Extracting asymptotics from series coefficients.

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INTRODUCTION.

- We are trying to extract asymptotics from counting sequences.
- We consider three different types of behaviour:
  (i) Simple power law, $a_n \sim C \cdot \mu^n \cdot n^g$,
  (ii) Power law plus logarithms, $a_n \sim C \cdot \mu^n \cdot n^g \cdot (\log n)^\beta$,
  (iii) Stretched exponential, $a_n \sim C \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g, \;\; 0 < \sigma < 1.$
- We will call μ the *growth constant,* usually the reciprocal of the radius of convergence, and g the *exponent*. (Of course β and σ are also exponents.)
- We study these using variants of two methods:
  (i) The behaviour of the ratios $r_n = a_n/a_{n-1}$,
  (ii) The solution of a family of D-finite ODEs, constructed from the known coefficients.

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Series analysis 101.

- Given $f(z) = \sum c_n z^n$, the Cauchy-Hadamard theorem tells us that the growth constant is given by
$$\mu = \limsup_{n\to\infty} |c_n|^{1/n}.$$
- Alternatively, the ratio test tells us that
$$\mu = \lim_{n\to\infty} \left| \frac{c_n}{c_{n-1}} \right|.$$
- $$\text{If } f(z) \sim C \cdot \Gamma(\gamma) \cdot (1 - \mu \cdot z)^{-\gamma}, \text{ then } c_n \sim C \cdot \mu^n \cdot n^{\gamma-1}.$$
- $$|c_n|^{1/n} \sim C^{1/n} \cdot \mu \left(1 + \frac{(\gamma-1)\log n}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right).$$

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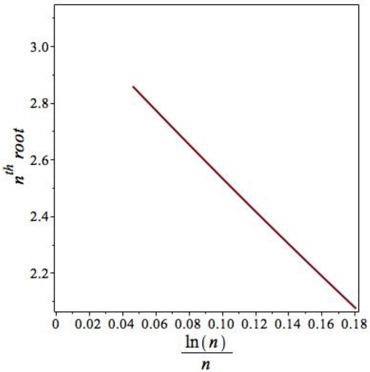
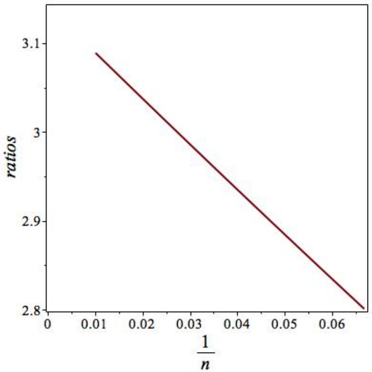
Ratio method.

- $r_n = \frac{c_n}{c_{n-1}} = \mu\left(1 + \frac{\gamma - 1}{n} + \mathrm{o}(\frac{1}{n})\right).$
- Test series $f(z) = \exp(-z) \cdot (1 - \pi \cdot z)^{2/3}$.





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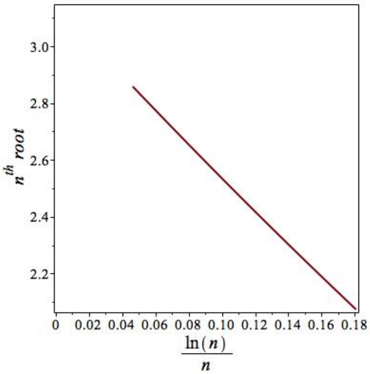
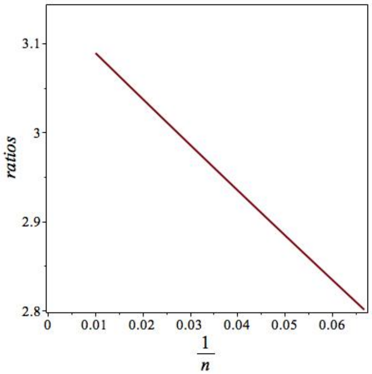
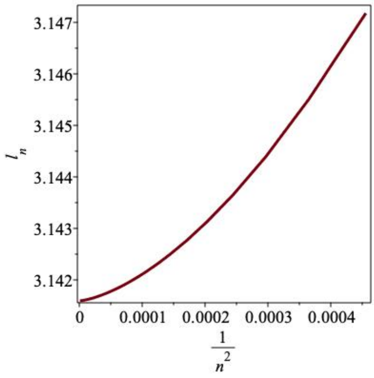
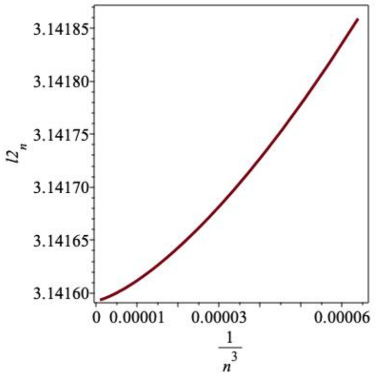




