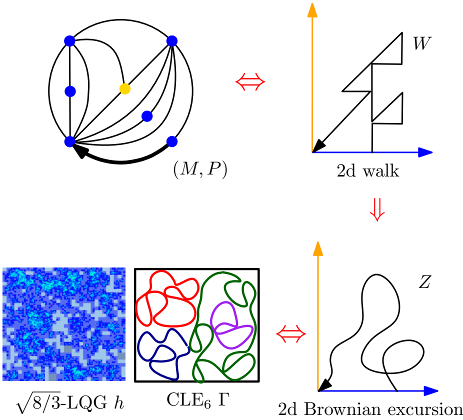


Scaling limit via Kreweras walks

(M, P) $\overset{\text{bijection}}{\Leftrightarrow}$ walk $(W_k)_{k\in[3n]}$

$\Downarrow$ convergence of walk and observables encoded by walk

(h, Γ) $\overset{\text{“bijection”}}{\Leftrightarrow}$ Brownian excursion $(Z_t)_{t\in[0,1]}$

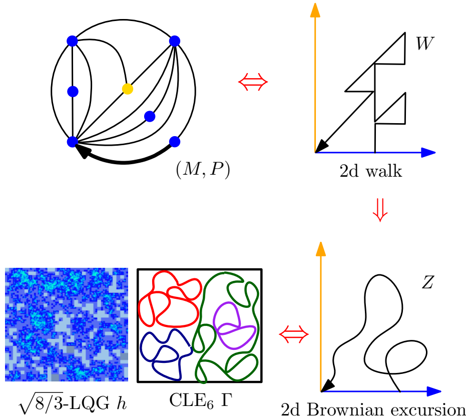


Scaling limit via Kreweras walks

(M, P) $\overset{\text{bijection}}{\Leftrightarrow}$ walk $(W_k)_{k \in [3n]}$

$\Downarrow$ convergence of walk and observables encoded by walk

(h, Γ) $\overset{\text{“bijection”}}{\Leftrightarrow}$ Brownian excursion $(Z_t)_{t \in [0,1]}$

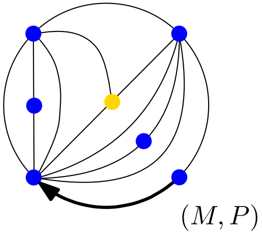


Bijection between percolated maps and walks

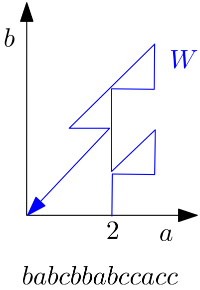
Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$



$\Leftrightarrow$

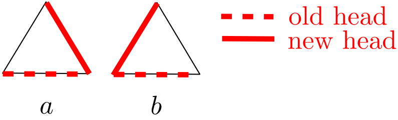


Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$

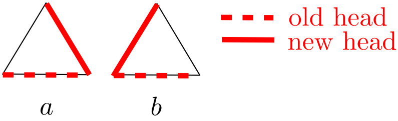


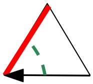
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n + 1$ edges

(2) cone excursion W length n, steps $a = (1, 0)$, $b = (0, 1)$, $c = (-1, -1)$





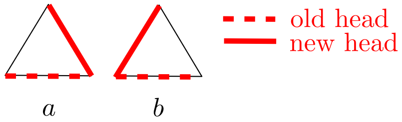
b

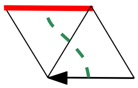
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$





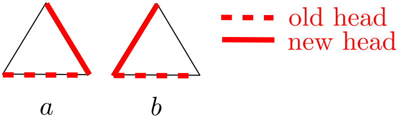
ba

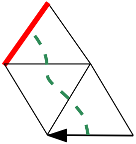
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$





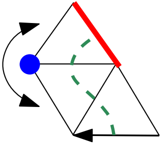
$$bab$$

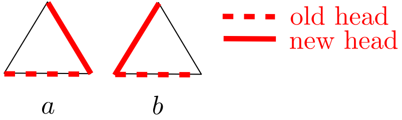
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1, 0)$, $b = (0, 1)$, $c = (-1, -1)$





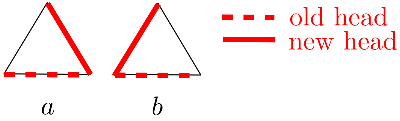
$$babc$$

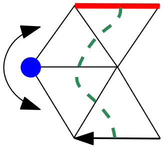
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$





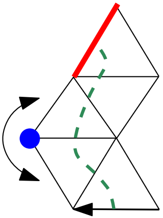
$babcb$

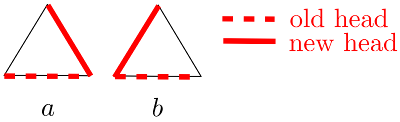
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$





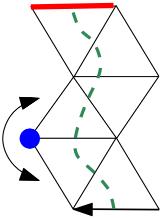
$$babcbb$$

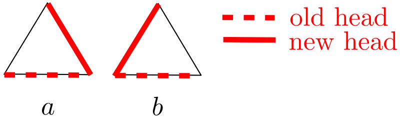
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$