Ratio method refinement.

- $l_n = n \cdot r_n - (n-1) \cdot r_{n-1} = \frac{1}{z_c}\left(1 + o(\frac{1}{n})\right).$
- If $o(\frac{1}{n}) = O(\frac{1}{n^2})$, $l2_n = \frac{n^2 \cdot l_n - (n-1)^2 \cdot l_{n-1}}{2n-1} = \frac{1}{z_c}\left(1 + o(\frac{1}{n^2})\right).$





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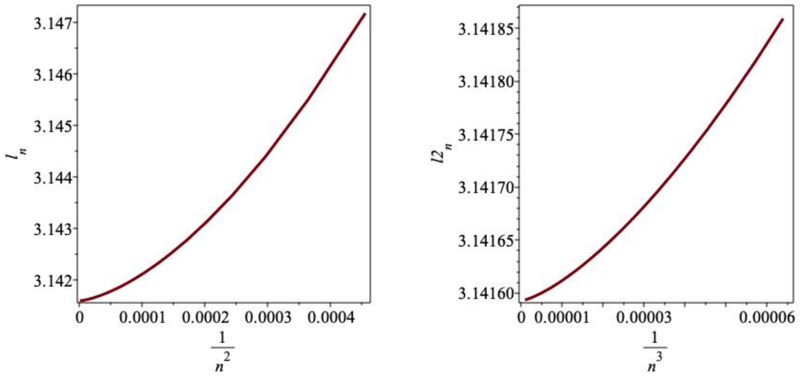
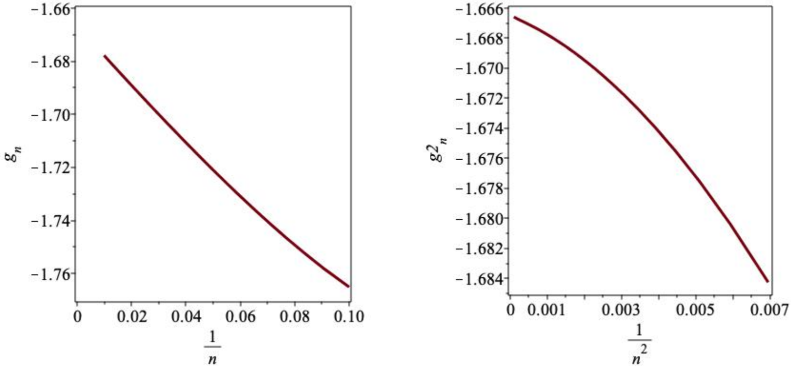


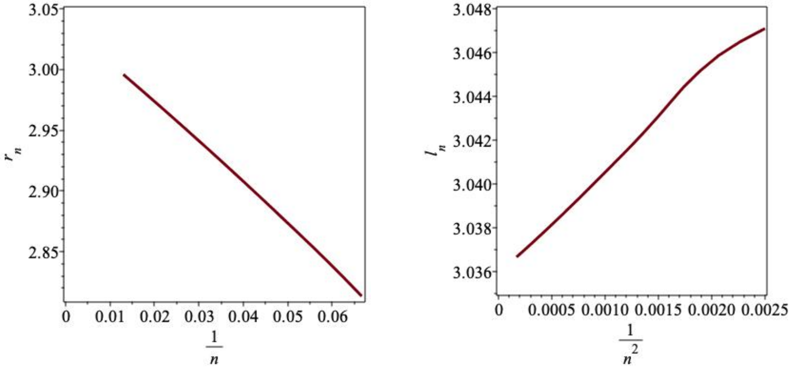
ESTIMATING THE EXPONENT.

- Estimating $\mu = 3.14159$, we can estimate exponent $\gamma - 1 = g_n = \left(\frac{r_n}{\mu} - 1\right) \cdot n$, and we can eliminate $O(1/n)$ term similarly.

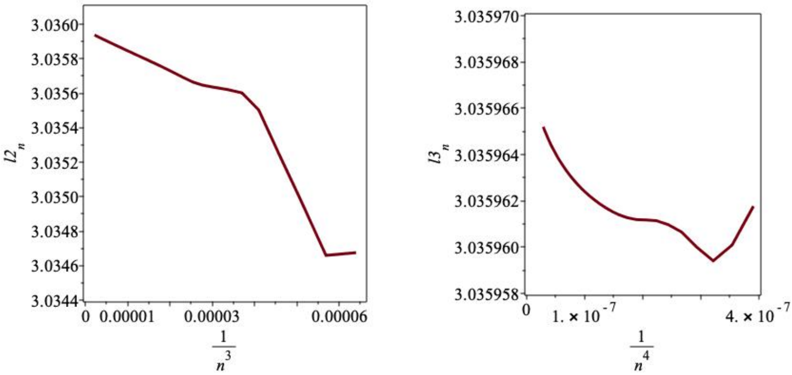


TRIANGULAR POLYOMINOES.

- $\mu = 3.0359688(3)$; logarithmic singularity.



Triangular polyominoes II.



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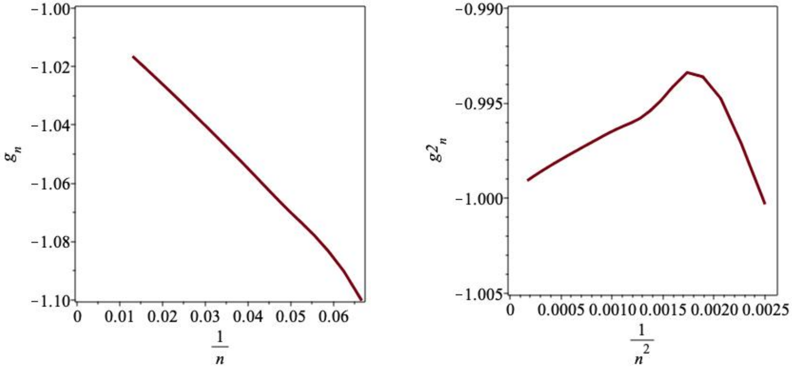

Extracting the asymptotics II.

- An alternative method is the *method of differential approximants*.
- Fit available coefficients to many ODEs, using most/all coefficients. E.g.

$$Q_2(z)F''(z) + Q_1(z)F'(z) + F(z) = P(z),$$

where $Q_k(z)$ and $P(z)$ are polynomials. Vary their degree until all known coefficients are used.

- Asymptotics can be extracted from the ODEs. For a power-law, this usually gives better precision than the ratio method.

Critical point and exponent estimates for self-avoiding polygons.

L	Second order DA		Third order DA	
	x_c^2	$2-\alpha$	x_c^2	$2-\alpha$
0	0.29289321854(19)	1.50000065(41)	0.29289321865(12)	1.50000040(28)
5	0.29289321875(21)	1.50000010(59)	0.29289321852(48)	1.50000041(99)
10	0.29289321855(23)	1.50000060(48)	0.29289321878(32)	1.49999999(97)
15	0.29289321859(19)	1.50000054(43)	0.29289321861(37)	1.50000035(67)
20	0.29289321866(15)	1.50000038(33)	0.29289321860(21)	1.50000049(43)

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EXTENDING THE KNOWN SEQUENCES **approximately**.

- The differential approximants reproduce all known coefficients, and approximate all subsequent coefficients.
- We average over dozens of DAs and calculate the mean and s.d. of many subsequent coefficients.
- We accept the coefficients as long as the s.d. is $\leq 10^{-6}$ the value of the coefficient. (So we'll have typically 6 sig. digits).
- In this way, we will typically gain an extra 50-1000 coefficients estimated with sufficient accuracy to use the ratio method.
- Example: 3-stack-sortable permutations (Defant, Elvey Price and G.). Defant found 174 coefficients. We used these to predict the next 327 coefficients. Elvey Price subsequently wrote an improved algorithm, generating 1000 terms. All our predicted terms were accurate to 29 significant digits. (EJC **28**(2) #P2.49).