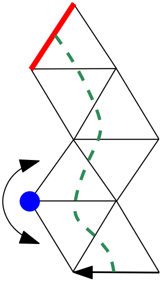
$babcbba$

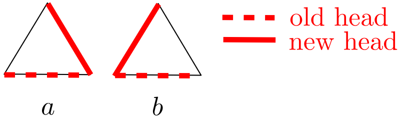
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n + 1$ edges

(2) cone excursion W length n, steps $a = (1, 0)$, $b = (0, 1)$, $c = (-1, -1)$





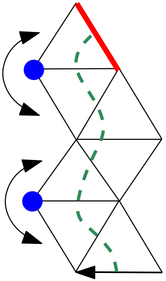
$$babcbbab$$

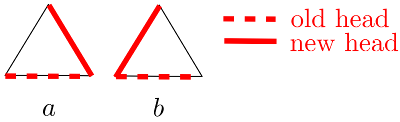
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$





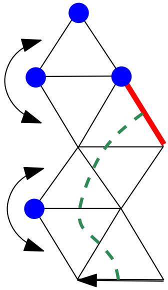
$$babcbbabc$$

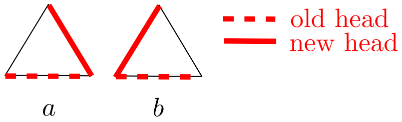
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$





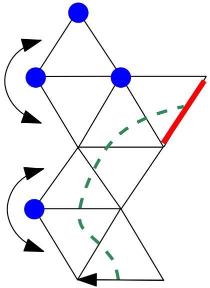
$babcbbabcc$

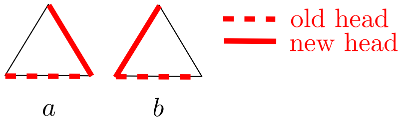
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1, 0)$, $b = (0, 1)$, $c = (-1, -1)$





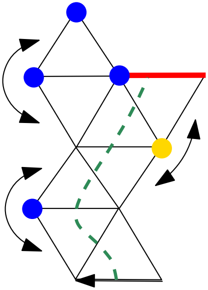
$babcbbabcca$

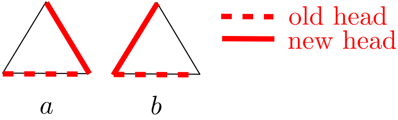
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$





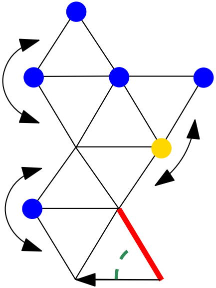
$$babcbbabccac$$

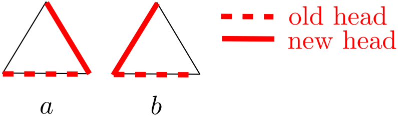
Bijection between percolated maps and walks

Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

(2) cone excursion W length n, steps $a = (1,0)$, $b = (0,1)$, $c = (-1,-1)$



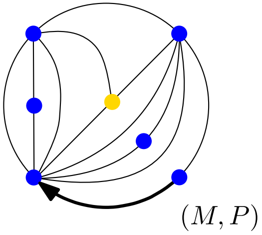


$$babcbbabccacc$$

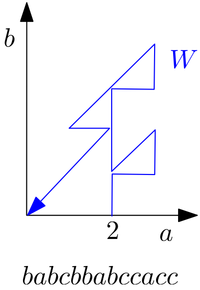
Bernardi'07, Bernardi-H.-Sun'17: Bijection between

(1) site-percolated rooted triangulation (M, P) of a disk with $n+1$ edges

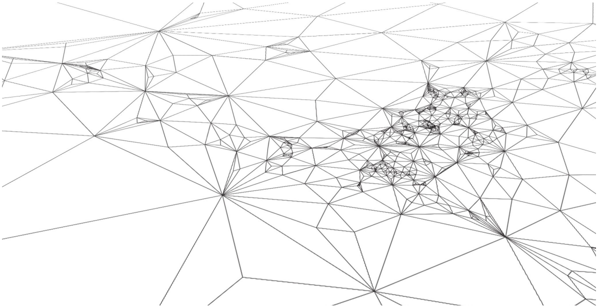
(2) cone excursion W length n, steps $a=(1,0)$, $b=(0,1)$, $c=(-1,-1)$



$\Leftrightarrow$

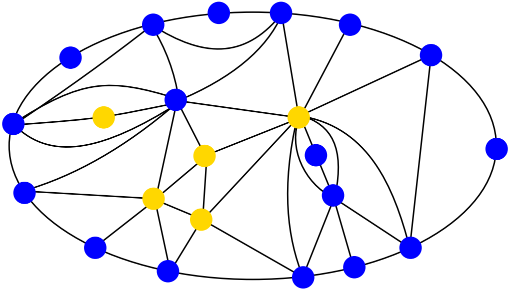


Infinite-volume bijection



I. Kortchemski

- Infinite-volume bijection:
  - Percolated uniform infinite planar triangulation (UIPT)
  - walk with i.i.d. increments a, b, c.
- Allows to relate properties of the UIPT and LQG:
  - random walk on the UIPT (Gwynne-Hutchcroft'18, Gwynne-Miller'17)
  - dimension of γ-LQG (Ding-Gwynne'18, Gwynne-H.-Sun'17)



Thanks!