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STRETCHED-EXPONENTIAL ASYMPTOTICS.

- Many problems have more complex asymptotics.

$$b_n \sim B \cdot \mu^n \cdot \exp(-c \cdot n^\sigma) \cdot n^{\gamma-1}, \ \ c > 0, \ 0 < \sigma < 1. \ (1)$$

- Or $f(x) = (1 - \mu \cdot x)^\alpha \left(\frac{1}{\mu \cdot x} \log \frac{1}{1 - \mu \cdot x} \right)^\beta. \ (2)$

- For (1), $r_n = \mu \left(1 + \frac{\sigma \log \mu_1}{n^{1-\sigma}} + \frac{\gamma - 1}{n} + \mathrm{O}(\frac{1}{n^{2-2\sigma}}) \right).$

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L-CONVEX POLYGONS.

- An *L-convex polyomino* is one in which any two cells may be joined by an L-shaped path.
- Introduced by Castiglione et al. in 2007.
- The perimeter generating function coefficients satisfy $c_{n+2} = 4c_{n+1} - 2c_n$, so the ogf is algebraic.
- The area generating function is unknown.
- From their appearance, they can be considered as the gluing together of two stack polyominoes. The area of stack polyominoes grows as

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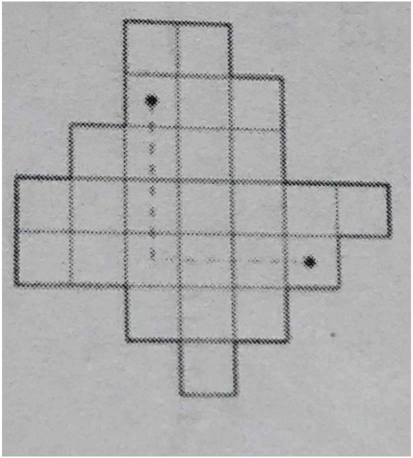

L-CONVEX CONTINUED.

- Plot ratios against $1/n$ and $1/\sqrt{n}$. Suggests $\mu_1^{\sqrt{n}}$ behaviour.
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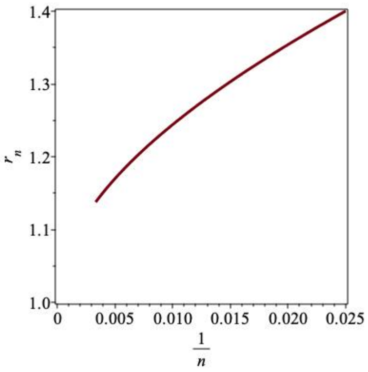

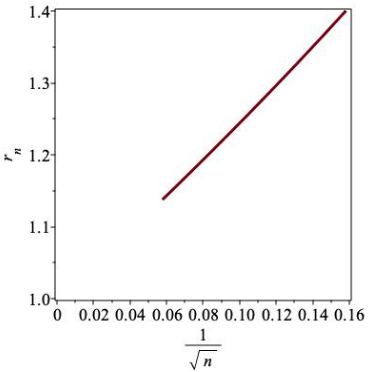

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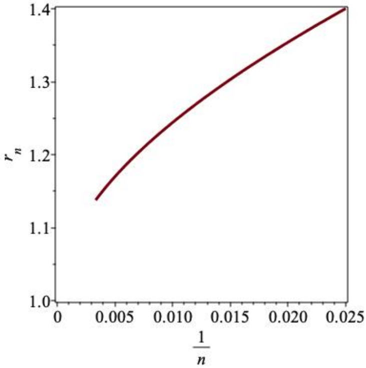
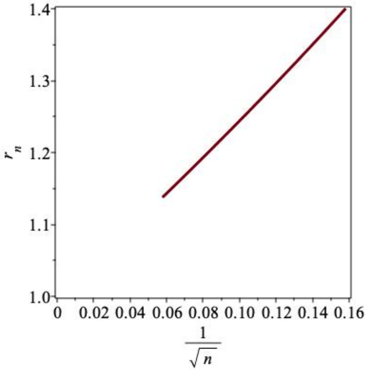
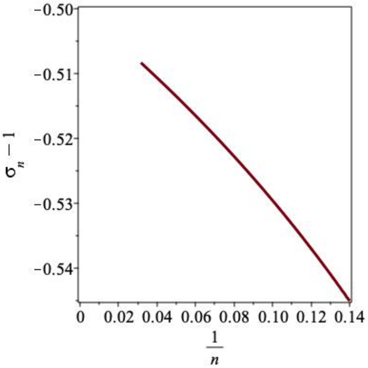
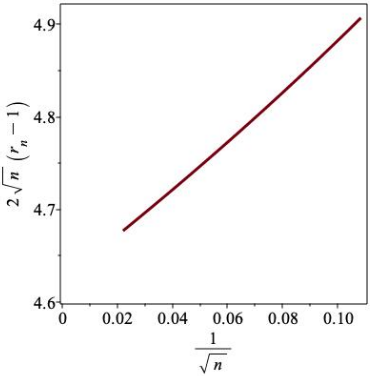




L-CONVEX CONTINUED.

- Recall $(r_n - 1) \sim \frac{\log \mu_1}{2\sqrt{n}}$,
- So $2\sqrt{n} \cdot (r_n - 1) \sim \log \mu_1$,





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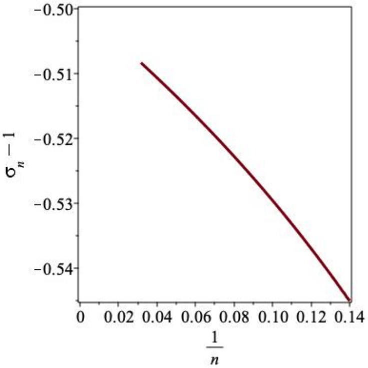

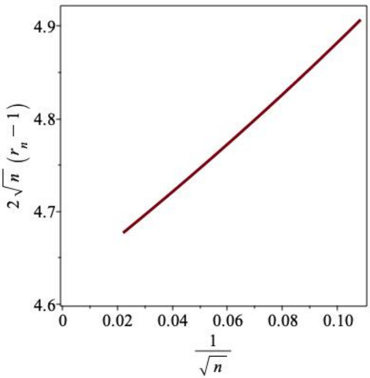

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- Plot implies $\log \mu_1 \approx 4.62$. Linear interpolation gives 4.624.
- Recall stack polyoms. grow as $\exp(2\pi\sqrt{n/3})$, so guess these grow as $\exp(\pi\sqrt{\alpha n})$.
- So $\pi\sqrt{\alpha} \approx 4.624$, implies $\alpha \approx 2.1664$. Guess $\alpha = 13/6$.
- So we have

$$l_n \sim C \cdot \frac{\exp(\pi\sqrt{13n/6})}{n^g}.$$

- To estimate g, set $m = n^2$, so

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This is of the form $C\mu^n n^g$, so we can use the usual ratio method.

- In this way, we find $g \approx -1.5$, so $l_n \sim C \cdot \frac{\exp(\pi\sqrt{13n/6})}{n^{3/2}}$.

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- To estimate C, we divide the known coefficients by $\frac{\exp(\pi\sqrt{13n/6})}{n^{3/2}}$ and extrapolate the sequence (Bulirsch-Stoer).
- In this way one finds $C \approx 0.0239385108214$.
- Try and identify this with PSLQ$(0.0239385108214, \sqrt{2}, 1)$ and one finds $C = 13\sqrt{2}/768$.
- So we conjecture

$$l_n \sim 13\frac{\exp(\pi\sqrt{13n/6})}{3 \cdot 2^{15/2} n^{3/2}}$$

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Counting Eulerian orientations: Logarithms.

- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree. In joint work with Andrew Elvey Price we generated 100 terms.
- Using series extension: 1000 further terms.
- Using the ratio method, we found a confluent logarithm.

$$E(x) \sim const.(1-\mu z)/\log(1-\mu z).$$

- We estimated $\mu \approx 12.56637$, which I recognised as 4π.
- Subsequent discussion with MBM led to the identification of this with a previously solved problem, and she and Andrew EP completed the proof of this.

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C'est tout. Merci